AN INTRODUCTION

TO THE

DIFFERENTIAL AND INTEGRAL CALCULUS

AND

DIFFERENTIAL EQUATIONS

BY

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PREFACE

This work professes to give a fairly full treatment of the more elementary parts of the Differential and Integral Calculus, together with a shorter treatment of Ordinary Differential Equations.

I have found it more convenient to separate the first two subjects than to take them together as is frequently done; but the student need not necessarily keep to the order of the book. It is generally admitted nowadays that, in learning the Calculus, integration should immediately follow differentiation; and therefore the student is advised to read, say, the first six chapters on Differentiation, and then proceed to the first two chapters of the Integral Calculus.

In both of these subjects I have made a point of giving a few practical applications as early as possible, in order that the beginner may have some idea of the uses to which the Calculus may be put. I have also introduced elementary curve-tracing as early as in the third chapter for the same reason.

A separate chapter has been devoted to the Hyperbolic Functions and their differentiation; this may be omitted by the beginner at first, though I have made free use of these functions in the Integral Calculus.

There are many questions which, although they may be treated by the Differential Calculus, can be treated equally well by algebraical or geometrical methods. I have therefore given the latter due prominence, so that the student may be able to make a comparison, and judge for himself as to which is the most suitable

for any particular question. He will thus have a fair appreciation of the place which the Differential Calculus holds among other mathematical methods.

Towards the end of the first section the subject of Curves and Curve Tracing has been treated as fully as space would allow. I have also included Envelopes, and their application to Evolutes and Caustics.

In the Integral Calculus I have collected into one chapter the various elementary methods of dealing with expressions which cannot be integrated at sight; among these methods I have included that of Integration by Parts. This is followed by the consideration of the standard integrals, and the routine methods of Partial Fractions and Successive Reduction.

Among the applications of the Integral Calculus, it will be seen that I have included the determination of Centroids and Moments of Inertia.

The third section deals with the elementary methods of solving Ordinary Differential Equations of the first and second orders. Partial Differential Equations are necessarily excluded, but enough of the subject has been given to enable the reader to apply it to the study of many problems in Physics; and, in fact, the requirements of the Physicist have been kept in view throughout the book.

Speaking generally, I may say that my aim has been to present each subject in a clear and simple manner, rather than to introduce any special novelties; and I have, at the same time, endeavoured to take advantage of every opportunity afforded me of investing the subject with interest.

I have to express my indebtedness to the various standard works on the Calculus from which information had been gained for teaching purposes, before I had any idea of preparing this book; and among these I must note specially the treatises of Todhunter, Williamson, and Edwards.

During the actual preparation of the book I have consulted several works, including, among others, Chrystal's *Algebra*, Frost's *Curve Tracing*, Besant's *Particle Dynamics*, De Morgan's large

treatise on the *Calculus*, and Greenhill's *Differential and Integral Calculus*. In respect to the third section, I am indebted to the works of Boole, Forsyth, and Murray.

The book contains a very large number of examples, which have been carefully selected and graduated. Many of these are taken from the examination papers of the London University, the Science and Art Department, and Woolwich Academy. In dealing with so many examples it is extremely difficult to avoid making slips; and, in spite of all care, I fear that many must still remain undetected. I shall feel grateful to any one who will inform me of those that he may have come across.

I have to thank various friends at this College for kindly help given to me at different times, and in particular my friend Mr. W. H. Everett, B.A., B.E., for reading several of the proof sheets, and making valuable corrections and suggestions.

Finally, I must not omit to gratefully acknowledge the courtesy shown to me by both publishers and printers during the time that the work has been in their hands.

F. GLANVILLE TAYLOR.

UNIVERSITY COLLEGE, NOTTINGHAM,
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PREFACE TO SECOND EDITION

In this edition, thanks to friends and correspondents, various corrections have been made, and I hope that the book is now practically free from error. No alteration has been made in the text, with the exception of p. 185, which has been recast.

Attention might be called to the Index at the end of the book.

F. G. T.

November, 1905.

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MISCELLANEOUS DEFINITIONS AND THEOREMS IN ALGEBRA AND TRIGONOMETRY

THE following can be obtained from *Hall and Knight's Higher Algebra*, and *Lock's* (or *Loney's*, or *Levett and Davison's*, etc.) *Trigonometry*, under the headings given in italics below :—

(1) ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$. [*Combinations and Permutations.*]

(2) An infinite series is *convergent* when the sum of the first n terms cannot numerically exceed some finite quantity, however great n may be; and *divergent* when the sum can be made numerically greater than any finite quantity, by taking n sufficiently great. [*Convergency and Divergency.*]

(3) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$; convergent if $x < 1$.

[*Binomial Theorem.*]

(4) $a = e^{\log a}$ (by definition of logarithm); $\log_a e = \frac{1}{\log_e a}$.

$$a^x = 1 + x\log a + \frac{x^2\overline{\log a}|^2}{2!} + \frac{x^3\overline{\log a}|^3}{3!} + \ldots;\ \text{always convergent.}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots;\ \text{always convergent.}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots;\ \text{convergent if } x < 1.$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots;\ \text{convergent if } x < 1.$$

$$\underset{n=\infty}{lt}\left(1 + \frac{1}{n}\right)^n = e.$$ [*Exponential and Logarithmic Theorems.*]

(5) $\operatorname{cosec}\theta + \cot\theta = \frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$.

$$\operatorname{cosec}\theta - \cot\theta = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}.$$

Changing θ into $\frac{\pi}{2} + \theta$ in the latter, we get

$$\sec\theta + \tan\theta = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

Also $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$; $\sqrt{\frac{1}{1}}$ $\tan\frac{\theta}{2}$.

[*Multiple and Submultiple Angles.*]

(6) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$. [*Relations between Sides and Angles of a Triangle.*]

(7) $\sin^{-1} a \pm \sin^{-1} b = \sin^{-1}(a\sqrt{1-b^2} \pm b\sqrt{1-a^2})$;

$\cos^{-1} a \pm \cos^{-1} b = \cos^{-1}(ab \mp \sqrt{1-a^2}.\sqrt{1-b^2})$;

$\tan^{-1} a \pm \tan^{-1} b = \tan^{-1}\frac{a \pm b}{1 \mp ab}$

[*Inverse Circular Functions.*]

(8) $\underset{\theta=0}{lt}\left(\frac{\sin\theta}{\theta}\right) = 1$; $\underset{x=0}{lt}\left(\frac{\sin^{-1}x}{x}\right) = 1$. [*Lock's Elem. Trig.*, p. 247. See also Art. 14, below.]

(9) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. [*Demoivre's Theorem.* See "Abbreviations" below.]

$$\left.\begin{aligned} \sin\theta &= \frac{e^{\theta i} - e^{-\theta i}}{2i} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \\ \cos\theta &= \frac{e^{\theta i} + e^{-\theta i}}{2} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \end{aligned}\right\} \text{convergent.}$$

[*Exponential Values of Sine and Cosine; Expansion of Sine and Cosine.*]

(10) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$; convergent. [*Gregorie's Series.*]

(11) If $f(x)$, any function of x, vanishes when $x = a$, then in general $x - a$ is a factor of $f(x)$.

If $f(x)$ vanishes when $x = 0$, then x is a factor. [*Hall and Knight's Elem. Alg.*, Art. 236, and their *Higher Algebra*, Art. 514, Cor., p. 433.]

ABBREVIATIONS.

$\equiv$ means "is identically equal to," or "stands for."

i „ $\sqrt{-1}$

$\approx$ „ "approximately equal to."

$+^{ve}$ „ "positive."

$-^{ve}$ „ "negative."

d.c. „ "differential coefficient."

(*p.i.*) „ "particular integral."

(*c.f.*) „ "complementary function."

An asterisk (*) is placed against articles or examples which may be omitted on a first reading.

DIFFERENTIAL CALCULUS

CHAPTER I.

PRELIMINARY ILLUSTRATIONS.

1. The differential calculus is a method by which we deal with *variable magnitudes or quantities*, i.e. quantities which are supposed to be in a state of gradual change or growth.

When two such quantities are so connected that a value given to one produces a definite corresponding value in the other (*e.g.* the side of a square, and its area), the latter is said to be a *function* of the former; and it is a fundamental part of the subject to consider how a *small change* in the value of one affects the other. Generally speaking, a small change in the one produces a small change in the other, and these changes, or *increments*, are compared by finding their ratio, *when both are indefinitely small.*

Before giving further definitions we shall illustrate the subject by a few simple examples.

2. Example 1.

Find the increase in the area of a square as compared with a (very small) increase in the length of a side.

Let x = the length of a side in feet, y = the area in square feet.

Then $y = x^2$.

Let $x + h$ be the increased value of x, and $y + k$ the correspondingly increased value of the area.

Then $y + k = (x + h)^2$

$$\therefore k = (x+h)^2 - x^2 = 2xh + h^2$$

$$\therefore k/h = 2x + h = 2x \text{ nearly, since } h \text{ is small compared with } x.$$

Hence the ratio of the increments is very nearly $2x$.

3. Now when h is so small as to be absolutely insignificant compared with x, the ratio k/h becomes as near to $2x$ as we please, so that *we may pass from an approximate result to an exact one* by imagining h (and therefore k) to dwindle down to nothing.

If we consider the square to be growing in size, as would appear to a person moving towards it, then at any instant k/h gives us a comparison of the rates at which x^2 and x are growing; for x^2 is increased by k in the same time that x is increased by h.

Or, again, consider a metallic bar of square section expanding equally in all directions, under the action of heat. The superficial expansion is k/y (*see note below*), i.e. $\dfrac{2xh}{x^2}$, or $\dfrac{2h}{x}$. The corresponding linear expansion is h/x. Hence, *the superficial expansion is twice the linear expansion.*

Again, k is the error produced in the computed area of a square, due to an error, h, in the estimated length of the side, y and x being their correct values. Now the percentage error in $x + h$ is $\dfrac{100h}{x}$, and the percentage error in $y + k$ is $\dfrac{100k}{y}$ or $2 \cdot \dfrac{100h}{x}$ (since k was $2xh$), or twice the percentage error in x.

Note.—Errors are always estimated by comparing them with (*i.e.* finding their ratio to) the correct value; also the "expansion" of a bar, referred to above, is the ratio of the actual increase in length, area, or volume, to the original value of these. The "*coefficient of expansion*" is the expansion for 1° C. rise of temperature.

4. Example 2.

Given that, when a body falls from rest under the action of gravity, the depth is proportional to the square of the time from rest: find the velocity at any instant.

Let s be the depth at the time t; then $s \propto t^2$, the actual formula being

$$s = \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

g being fixed while s and t are variable.

If at the time $t + h$, s has increased to $s + k$, then k represents the very small distance through which the body has fallen in the very small interval h, so that k/h is the mean velocity during the interval; also we have, instead of (1),

$$s + k = \tfrac{1}{2}g(t + h)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By subtraction,

$$\therefore \quad k = \tfrac{1}{2}g[(t + h)^2 - t^2] = \tfrac{1}{2}g(2ht + h^2).$$

$$\therefore \text{ mean velocity during the interval } h = \frac{k}{h} = \tfrac{1}{2}g(2t + h)$$
$$= gt + \tfrac{1}{2}gh.$$

Now, by diminishing the interval h until it becomes an instant of time, the mean velocity becomes the velocity at that instant. In this case the term $\frac{1}{2}gh$ vanishes, and we get

$$\text{vel. at time } t \text{ secs. after starting} = gt;$$

and it evidently increases with t in direct proportion.

5. Differential Coefficient.—The ratio k/h, in its ultimate form, is called the *differential coefficient* of the function y (or s) with respect to the variable x (or t); and the process of finding this differential coefficient is called *differentiation.*

NOTE 1.—The terms *derived function* and *derivative* are sometimes used instead of the above.

NOTE 2.—We shall often write *diff. co.* or *d.c.* for the above term.

6. Example 3.

Differentiate $(2x + 1)^3$ *with respect to* x.

Let $y = (2x + 1)^3$, and suppose x to increase to $x + h$, *in consequence of which* y increases to $y + k$; then

$$y + k = \{2(x + h) + 1\}^3 = \{(2x + 1) + 2h\}^3$$

$$\begin{aligned}\therefore\ k &= \{(2x + 1) + 2h\}^3 - (2x + 1)^3 \\ &= [(2x + 1)^3 + 3(2x + 1)^2 . 2h + 3(2x + 1) . 4h^2 + 8h^3] - (2x + 1)^3 \\ &= 6h(2x + 1)^2 + 4h^2(6x + 3 + 2h).\end{aligned}$$

$\therefore\ k/h = 6(2x + 1)^2 +$ a term which vanishes with h.

Hence the required *d.c.* is $6(2x + 1)^2$.

EXAMPLES I.

1. Find the percentage increase in the volume of a cube as compared with that of a side, when the increase is indefinitely small.

2. Show that the coefficient of cubical expansion is three times that of linear expansion for the same substance, supposed perfectly homogeneous.

3. Show that a small percentage error, made in measuring the diameter of a sphere, produces a percentage error in the volume equal to three times the former.

4. Use the last question to find what percentage error will be produced in the computed diameter of a globe whose volume (measured, say, by the weight of water which it will hold) is incorrect by x per cent.

5. Give a geometrical solution of Ex. 1, Art. 2.

6. It is required to construct a rod of uniform thickness, such that the weight of any length measured from one end is proportional to the square of that length. Show that the density at any point must vary as the length.

7. Show that if the side of a square increase at a uniform rate, the area increases at an increasing rate, which is proportional to the length of the side.

8. Show that if the side of a cube increase at a uniform rate, the volume increases at an increasing rate which is proportional to the area of a face.

9. Differentiate with respect to x:—

(1) $3x - 5$.	(2) $(x + 1)^2$.	(3) $(2x - 3)^2$.
(4) $x^3 - 2x$.	(5) $(5x - 2)^3$.	(6) $(ax + b)^3$.

10. Given that $\pi = 3\frac{1}{7}$, true to 1 in 3000: about what error should we make in substituting $3\frac{1}{7}$ for π in (1) π^2, (2) $\sqrt{\pi}$, (3) π^3, (4) $\sqrt[3]{\pi}$?

ANSWERS.

1. As 3 to 1. **4.** $\frac{1}{3}x$ %.

9. 3; $2(x + 1)$; $4(2x - 3)$; $3x^2 - 2$; $15(5x - 2)^2$; $3a(ax + b)^2$.

10. $\frac{1}{1500}$; $\frac{1}{6000}$; $\frac{1}{1000}$; $\frac{1}{9000}$.

CHAPTER II.

FUNCTIONS AND LIMITS.

7. One variable quantity, or *variable*, is said to be a *function* of another when, any value being given to the latter, the former assumes a definite corresponding value. Thus (1) the area of a square is a function of its side; and (2) the attraction of two given masses on each other is a function of the distance between them. For in (1), if we fix on a length of side the area will be a definite area—that is, will not be one thing at one time and another thing at another time; and similarly for (2).

A quantity may be a function of several variables; thus, the pitch of a note uttered by a stretched string (as measured by the *number* of vibrations per second) is a function of the *tension*, the *mass per unit length*, and the *length*; the actual law being shown by the equation

$$n = \frac{1}{2l}\sqrt{\frac{t}{m}}$$

the letters being the initials of the terms in italics.

NOTE.—Any mathematical expression involving a variable x is called a function of x; similarly for several variables.

8. Functions are divided into two classes, *algebraical* and *transcendental.*

An *algebraical function* is one which involves only the first four rules, with powers and roots; *e.g.* $2x^2 - 3x + 4$, $x(x - 2)$, $\frac{\sqrt{x-1}}{x^2+4}$.

A *transcendental function* is one which involves special definitions; *e.g.* 2^x (in which x itself is an index), $\log x$, $\sin x$, $\sinh^{-1} x$, $\mathrm{gd}\, x$, etc.; or *any* compounds of these, *e.g.* $\log(\sin x)$.

A quantity whose value is fixed permanently or temporarily is called a *constant*.

Sometimes the same quantity is regarded as constant at one time and variable at another. Thus, in considering one of a system, or *family*, of circles passing through two points, the radius will be variable, though constant for each circle.

9. When one variable y is expressed in terms of another, x, in such a way that y appears *singly and alone* on the left-hand side of the equation, and not on the right-hand side, then y is said to be an *explicit function* of x; as in the equations $y = 2x(x^2 - 1)$; $y = x\sin^{-1}x + \dfrac{1}{\sqrt{1-x^2}}$. Otherwise, when x and y are together on the same side of the equation, y is said to be an *implicit function* of x; as in the equations $xy - x^2(y^2 - 1) = 2y + 1$; $y\log x + x\log(y^2 - 1) = xy$.

10. Notation.—Any function of x is usually expressed by the notation $f(x)$, $\phi(x)$, $\chi(x)$, $\psi(x)$, $F(x)$, etc.,

or fx, ϕx, χx, ψx, Fx, etc.,

when there is no likelihood of confusion.

Any function of several variables $x_1, x_2, x_3, \ldots$ will be written $f(x_1, x_2, x_3 \ldots)$, etc.

NOTE.—*If the same letter is used throughout a statement, the same function is meant.* Thus if $f(x) = \sin x$, then $f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$; also $f(0) = 0$.

11. Dependent and Independent Variables—Graphs of Functions.—If y be a function of x, the latter is called the *independent*, and the former the *dependent, variable*; since while we choose *any* value of x, y will have a value *depending* on that given to x.

In co-ordinate geometry an equation connecting two variables, x and y, is represented by a plane curve called the *locus of the equation*. If y be expressed as an explicit function of x, then the curve $y = f(x)$ is called the *graph of the function* $f(x)$.

12. When the equation is in the latter form, the graph is generally easy to trace, since by giving x in succession a number of values chosen at will, we can obtain a like number of corresponding values of y, and so plot a series of points forming a curve.

We might, however, choose the values of y at will; when, by solving the equation $y = f(x)$, we can obtain the corresponding values of x. In this case we should be making y the independent, and x the dependent, variable. And we should then regard x as a function of y. Hence, *if* x *and* y *be connected by an equation, each is a function of the other, and the choice of the independent variable is optional.*

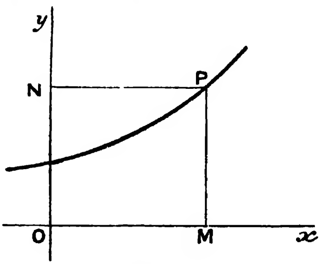

FIG. 1.

The actual finding of x from y may be difficult or impossible by algebraical methods. But if we plot the curve by making x the independent variable, we can use the graph to obtain x from y by direct measurement. Thus, given y, we can find x by measuring a distance ON, equal to y, along Oy, and drawing NP parallel to Ox to meet the curve in P. Then NP is evidently equal to OM or x.

13. The Use of Graphs in Physics.

Whenever a law exists between two physical quantities, the law can be exhibited by means of a graph or curve, by taking for abscissæ the different values of one of the quantities, and for ordinates the corresponding values of the other quantity.

We give a few examples in physics.

1. *Boyle's Law* is, that for a constant temperature the pressure of a gas is inversely proportional to its volume, or $pv =$ a constant (c, say). The equation $xy = c$ is a rectangular hyperbola.

2. *Time of oscillation of a pendulum.* If T be the time, and l the length, we have $T = 2\pi\sqrt{\dfrac{l}{g}}$.

Since π and g are constant we may write $T^2 = Al$, where $A = \frac{4\pi^2}{g}$. The curve $y^2 = Ax$ is a parabola.

3. *Charles's Law*, connecting the volume of a gas, under constant pressure, with the temperature, is $V_t = V_0(1 + \alpha t)$, t being the number of degrees C. above zero. V_0 and α are *constant*, being the volume at 0° C. and the coefficient of expansion respectively, while V_t is the volume at t° C., and varies with t.

The curve $y = V_0(1 + \alpha x)$ is a straight line.

14. Limit of a Function.—Referring to Art. 5, it will be seen that we have used the expression, "in its ultimate form," as something to which the ratio k/h approaches as h and k diminish indefinitely. We now give two formal definitions.

Def. 1.—If y be a function of x, the *limiting value* or *limit* of y for a given value (a) of x, is that value towards which y continually approaches, and from which it may be made to differ by less than any assignable quantity however small, as x approaches a.

Thus if a regular polygon be inscribed in, or circumscribed about, a circle, the limit of the perimeter of the polygon, as the number of sides increases indefinitely, is the circumference of the circle.

Def. 2.—When two related quantities are both diminishing (or both increasing) indefinitely, their *limiting ratio* is that value to which their ratio continually approaches, and from which it can be made to differ by less than any assignable quantity however small, by sufficiently diminishing (or increasing) each quantity.

Thus the limiting ratio of sin θ and θ, when θ (and therefore sin θ) diminishes indefinitely is unity. This is written

$$\underset{\theta=0}{lt}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

If we were to make $\theta = 0$ at once, instead of approaching it,

we should have the fraction $\frac{0}{0}$, which is meaningless. We find, however, that as θ diminishes $\dfrac{\sin\theta}{\theta}$ approaches indefinitely near to 1; we therefore call 1 the limiting value.

15. It is important to notice that two quantities can only be said to be equal when their difference is infinitely small in comparison with either quantity.

Thus suppose $$x - a = h, \text{ then}$$
$$\frac{x}{a} = 1 + \frac{h}{a}.$$

Now, so long as x and a are finite, we may put $x/a = 1$, when h vanishes; but suppose that when h diminishes, x and a also diminish, then we can only put $x/a = 1$, or $x = a$, *when the ratio of* h *to* a *diminishes indefinitely.*

For example, the chord of a circle and the arc which it cuts off become ultimately equal when both are indefinitely diminished, but only because the ratio of their difference to either of them becomes indefinitely diminished also.

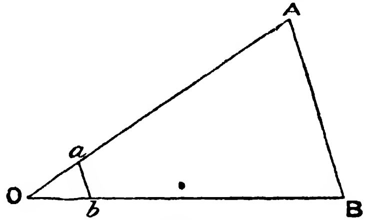

FIG. 2.

But suppose OAB to be a *scalene* triangle and ab a straight line parallel to AB, which moves towards O. Then $Oa - Ob$ diminishes without limit, yet Oa and Ob will not be ultimately equal, since $\dfrac{Oa - Ob}{Oa} = \dfrac{OA - OB}{OA}$ which is finite. In fact, the ratio of Oa to Ob is always the same, viz. equal to the ratio of OA to OB.

Similarly, if x and a *increase* indefinitely, then if h remain finite, h/a diminishes indefinitely, and ultimately we have $x/a = 1$. In this sense we may put $x = a$, the difference h being absolutely insignificant compared with x and a. It appears from the above remarks that the true criterion of the equality of two quantities *in all cases* is, not that their difference is infinitely small, but that it is infinitely small in comparison with either of them; or, that their ratio is unity.

16. Rational Algebraical Functions.†—

Suppose we wish to find $\underset{x=a}{lt}\dfrac{f(x)}{\phi(x)}$, where $f(x)$ and $\phi(x)$ are rational algebraical functions, each of which vanishes when $x = a$.

By (11) [see *Miscellaneous Theorems* above], $x-a$ is a common factor; hence we can divide above and below by $x - a$ *before proceeding to make* $x = a$, and the ratio will be unaltered; moreover, generally speaking, the quotients will no longer vanish when $x = a$. If they do, then $x - a$ will again be a common factor which can be again removed.

Ex. 1 $\underset{x=a}{lt}\dfrac{x^3-a^3}{x^2-a^2} = lt\dfrac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)} = lt\dfrac{x^2+ax+a^2}{x+a} = \dfrac{3a^2}{2a} = \dfrac{3}{2}a.$

NOTE.—Here $x - a$ is called the *vanishing factor*.

Ex. 2. Find $\underset{x=0}{lt}\dfrac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x(\sqrt{a+x}-\sqrt{a-x})}$.

We can rationalize both numerator and denominator by multiplying above and below by the conjugate surd in each case.

Thus we shall get

$$\frac{(a^2+x^2)-(a^2-x^2)}{(a+x)-(a-x)}\cdot\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}\cdot\frac{1}{x} = \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}},$$

having divided out by $2x^2$.

Now put $x = 0$, and we get $\dfrac{2\sqrt{a}}{2a} = \dfrac{1}{\sqrt{a}}$, as the limit.

Ex. 3. Find $\underset{x=1}{lt}\dfrac{x^m-1}{x-1}$, where m is any quantity.

Put $x = 1 + h$, then

$$\frac{x^m-1}{x-1} = \frac{(1+h)^m-1}{h} = \frac{(1+mh+\text{higher powers of } h)-1}{h}$$
$$= m + \text{terms containing } h \text{ as a factor.}$$

† A *rational algebraical function* of x is one which contains only integral powers of x, or of expressions involving x, as $x^3+\dfrac{a^3}{x^3}+\dfrac{x^2(x+a)}{a^2-x^2}$. When, in addition to this, x and its powers do not appear as fractions, the function is called a *rational integral algebraical function*; as, $a+\dfrac{b}{c}x-\dfrac{c-d}{e+f}x^2$; and $p_0+p_1x+p_2x^2+\ldots+p_nx^n$, where p_0, p_1, etc., do not contain x.

The expansion of $(1+h)^m$ is convergent if $h<1$; hence when $x=1$, *i.e.* when $h=0$, the limiting value becomes m.

Ex. 4. Find $\underset{x=\infty}{lt}\dfrac{ax^3+bx^2+cx+d}{a'x^3+b'x^2+c'x+d'}$.

Putting $x=1/y$, the expression, *which is of the same degree above and below*, becomes, on clearing of fractions,

$$\underset{y=0}{lt}\frac{a+by+cy^2+dy^3}{a'+b'y+c'y^2+d'y^3}=\frac{a}{a'}.$$

We shall consider the subject further in the chapter on Indeterminate Forms.

EXAMPLES

1. If $f(x)=\dfrac{1}{1-x}$, prove that $f\left(\dfrac{1}{1+x}\right)=\dfrac{1+x}{x}$, and $f\left(\dfrac{1+x}{x}\right)=-x$.

2. If $fx=\dfrac{x+1}{x-1}$, and $f(fx)$ is denoted by f^2x, $f[f(fx)]$ by f^3x, etc., show that $f^2x=x$, $f^3x=fx$, $f^4x=x$, etc.

3. If $f(x)=x^3+3x^2-2x+1$, show that $\dfrac{f(x)-f(2)}{x-2}=x^2+5x+8$.

4. Find the limits of the following functions:—

(1) $\dfrac{x^5-a^5}{x^3-a^3}\,(x=a)$. (2) $\dfrac{1-\cos 2x}{\sin 2x}\,(x=0)$.

(3) $\dfrac{x-2}{x+1}\,(x=0$, and $x=\infty)$. (4) $\dfrac{(x-1)(2-3x)}{(3x+2)(1+2x)}\,(x=\infty)$.

(5) $\dfrac{(x+4)^3-(x-8)^2}{x(x-2)(x-4)}\,(x=0)$. (6) $\dfrac{2x^2+3}{x^2-x}+\dfrac{3(x^2+1)}{2x^2+x}\,(x=0)$.

(7) $\dfrac{x^3+2x+1}{x+1}-\dfrac{4x^4-3x^2+1}{(2x+1)^2}\,(x=\infty)$.

(8) $\dfrac{3x^3-x-2}{2x^3+3x^2-5}\,(x=1)$. (9) $\dfrac{x^4-4x+3}{x^4+2x^3-5x^2+2}\,(x=1)$.

(10) $\dfrac{x^m-a^m}{x^n-a^n}\,(x=a)$. (11) $\dfrac{\sqrt{x^2+a^2}-\sqrt{2a^2}}{\sqrt{x+a}-\sqrt{2a}}\,(x=a)$.

(12) $\dfrac{(x+a)^{\frac{1}{2}}-a^{\frac{1}{2}}}{x}\,(x=0)$. (13) $\dfrac{x}{(x+a)^{\frac{1}{3}}-a^{\frac{1}{3}}}\,(x=0)$.

5. Find the value of $\dfrac{x^3 - 2^3}{x^2 - 2^2}$ when $x = 2{\cdot}001$ and when $x = 2{\cdot}0001$; find also the limit when $x = 2$.

6. Through a point, P, two chords are drawn parallel to two given straight lines, to meet a circle in AB and CD respectively. Prove that as P moves towards, and ultimately lies on, the circumference, the ratio of the vanishing segments PA, PC becomes equal to the ratio of PD to PB.

ANSWERS.

4. (1) $\frac{7}{3}a^2$. (2) 0. (3) -2, 1. (4) $-\frac{1}{2}$. (5) 8. (6) -9.

(7) 3. (8) $\frac{2}{3}$. (9) $\frac{6}{7}$. (10) $\dfrac{m}{n}a^{m-n}$. (11) $2\sqrt{a}$. (12) $\dfrac{1}{2\sqrt{a}}$.

(13) $3a^{\frac{2}{3}}$. See footnote in Examples III.

5. $3{\cdot}00075\ldots$; $3{\cdot}000075\ldots$; 3.

CHAPTER III.

NOTATION—DIFFERENTIALS—GRAPHS.

17. Notation.—We shall now express in quite general terms the d.c. of a given function, the statement of which may be taken as a formal definition.

Let $y = f(x)$, the given function of x.

It is usual to denote by Δx, δx, or h, a small but finite increase in the value of x, called the *increment of x*; and by Δy, δy, or k respectively, the consequent increase in y, called the *increment of y*. In practice we shall generally use h and k, as they are the shortest.

Now $y + \Delta y$ is the same function of $x + \Delta x$ as y is of x,

therefore $$y + \Delta y = f(x + \Delta x) \text{ or } f(x + h)$$

and $$y = f(x),$$

$$\therefore \Delta y = f(x + h) - f(x);$$

and since $$\Delta x \equiv h$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{h}.$$

In the limit when Δx (or h), and therefore Δy (or k), have vanished altogether, these expressions become the diff. co. of y with respect to x, the notation for which is $\frac{dy}{dx}$, or dy/dx.

Hence—

$$\frac{dy}{dx} = \underset{h=0}{lt} \frac{f(x + h) - f(x)}{h}.$$

Ex. If $y = x^2$, we have seen that the d.c. is $2x$; this is written $\frac{dy}{dx} = 2x$.

18. Other notations for $\frac{dy}{dx}$ are $f'(x)$, $\frac{d.f(x)}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}\cdot f(x)$, $\frac{d}{dx}\cdot y$, p, y', y_1, $\dot{y}$, Dy, etc. Each has its advantages as will be seen hereafter. Thus y', y_1, $\dot{y}$, Dy have the advantage of brevity, but cannot be used if there are two or more independent variables. In the case of $\dot{y}$, time is usually the independent variable; d/dx is usually prefixed to a compound expression; p is used in differential equations, as well as Dy; and so on.

The symbols d/dx or D placed before y or $f(x)$ denote the *operation* of differentiating it with respect to x, and either symbol (the latter being generally used) is termed an *operator*. [See Art. 509.]

Examples.

1. If $f(x) \equiv x^2$, then $f'(x) = 2x$.
2. If $y = x^2$, $\therefore y_1 = 2x$; or $y' = 2x$.
3. Again, $\frac{d.x^2}{dx} = 2x$; or $\frac{d}{dx}\cdot x^2 = 2x$.
4. $\dot{s} \equiv \frac{ds}{dt}$.
5. $\frac{d}{dt}\left(\frac{1}{2}gt^2\right) = gt$. [Art. 4.]

19. Differential Coefficients and Differentials.—If we regard Δy and Δx *as having vanished altogether*, then dy/dx is no longer the ratio of two existent quantities, but *a single quantity*,—the limit, in fact, towards which $\Delta y/\Delta x$ has approached and finally attained. We may, however, regard Δy and Δx as infinitely small, or *infinitesimal*, but yet existent. [See Art. 21.] They are then called the *differentials* of y and x respectively, and are denoted by dy and dx.

The ratio of dy to dx differs from the limiting ratio, or differential coefficient, by a quantity which may be made less than any assignable quantity, however small; hence we make an *absolutely insignificant error* in saying that

$$dy \div dx = \text{the single symbol } \frac{dy}{dx}.$$

In other words, we may regard dy/dx either as the ratio of two infinitely small quantities, or as a limiting ratio as defined in Art. 14.

Ex. If $y = x^2$, then $\dfrac{dy}{dx} = 2x$, $\therefore dy = 2xdx$;

or we may write $d(x^2) = 2x \,.\, dx$.

Generally, *the differential of a function of a variable is the product of its differential coefficient with respect to the variable, into the differential of the variable.*

Or, $$d\,.f(x) = f'(x)\,.\,dx.$$

Note 1.—Δy and Δx (or δy and δx) are used to denote small but finite increments; dy and dx to denote infinitely small increments.

Note 2.—If x be incorrect by a *small* error δx, then the error produced in $f(x)$ is approximately $f'(x)\,.\,\delta x$.

20. Geometrical Illustration.—Let P be some point on a curve whose equation is $y = f(x)$.

If (x, y) be taken as the co-ordinates of P, those of Q, a point near to P, may be denoted by $(x + \Delta x,\ y + \Delta y)$, so that $PS = MN = \Delta x$, $QS = \Delta y$.

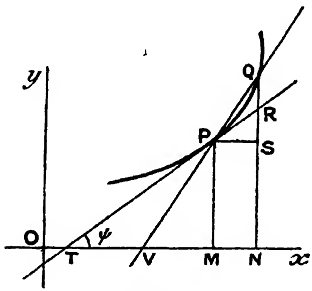

Fig. 3.

Since Q and P are on the same curve, QN is the same function of ON as PM is of OM, so that—

$QN = f(ON)$, or $y + \Delta y = f(x + \Delta x)$,
$PM = f(OM)$, or $y = f(x)$;

whence, at the point P,

$$\frac{dy}{dx} = \underset{\Delta x = 0}{lt}\frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ as before.}$$

Now $$\frac{\Delta y}{\Delta x} = \frac{QS}{PS} = \tan QPS = \tan PVx,$$

PVx being the inclination of the *secant* through P to the axis of x. As Q moves towards P, both Δy and Δx diminish; and in the limit, when Q has coincided with P, *i.e.* when the *secant*

through P has become the *tangent* at P, Δy and Δx have vanished altogether, and the ratio $\Delta y/\Delta x$ has reached the limiting value dy/dx, while the angle PVx is now the angle, PTx, *between the tangent at P and the axis of x*.

Hence at any point (x, y) of the curve $y = f(x)$, $dy/dx = \tan\psi$, where ψ is the inclination of the tangent at P to the axis of x.

21. There are two ways (among others) of regarding the tangent at a point:—

(1) As the secant through two infinitely near, or (as it is sometimes expressed) two consecutive points on a curve.

(2) As the limit of a secant through two points when they have become coincident.

In (1) the direction of the line is perfectly determinate, since there are two distinct points.

In (2) the two points are ultimately coincident—that is, have become *one* point. And through a single point an infinite number of straight lines can be drawn. Hence, unless we consider the relative positions of the two points *just before they have coincided*, or the path along which the second point approaches the first, we shall have nothing to tell us in what direction to draw the straight line through them when they are coincident.

But having ascertained the position to which the line has settled down as the two points are about to coincide, there is nothing to prevent us from considering the points as absolutely coincident.

The indeterminateness referred to above corresponds exactly to the result of putting $h = 0$ in the expression $\dfrac{f(x+h) - f(x)}{h}$ without having examined its value *just before* h has vanished, *i.e.* without having examined the function $f(x)$.

22. Again, the tangent at P gives the direction of the curve at P; that is, the direction in which a particle moving along the curve from Q to P would move if it were suddenly released at the point P and allowed (by Newton's first law) to move in a straight line. It is also, of course, the direction in which the particle would be moving at the *instant* it coincided with P. This must be completely determinate, for a particle can only move in one absolute direction at a time. We are assuming that only one branch of the curve passes through P.

23. Velocity.—Let s ft. be the distance of a moving particle from a given point on its path, t secs. after passing through the given point. If the path be curvilinear, s is the distance measured along the curve. If in the small interval of time Δt the particle move through the small distance Δs, then

$$\frac{\Delta s}{\Delta t} = \text{mean velocity during the interval } \Delta t.$$

Hence ds/dt is the velocity at the time t, as explained in Art 4. It can be similarly shown that if v be the velocity at the time t, dv/dt is the acceleration at that time.

24. Referring to the figure in Art. 20, if we imagine P to be moving along the curve, then dx/dt is the time-rate of increase of x, and dy/dt is the time-rate of increase of y.

Also, since dt is the same in both cases,

$$\frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{dy}{dx}. \quad \text{[See Art. 38, Cor.]}$$

Hence dy/dx is the rate of increase of y as compared with that of x; or the rate at which the function of x increases, compared with the rate at which x increases. It is also the ratio of the vertical to the horizontal velocity of P.

NOTE 1.—The steepness of the curve at any point gives us an idea of the vertical, as compared with the horizontal, velocity; for $\dfrac{dy}{dt}\bigg/\dfrac{dx}{dt} = \dfrac{dy}{dx} = \tan\psi$, which increases as the curve gets steeper.

NOTE 2.—The case in which dy/dx is negative will be discussed hereafter (Art. 113), but at present we may note that in this case $\tan\psi$ will be negative, hence the curve will be in a *downward* direction.

25. Examples.

We add examples of differentiation by first principles.

Ex. 1. Let $y = \sqrt{x}$.

$$\text{Then } k = \sqrt{x+h} - \sqrt{x}.$$

$$\therefore \frac{k}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

To find the factor h in the numerator (see Art. 16), we may either expand by the Binomial Theorem, or rationalize by multiplying above and below by the conjugate surd $\sqrt{x+h}+\sqrt{x}$.

Adopting the latter method, we get

$$\frac{k}{h}=\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

$$\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{x}}.$$

Ex. 2. $y=\dfrac{x^2-2x-3}{x+4}$.

Here $y=x-6+\dfrac{21}{x+4}$;

$$\therefore y+k=x+h-6+\frac{21}{x+h+4}.$$

$$\therefore k=h+21\left\{\frac{1}{x+h+4}-\frac{1}{x+4}\right\}=h-\frac{21h}{(x+4)(x+h+4)},$$

$$\therefore \frac{dy}{dx}=\underset{h=0}{Lt}\frac{k}{h}=1-\frac{21}{(x+4)^2}=\frac{x^2+8x-5}{(x+4)^2}.$$

Ex. 3. $y=\sqrt{\dfrac{2-x}{x+3}}$.

$$k=\sqrt{\frac{2-x-h}{x+h+3}}-\sqrt{\frac{2-x}{x+3}}$$

$$=\frac{\sqrt{(2-x-h)(x+3)}-\sqrt{(x+h+3)(2-x)}}{\sqrt{x+h+3}\,\sqrt{x+3}}$$

$$=\frac{\{(2-x)-h\}(x+3)-\{(x+3)+h\}(2-x)}{\sqrt{(x+h+3)(x+3)}\{\sqrt{(2-x-h)(x+3)}+\sqrt{(x+h+3)(2-x)}\}}$$

The numerator $=-h\{(x+3)+(2-x)\}=-5h$.

Hence, dividing by h and putting $h=0$, we get

$$\frac{dy}{dx}=\frac{-5}{2(x+3)\sqrt{(2-x)(x+3)}}=-\frac{5}{2(2-x)^{\frac{1}{2}}(x+3)^{\frac{3}{2}}}.$$

EXAMPLES III.

Differentiate from first principles :—

1. $2x^2 + 3x + 4.$ 2. $3x^3 - x + 4.$ 3. $(3x - 5)^3.$

4. $(2x - 3)(x + 4)^2.$ 5. $\dfrac{2}{x}.$ 6. $\dfrac{2}{3x - 4}.$

7. $x + \dfrac{1}{x}.$ 8. $\dfrac{x - 2}{x + 3}.$ 9. $\dfrac{x^2 + 3x + 5}{x + 3}.$

10. $\sqrt{ax}.$ 11. $\sqrt{x - a}.$ 12. $\sqrt{1 - x^2}.$

13. $\sqrt[3]{x}.$† 14. $x^{\frac{3}{2}}.$ 15. $\sqrt{\dfrac{x + 1}{x - 1}}.$

16. $\sqrt{\dfrac{2x - 3}{2x + 3}}.$ 17. $x + \sqrt{a^2 + x^2}.$ 18. $\dfrac{1}{\sqrt{a^2 + x^2} - x}.$

19. $(\sqrt{x} - 3)(3\sqrt{x} + 7).$ 20. $\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}.$

Answers.

1. $4x + 3.$ 2. $9x^2 - 1.$ 3. $9(3x - 5)^2.$ 4. $2(x + 4)(3x + 1).$

5. $-\dfrac{2}{x^2}.$ 6. $-\dfrac{6}{(3x - 4)^2}.$ 7. $1 - \dfrac{1}{x^2}.$ 8. $\dfrac{5}{(x + 3)^2}.$

9. $\dfrac{x^2 + 6x + 4}{(x + 3)^2}.$ 10 $\dfrac{1}{2}\sqrt{\dfrac{a}{x}}.$ 11. $\dfrac{1}{2\sqrt{x - a}}.$ 12. $-\dfrac{x}{\sqrt{1 - x^2}}.$

13. $\dfrac{1}{3\sqrt[3]{x^2}}.$ 14. $\frac{3}{2}x^{\frac{1}{2}}.$ 15. $-\dfrac{1}{(x + 1)^{\frac{1}{2}}(x - 1)^{\frac{3}{2}}}.$

16. $\dfrac{6}{(2x - 3)^{\frac{1}{2}}(2x + 3)^{\frac{3}{2}}}.$ 17. $\dfrac{x + \sqrt{a^2 + x^2}}{\sqrt{a^2 + x^2}}.$

18. $\dfrac{x + \sqrt{a^2 + x^2}}{a^2\sqrt{a^2 + x^2}}.$ 19. $3 - \dfrac{1}{\sqrt{x}}.$ 20. $\dfrac{1}{\sqrt{x}(1 - \sqrt{x})^2}.$

---

† We have—

$$\frac{k}{h} = \frac{\sqrt[3]{x + h} - \sqrt[3]{x}}{h}.$$

To rationalize $\sqrt[3]{a} - \sqrt[3]{b}$, we must choose such a factor as will give the difference of the *cubes* of $\sqrt[3]{a}$ and $\sqrt[3]{b}$; the factor is—

$$a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}},\ \textit{i.e.}\ (x + h)^{\frac{2}{3}} + (x + h)^{\frac{1}{3}}.x^{\frac{1}{3}} + x^{\frac{2}{3}}.$$

Multiplying above and below by this we get—

$$\frac{k}{h} = \frac{(x + h) - x}{h[(x + h)^{\frac{2}{3}} + (x + h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}]};\ \text{etc.}$$

26. Graphs of Functions—Elementary Curve Tracing

The great advantage of obtaining the graph of a function is that we can see at a glance how the latter varies from point to point, as we vary x.

We shall give a few simple examples of tracing curves by *plotting points*, i.e. by giving x different values, and finding the corresponding values of y. We shall also find the value of dy/dx at these points, and so determine the direction of the curve, as given by $\tan\psi$. For the construction of the angle, see *Lock's Elem. Trig.*, p. 80, Ex. 3.

We give these values in tabular form ; the arrows in Ex. 1 show the directions of the tangents.

Ex. 1. *Trace $xy = 1$, or $y = 1/x$.*

The student can easily show that $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$, which is $-^{ve}$ for *all* values of x.

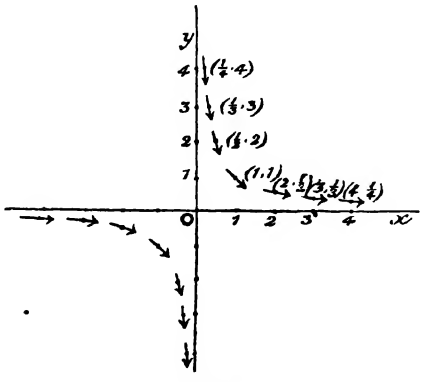

FIG. 4.

x	y	$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$	ψ
∞	0	0	0
4	$\frac{1}{4}$	$-\frac{1}{16}$	$-^{ve}$
3	$\frac{1}{3}$	$-\frac{1}{9}$	,,
2	$\frac{1}{2}$	$-\frac{1}{4}$	,,
1	1	-1	,,
$\frac{1}{2}$	2	-4	,,
$\frac{1}{3}$	3	-9	,,
$\frac{1}{4}$	4	-16	,,
0	∞	∞	$-\frac{\pi}{2}$
$-\frac{1}{4}$	-4	-16	$-^{ve}$
$-\frac{1}{3}$	-3	-9	,,
$-\frac{1}{2}$	-2	-4	,,
-1	-1	-1	,,
-2	$-\frac{1}{2}$	$-\frac{1}{4}$	,,
-3	$-\frac{1}{3}$	$-\frac{1}{9}$	,,
-4	$-\frac{1}{4}$	$-\frac{1}{16}$	,,
$-\infty$	0	0	0

As x varies gradually, y or $1/x$ varies gradually, as also $-1/x^2$ or $\tan\psi$ (and therefore ψ). Hence we have a continuous curve through the above points, as in the second figure (Fig. 5).

This curve is of course a rectangular hyperbola whose asymptotes are the axes of co-ordinates.

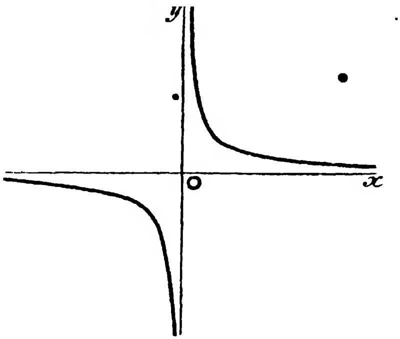

FIG 5

Ex. 2. *Obtain the general form of the curve* $y = x(x-1)(x-2)$.

We have $y = x^3 - 3x^2 + 2x$;

$$\frac{dy}{dx} = 3x^2 - 6x + 2 = 3(x-1)^2 - 1$$

$$= (\sqrt{3}x - \sqrt{3} + 1)(\sqrt{3}x - \sqrt{3} - 1)$$

$$= 3(x - 0{\cdot}42)(x - 1{\cdot}6) \text{ nearly.}$$

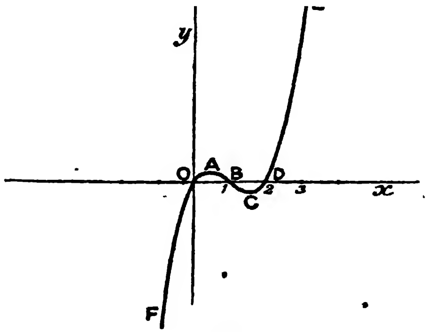

FIG. 6.

The special points to notice are the values of x which make y and dy/dx vanish; and also their signs for other values of x.

Thus, when $x = 1{\cdot}6$, $\dfrac{dy}{dx} = 0$, *i.e.* $\psi = 0$, or the tangent is parallel to the axis of x at this point. Here also $x(x-1)(x-2)$, and therefore y, is negative; *i.e.* the curve is below the axis of x at this point.

Similarly for other points.

When the general form of the curve has been obtained we may, by giving x special values, obtain those of y to correspond, and so form a more correct figure.

		$\frac{dy}{dx}$
∞	∞	
>2	$+$	$+$ (F in figure)
2	0	$+$ (D)
$1{\cdot}6$	$-$	0 (C)
$<1{\cdot}6$ and >1	$-$	
1	0	$-$ (B)
$0{\cdot}42$	$+$	0 (A)
0	0	$+$ (O)
$-$		$+$ (F)
$-\infty$		$+\infty$

Ex. 3. *Trace* $y^2 = x^3$.

We have $y = \pm x^{\frac{3}{2}}$: hence any value of x gives two equal and opposite values to y, which shows that the curve is symmetrical with respect to the axis of x. Also x cannot be negative, for real values of y.

$\dfrac{dy}{dx}$ will be found to be $\pm \dfrac{3}{2}x^{\frac{1}{2}}$. Therefore $\tan\psi = 0$ at the origin, and increases numerically with x.

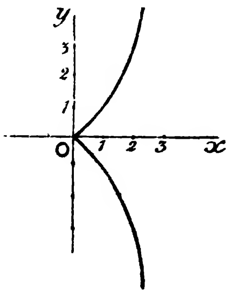

FIG. 7.

x	y	$\frac{dy}{dx}$
∞	∞	∞
1	1	$\frac{3}{2}$
0	0	0
$-$	imag^y.	imag^y.

EXAMPLES IV.

[The student should only take a few of these on a first reading, or at least until he has read the next chapter. A very few should be traced carefully; in the others, only the general form of the curve need be made, distances being roughly measured by the eye.]

Trace the following curves, finding dy/dx from first principles.

1. $y = x.$
2. $y + x = 0.$
3. $2y = 3x - 4.$
4. $\frac{x}{2} + \frac{y}{3} = 1.$
5. $y = x^2.$
6. $y = (x - 1)^2.$
7. $y = x(x - 1).$
8. $y = (x - 1)(x - 2).$
9. $y = (x - 1)(x - 2)(x - 3).$
10. $y = (x - 1)(x - 2)^2.$
11. $y = (x - 1)^3.$
12. $y = 1 + x + x^2.$
13. $y = x + x^2 + x^3.$
14. $x^2 + xy - x + 1 = 0.$
15. $(x - 2)(x - 2y) = 1.$
16. $x^2 + y^2 = 1.$
17. $4x^2 + y^2 = 4.$
18. $y^2 = x.$
19. $y^2 = 2x - 3.$
20. $y^2 = x^2 + 1.$
21. $y = \frac{1}{x - 1}.$
22. $y = x + \frac{1}{x -}$
23. $y = \frac{1}{x - 1} - \frac{1}{x}.$
24. $(y - 1)^2 = x^3.$
25. $(y + 1)^2 = x^3.$
26. $y^2 = (x - 1)^3.$
27. $y^2 = (x + 1)^3.$
28. $y(x^2 + 1) = 1.$
29. $y(x^2 - 1) = 1.$
30. $y = x(1 - x).$
31. $y = x^2(1 - x).$
32. $y = x^3(1 - x).$
33. $y = x^2(1 - x)^2.$

CHAPTER IV.

RULES FOR DIFFERENTIATION OF COMPOUNDS OF FUNCTIONS.

27. Prop.—*The d.c. of a constant is zero.*

For, from the nature of a constant, *no change whatever* is caused by any change in x.

Hence, if $y = c$ (a constant),

then $y + k = c$, or $k = 0$.

$$\therefore \frac{k}{h} = 0 \text{ for all values of } h, \text{ however small.}$$

Hence, in the limit, $$\frac{dy}{dx} = 0.$$

NOTE.—The "curve" $y = c$ is a straight line parallel to Ox, so that $\tan\psi = 0$ at all points; hence $dy/dx = 0$.

28. Prop.—*A constant factor may be placed outside the sign of differentiation.*

Thus, if $y = c.f(x)$,

$$k = c.f(x+h) - c.f(x) = c.\{f(x+h) - f(x)\}$$

$$\therefore \frac{k}{h} = c.\frac{f(x+h) - f(x)}{h}.$$

Hence, proceeding to the limit,

$$\frac{dy}{dx} = c.\frac{du}{dx}$$

i.e. $$\frac{d}{dx}.c.f(x) = c\frac{d}{dx}f(x)$$

or $$\frac{d}{dx}cf(x) = cf'(x).$$

So also for differentials, $d.cf(x) = cf'(x)dx$.

29. The following preliminary table of differential coefficients should be verified by the student and learnt by heart.

y	c	x	x^2	x^3	$\sqrt{x}$	$1/x$
$\dfrac{dy}{dx}$	0	1	$2x$	$3x^2$	$\dfrac{1}{2\sqrt{x}}$	$-\dfrac{1}{x^2}$

Ex. 1. $y = 7x^2$; $\dfrac{dy}{dx} = 7 . 2x = 14x.$

Ex. 2. $y = \dfrac{2\sqrt{x}}{3}$; $\dfrac{dy}{dx} = \dfrac{2}{3} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{3\sqrt{x}}.$

30. Sum or Difference.—If $y = f(x) + \phi(x) + \chi(x) + \dots$; then, supposing x increased by h, and y accordingly increased by k, both h and k afterwards being made to diminish indefinitely, we have

$$y + k = f(x+h) + \phi(x+h) + \chi(x+h) + \dots;$$

$$\therefore \frac{k}{h} = \frac{f(x+h) - f(x)}{h} + \frac{\phi(x+h) - \phi(x)}{h} + \frac{\chi(x+h) - \chi(x)}{h} + \dots$$

and in the limit $\quad \dfrac{dy}{dx} = f'(x) + \phi'(x) + \chi'(x) + \dots$

Similarly, if $y = f(x) \pm \phi(x) \pm \chi(x) \pm \dots$ we can show that

$$\frac{dy}{dx} = f'(x) \pm \phi'(x) \pm \chi'(x) \pm \dots$$

Rule.—*The d.c. of an expression is the algebraical sum of the d.c.'s of all the terms.*

Ex. 1.
$$y = 7x^3 + 2x^2 - 3x + 7.$$
$$\frac{dy}{dx} = 7 . 3x^2 + 2 . 2x - 3 + 0 = 21x^2 + 4x - 3.$$

Ex. 2.
$$y = (2\sqrt{x} - 5)(3 - 2\sqrt{x}) = -4x + 16\sqrt{x} - 15.$$
$$\frac{dy}{dx} = -4 + 16 . \frac{1}{2\sqrt{x}} = \frac{8}{\sqrt{x}} - 4.$$

EXAMPLES V.

Find the d.c.'s of—

1. $2x - 3$. **2.** $x^2 + 3ax - a^2$.

3. $7x^3 - 4x^2 + 3x$. **4.** $(2x - 3)(x + 5)$.

5. $3x - 2\sqrt{ax}$. **6.** $\sqrt{x}(\sqrt{x} - 2)$.

7. $(x + a)(x + b)(x + c)$. **8.** $\dfrac{x^2 - 3x + 5}{x}$.

9. $2\sqrt{x} - \dfrac{x^3}{3} - \dfrac{4}{x}$. **10.** $(2\sqrt{x} - 3)(3\sqrt{x} + 2)$.

11. $(3\sqrt{x} - 5)^2$.

12. $(\sqrt{x} + 3\sqrt{a})^2 + (\sqrt{x} + 3\sqrt{b})^2 + (\sqrt{x} + 3\sqrt{c})^2$.

ANSWERS.

1. 2. **2.** $2x + 3a$. **3.** $21x^2 - 8x + 3$. **4.** $4x + 7$.

5. $3 - \dfrac{}{\sqrt{x}}$ **6.** $1 - \dfrac{}{\sqrt{x}}$. **7.** $3x^2 + 2(a + b + c)x + bc + ca + ab$.

8 $1 - \dfrac{5}{x^2}$. **9.** $\dfrac{1}{\sqrt{x}} - x^2 + \dfrac{4}{x^2}$. **10.** $6 - \dfrac{5}{2\sqrt{x}}$. **11.** $9 - \dfrac{15}{\sqrt{x}}$

12. $\dfrac{3(\sqrt{x} + \sqrt{a} + \sqrt{b} + \sqrt{c})}{\sqrt{x}}$.

31. Product of Two Functions.

Let $y = f(x)\phi(x)$; then, reasoning as above,

$$k = f(x + h)\phi(x + h) - f(x)\,\phi(x)$$
$$= f(x + h)\{\phi(x + h) - \phi(x)\} + \phi(x)\{f(x + h) - f(x)\}$$

by adding and subtracting the term $\phi(x)f(x + h)$,

$$\frac{k}{h} = f(x + h)\frac{\phi(x + h) - \phi(x)}{h} + \phi(x)\frac{f(x + h) - f(x)}{h}$$

$\therefore$ in the limit, $$\frac{dy}{dx} = f(x)\phi'(x) + \phi(x)f'(x).$$

Or, if u and v be functions of x,

$$\frac{d.uv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx};$$

or again, $$\frac{1}{uv}\,\frac{d.uv}{dx} = \frac{1}{u}\,\frac{du}{dx} + \frac{1}{v}\,\frac{dv}{dx}.$$

32. Product of Several Functions.

If $y = uvw$; u, v, w being functions of x,

$$\frac{dy}{dx} = \frac{d}{dx}(uv \,.\, w) = uv\frac{dw}{dx} + w\frac{d.uv}{dx}, \quad \text{by Art. 31,}$$

$$= uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

i.e. $$\frac{d}{dx}(uvw) = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}.$$

Or, $$\frac{1}{uvw}\frac{d.uvw}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}.$$

This can be extended to the product of any number of functions.

Rule.—*Differentiate in turn each factor, writing down the others (left alone), and add together the results.*

Ex. 1. $y = (x-2)(2x+3)$. Here $u = x - 2$; $v = 2x + 3$.

$$\therefore \frac{dy}{dx} = (x-2).2 + (2x+3).1 = 4x - 1.$$

Ex. 2. $y = (\sqrt{x}+1)(\sqrt{x}+2)(\sqrt{x}+3)$.

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}(\sqrt{x}+2)(\sqrt{x}+3) + (\sqrt{x}+1)\frac{1}{2\sqrt{x}}(\sqrt{x}+3)$$
$$+ (\sqrt{x}+1)(\sqrt{x}+2)\frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}\{x + 5\sqrt{x} + 6 + x + 4\sqrt{x} + 3 + x + 3\sqrt{x} + 2\}$$
$$= \frac{3x + 12\sqrt{x} + 11}{2\sqrt{x}}.$$

Ex. 3. $y = (2x^2 - 5x - 1)^2$.

Here $y = (2x^2 - 5x - 1)(2x^2 - 5x - 1)$; i.e. u and v are equal.

$$\therefore \frac{dy}{dx} = (2x^2 - 5x - 1)(4x - 5) + (2x^2 - 5x - 1)(4x - 5)$$
$$= 2(4x - 5)(2x^2 - 5x - 1).$$

EXAMPLES VI.

1. Find the d.c. (by the rule for products) of—

(1) $(2x - 5)(7x - 3)$. (2) $(2x^2 - 3)(3x^3 - 2)$.

(3) $(x^2 - 3x - 7)(7x^2 - 3x - 1)$. (4) $x(x - 2\sqrt{x} - 1)$.

(5) $(2x - 2\sqrt{x} + 1)(x + 4\sqrt{x} + 3)$.

(6) $\left(x + \dfrac{1}{x}\right)\left(2x - \dfrac{3}{x}\right)$. (7) $(x + 1)(x + 1)(x + 1)$.

(8) $(2x - 3a)(3x + 4a)(4x - 5a)$. (9) $8x(2\sqrt{x} - 5)(3 + 4x)$.

(10) $(\sqrt{x} - \sqrt{a})(2\sqrt{x} + \sqrt{a})(3\sqrt{x} - \sqrt{a})$.

2. Use the same rule to find the d.c. of—

(1) x^2, x^3, x^4, x^n, n being a $+^{ve}$ integer.

(2) $(3x + 5)^2$, $(3x + 5)^3$, $(3x + 5)^4$, $(3x + 5)^n$.

(3) $(ax + b)^2$, etc., as above. (4) $(x^2 + a^2)^2$, etc., as above.

(5) $x^{\frac{3}{2}}$ (writing it in the form of $x\sqrt{x}$).

(6) $x^{\frac{5}{2}}$ (or $x^2\sqrt{x}$). (7) $\dfrac{1}{\sqrt{x}}$ (or $\dfrac{1}{x}\sqrt{x}$).

ANSWERS.

1. (1) $28x - 41$. (2) $30x^4 - 27x^2 - 8x$.

(3) $2(14x^3 - 36x^2 - 41x + 12)$. (4) $2x - 3\sqrt{x} - 1$.

(5) $4x + \dfrac{15}{2}\sqrt{x} - 5 - \dfrac{5}{2\sqrt{x}}$. (6) $2\left(2x + \dfrac{3}{x^3}\right)$.

(7) $3(x + 1)^2$. (8) $72x^2 - 68ax - 43a^2$.

(9) $8(20x\sqrt{x} - 40x + 9\sqrt{x} - 15)$. (10) $9\sqrt{x} - 5\sqrt{a} - \dfrac{a}{\sqrt{x}}$.

2. (1) $2x$, $3x^2$, $4x^3$, nx^{n-1}.

(2) $6(3x + 5)$, $9(3x + 5)^2$, $12(3x + 5)^3$, $3n(3x + 5)^{n-1}$.

(3) $2a(ax + b)$, $3a(ax + b)^2$, $4a(ax + b)^3$, $na(ax + b)^{n-1}$.

(4) $4x(x^2 + a^2)$, $6x(x^2 + a^2)^2$, $8x(x^2 + a^2)^3$, $2nx(x^2 + a^2)^{n-1}$.

(5) $\frac{3}{2}\sqrt{x}$. (6) $\frac{5}{2}x^{\frac{3}{2}}$. (7) $-\dfrac{1}{2x^{\frac{3}{2}}}$.

33. Quotient of Two Functions.

Let $y = \dfrac{f(x)}{\phi(x)}$; then

$$k = \frac{f(x+h)}{\phi(x+h)} - \frac{f(x)}{\phi(x)} = \frac{f(x+h)\phi(x) - \phi(x+h)f(x)}{\phi(x+h)\phi(x)},$$

which may be written, by adding and subtracting the intermediate term $\phi(x)f(x)$,

$$= \frac{\phi(x)\{f(x+h) - f(x)\} - f(x)\{\phi(x+h) - \phi(x)\}}{\phi(x+h)\phi(x)}.$$

$$\therefore \frac{k}{h} = \frac{1}{\phi(x+h)\phi(x)}\left\{\phi(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{\phi(x+h) - \phi(x)}{h}\right\}.$$

and, proceeding to the limit,

$$\frac{dy}{dx} = \frac{\phi(x)f'(x) - f(x)\phi'(x)}{[\phi(x)]^2}.$$

Or, if $\qquad y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

34. This might also have been deduced from Art. 31, thus:—

If $y = \dfrac{u}{v}$, then $u = vy$, where v and y are both functions of x, as also is u, of course.

Hence
$$\frac{du}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + \frac{u}{v}\frac{dv}{dx};$$

$$\therefore \frac{dy}{dx} = \frac{1}{v}\left\{\frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}\right\}$$

$$v\frac{du}{dx} - u\frac{dv}{dx} \text{ as before.}$$

Ex. 1. $y = \dfrac{2x+3}{3x+5}$; $\dfrac{dy}{dx} = \dfrac{(3x+5).2 - (2x+3).3}{(3x+5)^2} = \dfrac{1}{(3x+5)^2}$.

Ex. 2. $y = \dfrac{1}{x}$; $\dfrac{dy}{dx} = \dfrac{x.0 - 1.1}{x^2} \quad \dfrac{1}{x^2}$.

Ex. 3. $y = \dfrac{(2x-3)(x-1)}{x-2}$.

This may be written $2x - 1 + \dfrac{1}{x-2}$,

whence $\dfrac{dy}{dx} = 2 + \dfrac{(x-2).0 - 1.1}{(x-2)^2} = 2 - \dfrac{\text{[illegible]}}{(x-2)^2} = \dfrac{2x^2 - 8x + 7}{(x-2)^2}$.

EXAMPLES VII.

Find the d.c. (by the rule for quotients) of—

1. $\dfrac{\text{[illegible]}}{x+1}$.
2. $\dfrac{\text{[illegible]}}{3-2x}$.
3. $\dfrac{1}{x+1}$.
4. $\dfrac{\text{[illegible]}}{1-x}$.
5. $\dfrac{a+bx}{c+dx}$.
6. $\dfrac{x^2-1}{x^2}$.
7. $\dfrac{1}{x^2+1}$.
8. $\dfrac{x}{1-x^2}$.
9. $\dfrac{1}{\sqrt{x}}$.
10. $\dfrac{\text{[illegible]}}{\sqrt{x}+1}$.
11. $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$.
12. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.
13. $\dfrac{x^2-x+1}{x^2+x+1}$.
14. $\dfrac{(x-1)(x-4)}{(x-2)(x-3)}$.
15. [illegible]
16. $\left(\dfrac{a+bx}{a-bx}\right)^2$.

ANSWERS.

1. $\dfrac{\text{[illegible]}}{(1+x)^2}$.
2. $\dfrac{1}{(3-2x)^2}$.
3. $-\dfrac{1}{(x+1)^2}$.
4. $\dfrac{\text{[illegible]}}{(1-x)^2}$.
5. $\dfrac{bc-ad}{(c+dx)^2}$.
6. $\dfrac{2}{x^3}$.
7. $-\dfrac{2x}{(x^2+1)^2}$.
8. $\dfrac{1+x^2}{(1-x^2)^2}$.
9. $-\dfrac{1}{2x\sqrt{x}}$.
10. $-\dfrac{\text{[illegible]}}{2\sqrt{x}(\sqrt{x}+1)^2}$.
11. $\dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$.
12. $\dfrac{\sqrt{a}}{\sqrt{x}(\sqrt{a}-\sqrt{x})^2}$.
13. $\dfrac{2(x^2-1)}{(x^2+x+1)^2}$.
14. $\dfrac{2(2x-5)}{(x-2)^2(x-3)^2}$.
15. $-\dfrac{4}{(2x-3)^3}$.
16. $\dfrac{4ab(a+bx)}{(a-bx)^3}$.

35. Function of a Function.

Let $u = f(x)$ . . . . (1)

and $y = \phi(u)$. (2)

so that $y = \phi\{f(x)\}$. Required $\frac{dy}{dx}$.

Before proceeding, it should be noted that u may have more than one value corresponding to a given value of x; and y may also have more than one value corresponding to a given value of u. This will not affect the argument, provided that we fix upon one set of corresponding values and keep to them throughout.

Now let x be increased by Δx, and suppose u to be increased, in consequence, by Δu. In (2) *we may use this same increment* Δu *as if it were an arbitrary one*; let y be increased accordingly by Δy.

Then $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$, since Δu is supposed to be the same in both fractions; hence, in the limit, when Δx, Δu, and Δy have all vanished, we get—

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Or, if $y = \phi f x$,
$$\frac{dy}{dx} = \frac{d\phi}{df}\frac{df}{dx}.$$

36. This may be extended. Thus, if
$$u = f(x),$$
$$v = \phi(u),$$
$$y = \chi(v);$$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}.$$

And similarly for the general case.

Ex. 1. $y = (2x^2 - 3)^2$.

We have $y = u^2$ say, where $u = 2x^2 - 3$,

$$\therefore \frac{du}{dx} = 4x.$$

Also $\frac{dy}{du} = 2u$.

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u \cdot 4x = 8ux = 8(2x^2 - 3)x.$$

Ex. 2. $y = \frac{1}{\sqrt{1 - x^2}} = \frac{1}{u}$ say, where $u = \sqrt{1 - x^2} = \sqrt{v}$ say,

where $v = 1 - x^2$; $\therefore \frac{dv}{dx} = -2x$;

also $\frac{du}{dv} = \frac{1}{2\sqrt{v}}$, and $\frac{dy}{du} = -\frac{1}{u^2}$.

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} = -\frac{1}{u^2} \cdot \frac{1}{2\sqrt{v}} \cdot (-2x)$$

$$= \frac{x}{u^2\sqrt{v}} = \frac{x}{(1 - x^2)\sqrt{1 - x^2}} = \frac{x}{(1 - x^2)^{\frac{3}{2}}}.$$

Ex. 3. $y = \frac{2x - 3}{\sqrt{1 - x^2}} = \frac{2x - 3}{u}$ say, where $u = \sqrt{1 - x^2} = \sqrt{v}$ say,

where $v = 1 - x^2$; $\therefore \frac{dv}{dx} = -2x$.

Also $$\frac{du}{dv} = \frac{1}{2\sqrt{v}};$$

$$\therefore \frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx} = \frac{-2x}{2\sqrt{v}} = -\frac{x}{\sqrt{1 - x^2}}.$$

And, by the rule for quotients—

$$\frac{dy}{dx} = \frac{2u - (2x - 3)\frac{du}{dx}}{u^2} = \frac{2\sqrt{1 - x^2} - (2x - 3)\frac{-x}{\sqrt{1 - x^2}}}{(1 - x^2)}$$

$$= \frac{2(1 - x^2) + x(2x - 3)}{(1 - x^2)^{\frac{3}{2}}} = \frac{2 - 3x}{(1 - x^2)^{\frac{3}{2}}}.$$

EXAMPLES VIII.

Find the d.c. of—

1. $(3x^2 - 5)^2$. **2.** $(ax^2 + bx + c)^3$. **3.** $\left(x + \frac{1}{x}\right)^3$.

4. $\frac{1}{3x - 5}$. **5.** $\frac{1}{ax^2 + b}$. **6.** 1

7. $\sqrt{1 + x^2}$. **8.** $\sqrt{a^3 - x^3}$. **9.** $\frac{1}{\sqrt{a^2 + x^2}}$.

10. $(2x^2 - 3x + 5)^{-\frac{1}{2}}$. **11.** $\sqrt{\frac{a - x}{a + x}}$.

13. $\frac{a + x}{\sqrt{a^2 + x^2}}$. **14.** $\frac{x^2 - ax + a^2}{\sqrt{x^2 + a^2}}$.

ANSWERS.

1. $12x(3x^2 - 5)$.
2. $3(2ax + b)(ax^2 + bx + c)^2$.
3. $3\left(1 - \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right)^2$.
4. $-\frac{3}{(3x - 5)^2}$.
5. $-\frac{2ax}{(ax^2 + b)^2}$.
6. $-\frac{3}{x^4}$.
7. $\frac{x}{\sqrt{1 + x^2}}$.
8. $-\frac{3x^2}{2\sqrt{a^3 - x^3}}$.
9. $-\frac{x}{(a^2 + x^2)^{\frac{3}{2}}}$.
10. $\frac{-4x + 3}{2(2x^2 - 3x + 5)^{\frac{3}{2}}}$.
11. $-\frac{a}{(a + x)^{\frac{3}{2}}(a - x)^{\frac{1}{2}}}$.
12. $\frac{2x}{(x^2 + 1)^{\frac{1}{2}}(x^2 - 1)^{\frac{3}{2}}}$.
13. $\frac{a(a - x)}{(a^2 + x^2)^{\frac{3}{2}}}$.
14. $\frac{x^3 + a^2x - a^3}{(x^2 + a^2)^{\frac{3}{2}}}$.

37. Inverse Functions.

Let $$y = f(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then we have seen that x must be some function of y; this latter is called the *inverse function.*

As examples of direct and inverse functions, we have—

Direct	Inverse.
$y = x^2$	$x = \sqrt{y}$
$y = \sin x$	$x = \sin^{-1} y$
$y = \log x$	$x = e^y$
etc.	etc.

The last-named are called *logarithmic* and *exponential* functions respectively.

38. Let the inverse function be represented by $x = \phi(y)$. (2)

Remarks similar to those in Art. 35 apply here. In differentiating we must note that for a given value of x in (1) there may be many values of y. If we substitute these values in (2) we may get still more values of x, but amongst these values must be included the original given value of x. We must then fix upon that value of y, obtained from (1), which on substitution in (2) gives (among others) the original value of x.

On the above understanding, let x in (1) be increased by Δx, in consequence of which y is increased by Δy. Then it follows that in (2) this same increase, Δy, of y must produce the original increase, Δx, in x, since x and y are understood to be the same in (1) and (2).

Hence $\frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta y} = 1$ for all values of Δx and Δy (corresponding) however small. And, proceeding to the limit,

$$\frac{dy}{dx}\frac{dx}{dy} = 1$$

Or,
$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Ex. Let $y = \sqrt{x}$; then $x = y^2$;

$$\therefore \frac{dx}{dy} = 2y; \quad \therefore \frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}.$$

Other examples will follow. (See Arts. 52, etc.)

Cor.—It follows that the result of Art. 35 may be written—

$$\frac{dy}{dx} = \frac{dy}{du}\bigg/\frac{dx}{du}.$$

EXAMPLES IX.

Find the d.c. (by using the inverse function) of—

1. $\sqrt{1-x}$.
2. $\sqrt{1+x} + 1$.
3. $\frac{1}{\sqrt{a+x}}$.
4. $\sqrt{\frac{x}{1-x}}$.
5. $\sqrt[3]{x}$.
6. $\sqrt{\frac{a+x}{a-x}}$.

Answers.

1. $-\frac{1}{2\sqrt{1-x}}$.
2. $\frac{1}{2\sqrt{1+x}}$.
3. $-\frac{1}{2(a+x)^{\frac{3}{2}}}$.
4. $\frac{1}{2x^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$.
5. $\frac{1}{3x^{\frac{2}{3}}}$.
6. $\frac{a}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$.

CHAPTER V.

DIFFERENTIATION OF STANDARD FUNCTIONS.

39. All ordinary expressions in mathematics will be found to be built up from the following fundamental forms :—

(1) The algebraical function, x^n,
(2) The exponential function, e^x,
(3) The logarithmic function, $\log x$.
(4) The direct circular functions, $\sin x$, etc.
(5) The inverse circular functions, $\sin^{-1} x$, etc.,

to which we may add—

(6) The direct hyperbolic functions, $\sinh x$, etc.
(7) The inverse hyperbolic functions, $\sinh^{-1} x$, etc.

Hence, by obtaining the d.c.'s of these, we can (by the use of the rules for the combinations of functions given in the last chapter) find the d.c. of any ordinary expression.

The hyperbolic functions will be considered separately.

40. Differentiation of x^n.

Let $y = x^n$.

(i) Let n be a $+^{ve}$ integer. Then

$$k = (x + h)^n - x^n = nhx^{n-1} + \text{higher powers of } h$$
$$= nhx^{n-1} + Kh^2 \text{ say, } K\dagger \text{ being finite}$$

for finite values of x,

$$\therefore \frac{k}{h} = nx^{n-1} + Kh.$$

† We shall use K to denote "some multiple of," when we are not concerned with its actual value. Moreover, *the same symbol will be used for different multiples*, as, for instance, in case (ii) below.

Hence $$\frac{dy}{dx} = \underset{h=0}{Lt}\left(\frac{k}{h}\right) = nx^{n-1}.$$

(ii) Let n be negative or fractional.

Then $k = (x+h)^n - x^n = x^n(1 + h/x)^n - x^n$; and as long as x is finite, however small, we can make h smaller than x, so that the expansion of $(1 + h/x)^n$ is convergent or finite.

$$\therefore\ k = x^n\left\{1 + n\,.\frac{h}{x} + Kh^2\right\} - x^n = nhx^{n-1} + K'h^2,\ K' \text{ being finite.}$$

$$\therefore\ \frac{k}{h} = nx^{n-1} + K'h;\ \text{and, as before, } \frac{dy}{dx} = nx^{n-1}.$$

Hence, for all values of n, we have

$$\frac{d}{dx}x^n = nx^{n-1}.$$

41. The cases of fractional and negative powers may be also taken as follows:—

(1) Let n be a positive fraction $= p/q$ say, where p and q are +ve integers.

Then $$y = x^n = x^{\frac{p}{q}};$$
$$\therefore\ y^q = x^p.$$

$\therefore$ by Art 40 (1) [see also Art 35] $qy^{q-1}\dfrac{dy}{dx} = px^{p-1}$,

$$\therefore\ \frac{dy}{dx} = \frac{p}{q}x^{p-1} \div y^{q-1} = \frac{p}{q}x^{p-1} \div x^{\frac{p}{q}(q-1)} = \frac{p}{q}x^{\frac{p}{q}-1} = nx^{n-1}$$

(2) Let n be *any* negative quantity $= -m$ say.

Then $$y = x^n = x^{-m};$$
$$\therefore\ yx^m = 1.$$

$$\therefore\ y\,.\,mx^{m-1} + x^m\frac{dy}{dx} = 0 \text{ by (1), since } m \text{ is } +\text{ve}.$$

$$\therefore\ \frac{dy}{dx} = -\frac{my}{x} = -mx^{-m-1} = nx^{n-1}.$$

Hence the theorem is true for any value of n.

Cor.—If $y = u^n$, u being any function of x, then $\dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$.

Or, $\dfrac{d}{dx}u^n = nu^{n-1}\dfrac{du}{dx}$.

42. Examples.

Ex. 1. If $y = \sqrt{x} = x^{\frac{1}{2}}$;

$$\frac{dy}{dx} = \tfrac{1}{2}x^{\frac{1}{2}-1} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Ex. 2. $y = \dfrac{1}{\sqrt[3]{3x^2 - x + 1}} = u^{-\frac{1}{3}}$, where $u = 3x^2 - x + 1$.

$$\therefore \frac{dy}{dx} = -\tfrac{1}{3}u^{-\frac{1}{3}-1} \cdot \frac{du}{dx} = -\tfrac{1}{3}u^{-\frac{4}{3}}(6x - 1) = -\frac{6x-1}{3(3x^2 - x + 1)^{\frac{4}{3}}}.$$

Ex. 3. $y = \sqrt{1 + x^2} = \sqrt{u}$;

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx} = \frac{x}{\sqrt{1 + x^2}}.$$

Similarly, if $y = \sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}}$.

These two results should be learnt.

EXAMPLES X.

1. *Write down* the d.c. of–

(1) x^4, x^5, $3x^{10}$, x^{a+b}, $\dfrac{x^{n+1}}{n+1}$. (2) x^{-1}, x^{-2}, x^{-n}, $3x^{-(p+q)}$.

(3) $\dfrac{1}{x^3}$, $\dfrac{a}{x^5}$, $\dfrac{1}{10x^{10}}$, $\dfrac{1}{x^{n-1}}$, $\dfrac{q}{(1-q)x^{q-1}}$. (4) $x^{\frac{1}{3}}$, $x^{-\frac{2}{3}}$, $x^{\frac{n+1}{n-1}}$, $x^{-\frac{1}{n-1}}$.

(5) $\sqrt[3]{x}$, $\sqrt[4]{x}$, $\sqrt[n]{x}$, $2/\sqrt{x^3}$, $n/\sqrt[n]{x}$, $aq/\sqrt[q]{x}$.

2. Find the d.c. of—

(1) $(x^2 - a^2)^5$. (2) $3(5 - 4x^2)^7$. (3) $(ax^2 + 2hx + b)^n$.

(4) $(1 - x^2)^{-\frac{1}{2}}$. (5) $\sqrt{a^3 - x^3}$. (6) $(a^4 + x^4)^{\frac{3}{4}}$.

(7) $\dfrac{1}{x(2a - x)}$. (8) $\dfrac{1}{\sqrt{2ax - x^2}}$. (9) $(\sqrt{x} + \sqrt{a})^{2n}$.

(10) $\left(\dfrac{a^n}{a^n - x^n}\right)^{\frac{1}{n}}$. (11) $\sqrt[n]{x^{-1} + a^{-1}}$. (12) $(x^{n+1} + a^{n+1})^{\frac{n-1}{n+1}}$.

3. Find the d.c. of—

(1) $(1 - x^2)\sqrt{1 + x^2}$. (2) $\dfrac{\sqrt{1 - x^2}}{1 + x^2}$. (3) $\dfrac{x^2 - 2a^2}{\sqrt{a^2 - x^2}}$.

(4) $\dfrac{(x - a)^m}{(x + a)^n}$.

ANSWERS.

1. (1) $4x^3$, $5x^4$, $30x^9$, $(a+b)x^{a+b-1}$, x^n.

(2) $-x^{-2}$, $-2x^{-3}$, $-nx^{-n-1}$, $-3(p+q)x^{-(p+q+1)}$.

(3) $-\dfrac{3}{x^4}$, $-\dfrac{5a}{x^6}$, $-\dfrac{1}{x^{11}}$, $-\dfrac{n-1}{x^n}$, $\dfrac{q}{x^7}$.

(4) $\frac{1}{3}x^{-\frac{2}{3}}$, $-\frac{2}{3}x^{-\frac{5}{3}}$, $\dfrac{n+1}{n-1}x^{\frac{2}{n-1}}$, $\dfrac{1}{1-n}x^{\frac{n}{1-n}}$.

(5) $1/3x^{\frac{2}{3}}$, $1/4x^{\frac{3}{4}}$, $1/nx^{\frac{n-1}{n}}$, $-3/x^{\frac{5}{2}}$, $-1/x^{\frac{n+1}{n}}$, $-a/x^{\frac{q+1}{q}}$.

2. (1) $10x(x^2-a^2)^4$. (2) $-168x(5-4x^2)^6$.

(3) $2n(ax+h)(ax^2+2hx+b)^{n-1}$.

(4) $x(1-x^2)^{-\frac{3}{2}}$. (5) $-\dfrac{3x^2}{2\sqrt{a^3-x^3}}$. (6) $6x^3(a^4+x^4)^{\frac{1}{2}}$.

(7) $\dfrac{2(x-a)}{x^2(2a-x)^2}$. (8) $\dfrac{x-a}{(2ax-x^2)^{\frac{3}{2}}}$. (9) $\dfrac{n}{\sqrt{x}}(\sqrt{x}+\sqrt{a})^{2n-1}$.

(10) $ax^{n-1}/(a^n-x^n)^{\frac{n+1}{n}}$. (11) $-\dfrac{1}{nx^2}(x^{-1}+a^{-1})^{\frac{1-n}{n}}$.

(12) $(n-1)x^n(x^{n+1}+a^{n+1})^{-\frac{2n+1}{n+1}}$.

3. (1) $-\dfrac{x(1+3x^2)}{\sqrt{1+x^2}}$. (2) $\dfrac{x(x^2-3)}{(1+x^2)^2\sqrt{1-x^2}}$. (3) $-\dfrac{x^3}{(a^2-x^2)^{\frac{3}{2}}}$.

(4) $\dfrac{(x-a)^{m-1}}{(x+a)^{n+1}}\{(m-n)x+(m+n)a\}$.

43. Differentiation of e^x and a^x.

Let $y = e^x$.

Then, $k = e^{x+h} - e^x = e^x(e^h - 1)$

$$= e^x\left\{\left(1 + h + \frac{h^2}{2!} + \dots\right) - 1\right\}, \text{ a \textit{convergent series},}$$

$$= he^x + Kh^2;$$

$$\therefore \frac{dy}{dx} = \underset{h=0}{lt}\left(\frac{k}{h}\right) = e^x.$$

Again, let $y = a^x = (e^{\log a})^x$ [*Misc. Theorems* (4)] $= e^{x \log a} = e^z$ say, where $z = x \log a$,

$$\therefore \frac{dz}{dx} = \log a$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^z \log a = a^x \log a.$$

Hence $$\frac{d}{dx}e^x = e^x; \quad \frac{d}{dx}a^x = a^x \log a.$$

Cor.—If $y = e^u$, u being a function of x, then

$$\frac{dy}{dx} = e^u \frac{du}{dx}, \text{ or } \frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

44. Differentiation of $\log_e x$ and $\log_a x$.

Let $y = \log x$.

Then $k = \log(x + h) - \log x = \log \dfrac{x+h}{x} = \log\left(1 + \dfrac{h}{x}\right)$

$$= \frac{h}{x} - \frac{h^2}{2x^2} + \dots,$$

convergent for all finite values of x when $h/x < 1$.

$$\therefore \frac{k}{h} = \frac{1}{x} + Kh$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}.$$

Again, let $y = \log_a x = \log_a e \log_e x$

$$\therefore \frac{dy}{dx} = \frac{1}{x}\log_a e = \frac{1}{x \log a} \text{ [\textit{Misc. Theorems} (4)].}$$

Hence $$\frac{d}{dx}\log x = \frac{1}{x}; \quad \frac{d}{dx}\log_a x = \frac{1}{x \log a}.$$

Cor.—If $y = \log u$, u being a function of x, then

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}; \text{ or } \frac{d}{dx}\log u = \frac{1}{u}\frac{du}{dx}.$$

* **45.** The proofs in this article and the next do not involve expansion into series.

If $y = a^x$, we have,

$$\frac{k}{h} = \frac{1}{h}(a^{x+h} - a^x) = a^x \frac{a^h - 1}{h} \tag{1}$$

Now let $\dfrac{a^h - 1}{h}$ (which is undetermined when $h = 0$) $= m$.

$$\therefore\ a^h = 1 + mh\,. \tag{2}$$

and $h \log a = \log (1 + mh)$

$$= mh \log (1 + mh)^{\frac{1}{mh}}.$$

$$\therefore\ m = \frac{\log a}{\log (1 + mh)^{\frac{1}{mh}}}$$

But $\underset{h=0}{lt}(1 + mh)^{\frac{1}{mh}} = e$, since mh vanishes with h [as may be seen by putting $h = 0$ in (2)].†

$\therefore$ the limit of m, or $\dfrac{a^h - 1}{h}$, is

$$\frac{\log a}{\log e}, \quad i.e. \log a;$$

$\therefore$ in (1)

$$\frac{dy}{dx} = a^x \log a.$$

* **46.** If $y = \log_a x$,

$$\frac{k}{h} = \frac{1}{h}\log_a \frac{x+h}{x} = \frac{1}{h}\log_a\left(1 + \frac{h}{x}\right) = \frac{1}{x}\log_a\left(1 + \frac{h}{x}\right)^{\frac{x}{h}}.$$

But $\underset{h=0}{lt}\left(1 + \dfrac{h}{x}\right)^{\frac{x}{h}} = e$;

$$\therefore\ \frac{dy}{dx} = \frac{1}{x}\log_a e = \frac{1}{x \log a}.$$

Or, if $y = \log_a x$; $x = a^y$, $\dfrac{dx}{dy} = a^y \log a = x \log a$; $\therefore\ \dfrac{dy}{dx} = \dfrac{1}{x \log a}$.

---

† **This statement is necessary, for it might happen that m is infinite in the limit, in which case mh might not vanish with h.**

47. Examples.

Ex. 1. $y = e^{ax} = e^{u}$ say; where $u = ax$,

$$\therefore \frac{du}{dx} = a.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^{u}.a = ae^{ax}.$$

Or, at once, by Art. 43, Cor.,

$$\frac{dy}{dx} = e^{ax}.\frac{d}{dx}(ax) = ae^{ax}.$$

Ex. 2. $y = e^{\sqrt{1-x^2}} = e^{u}$ say; where $u = \sqrt{1-x^2}$.

$$\therefore \frac{du}{dx} = -\frac{x}{\sqrt{1-x^2}}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^{u}\cdot\frac{-x}{\sqrt{1-x^2}} = \frac{-xe^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}.$$

Or, at once, $\dfrac{dy}{dx} = e^{\sqrt{1-x^2}}\dfrac{d}{dx}(\sqrt{1-x^2}) = -\dfrac{x}{\sqrt{1-x^2}}e^{\sqrt{1-x^2}}.$

Similarly, if $y = a^{\sqrt{1-x^2}}$, $\dfrac{dy}{dx} = -\dfrac{x\log a}{\sqrt{1-x^2}}a^{\sqrt{1-x^2}}.$

Ex. 3. $y = \log ax = \log a + \log x.$

$$\therefore \frac{dy}{dx} = \frac{1}{x}.$$

Ex. 4. $y = \log(2x^2 - 5).$
We have, *at once*, by Art. 44., Cor.,

$$\frac{dy}{dx} = \frac{1}{(2x^2-5)}\frac{d}{dx}(2x^2-5) = \frac{4x}{2x^2-5}.$$

Similarly, if $y = \log_a(2x^2-5)$, $\dfrac{dy}{dx} = \dfrac{4x}{(2x^2-5)\log a}.$

Ex. 5. $y = \log\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}.$

Rationalizing the denominator,

$$y = \log\frac{(\sqrt{1+x^2}+x)^2}{1+x^2-x^2} = 2\log(\sqrt{1+x^2}+x) = 2\log u,$$

where $u = \sqrt{1+x^2}+x.$

$$\therefore \frac{du}{dx} = \frac{x}{\sqrt{1+x^2}} + 1 = \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{2}{u} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{2}{\sqrt{1+x^2}}.$$

Ex. 6. $y = \dfrac{x^{\frac{3}{2}}(1+x^2)^{\frac{3}{2}}}{(1-x^2)^{\frac{3}{2}}}$.

When products, quotients, powers, and roots occur exclusively, it is an advantage to take the logarithms of both sides, and then differentiate This process is called *logarithmic differentiation*.

Thus $\log y = \frac{3}{2}\log x + \frac{3}{2}\log(1+x^2) - \frac{3}{2}\log(1-x^2)$.

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{3}{2x} + \frac{3}{2}\frac{2x}{1+x^2} - \frac{3}{2}\frac{-2x}{1-x^2}$$

$$= \frac{3}{2x} + \frac{6x}{1-x^4} = \frac{3(1+4x^2-x^4)}{2x(1-x^4)}.$$

Multiplying up by y and substituting its value,

$$\frac{dy}{dx} = \frac{3(1+4x^2-x^4)}{2x(1-x^4)} \cdot \frac{x^{\frac{3}{2}}(1+x^2)^{\frac{3}{2}}}{(1-x^2)^{\frac{3}{2}}} = \frac{3x^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}}{2(1-x^2)^{\frac{5}{2}}}(1+4x^2-x^4).$$

Ex. 7. $\mathbf{y = u^v}$, *u and v being both functions of x*

We have $\log y = v \log u$.

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}\log u + \frac{v}{u}\frac{du}{dx} \text{ [Arts. 31; 44, Cor.]}$$

$$\therefore \frac{dy}{dx} = u^v\left\{\frac{dv}{dx}\log u + \frac{v}{u}\frac{du}{dx}\right\} = u^v \log u \cdot \frac{dv}{dx} + vu^{v-1}\frac{du}{dx}.$$

This is an important case, and the following rule should be noted :—

Rule.—(1) Differentiate on the supposition that u is constant.

(2) " " " v "

(3) Add the two results.

Thus,

(1) *Supposing* u constant, $\dfrac{d}{dx}u^v = \dfrac{d}{dv}u^v \cdot \dfrac{dv}{dx} = u^v \log u \cdot \dfrac{dv}{dx}$ [Art. 43].

(2) $\dfrac{d}{dx}u^v = \dfrac{d}{du}u^v \cdot \dfrac{du}{dx} = vu^{v-1}\dfrac{du}{dx}$ [Art. 40].

Adding, we get the same result as before.

See the Chapter on "Partial Differentiation."

EXAMPLES XI.

1. *Write down* the d.c. of—

(1) e^{2x}; a^{2x}; e^{x+3}; a^{2x+3}; e^{x^2}; a^{cx^2}.

(2) $e^{\sqrt{x}}$; $a^{c\sqrt{x}}$; $e^{ax+\frac{b}{x}}$; $a^{\sqrt{1+x^2}}$.

(3) 2^x; $m^{\sqrt{x}}$; $(a+b)^x$; $(2e)^x$; $(\frac{1}{2})^{x^2-1}$.

(4) $\log 2x$; $\log(2x-3)$; $\log(x^2+1)$; $\log\sqrt{x^2+1}$.

(5) $\log_m x$; $\log_{10} x$; $\log_a(ax^2+2hx+b)$; $\log_{ae} x$.

2. Find the d.c. of—

(1) $e^{x\sqrt{x-1}}$. (2) $e^{x\sqrt{1-x^2}}$. (3) $\log\{(x-2)(x-3)\}$.

(4) $\log\{\sqrt{x}(\sqrt{x}-1)\}$. (5) $\log(x+\sqrt{x^2-1})$.

(6) $\log\dfrac{x^2\sqrt{1-x^2}}{\sqrt{1+x^2}}$. (7) $\log\left(\sqrt{\dfrac{x+a}{x-a}}-\sqrt{\dfrac{x-a}{x+a}}\right)$.

(8) $x^2\sqrt{\dfrac{x-1}{x+1}}$. (9) $\sqrt{\dfrac{(1-x^2)(1+2x^2)}{(1+x^2)(1-2x^2)}}$.

(10) $\log\dfrac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1}$.

3. Find the d.c. of—

(1) x^x. (2) $(2x-3)^{2x-3}$. (3) $(x^2-1)^{\sqrt{x}}$.

(4) $(1+1/x)^x$. (5) u^v.

4. Trace the graphs of the functions e^x, $\log x$.

ANSWERS.

1. (1) $2e^{2x}$; $2a^{2x}\log a$; e^{x+3}; $2a^{2x+3}\log a$; $2xe^{x^2}$; $2cxa^{cx^2}\log a$.

(2) $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$; $\dfrac{c}{2\sqrt{x}}a^{c\sqrt{x}}\log a$; $\left(a-\dfrac{b}{x^2}\right)e^{ax+\frac{b}{x}}$; $\dfrac{x}{\sqrt{1+x^2}}a^{\sqrt{1+x^2}}\log a$.

(3) $2^x\log_e 2$; $\dfrac{1}{2\sqrt{x}}m^{\sqrt{x}}\log m$; $(a+b)^x\log(a+b)$; $(2e)^x(\log_e 2+1)$;
$-2x(\frac{1}{2})^{x^2-1}\log_e 2$.

(4) $\frac{1}{x}$; $\frac{2}{2x-3}$; $\frac{2x}{x^2+1}$; $\frac{x}{x^2+1}$.

(5) $\frac{1}{x\log m}$; $\frac{1}{x\log_e 10}$; $\frac{2(ax+h)}{(ax^2+2hx+b)\log a}$; $\frac{1}{x(1+\log a)}$.

2. (1) $\frac{3x-2}{2\sqrt{x-1}}e^{x\sqrt{x-1}}$. (2) $\frac{x(2-3x^2)}{\sqrt{1-x^2}}e^{x^2\sqrt{1-x^2}}$

(3) $\frac{2x-5}{(x-2)(x-3)}$. (4) $\frac{2\sqrt{x}-1}{2x(\sqrt{x}-1)}$. (5) $\frac{1}{\sqrt{x^2-1}}$.

(6) $\frac{2(1-x^2-x^4)}{x(1-x^4)}$. (7) $\frac{x}{a^2-x^2}$. (8) $\frac{\sqrt{x}(3x^2+2x-3)}{2(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$.

(9) $\frac{2x(1+2x^4)}{(1-x^2)^{\frac{1}{2}}(1+2x^2)^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}(1-2x^2)^{\frac{3}{2}}}$. (10) $\frac{3}{x\sqrt{1+x^3}}$.

3. (1) $x^x(1+\log x)$. (2) $2(2x-3)^{2x-3}\{1+\log(2x-3)\}$.

(3) $\frac{1}{2\sqrt{x}}(x^2-1)^{\sqrt{x}-1}\{(x^2-1)\log(x^2-1)+4x^2\}$.

(4) $\left(1+\frac{1}{x}\right)^x\left\{\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right\}$. (5) $u^u(1+\log u)\frac{du}{dx}$.

Differentiation of Direct Circular Functions.

48. Sin x and cos x.

Let $y = \sin x$.

Then $k = \sin(x+h) - \sin x = 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$.

$$\therefore \frac{dy}{dx} = \underset{h=0}{Lt}\left(\frac{k}{h}\right) = \underset{h=0}{Lt}\cos\left(x+\frac{h}{2}\right).\left(\sin\frac{h}{2}\Big/\frac{h}{2}\right) = \cos x.$$

Similarly we can show that if $y = \cos x$; then $dy/dx = -\sin x$.

Otherwise, $y = \cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$\therefore \frac{dy}{dx} = \cos\left(\frac{\pi}{2}-x\right).\frac{d}{dx}\left(\frac{\pi}{2}-x\right) = -\sin x.$$

Hence $\frac{d}{dx}\sin x = \cos x$; $\frac{d}{dx}\cos x = -\sin x$.

49. The following is a geometrical method:—

Let $\angle POM = x$; $POQ = \dfrac{arc\ PQ}{OP} = h.$

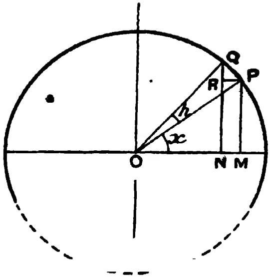

FIG. 8.

Then $\sin x = \dfrac{PM}{OP}$; $\sin(x+h) = \dfrac{QN}{OQ} = \dfrac{QN}{OP}.$

$$\therefore k = \frac{QN - PM}{OP} = \frac{QR}{OP}.$$

$$\therefore \frac{k}{h} = \frac{QR}{OP}\frac{OP}{\text{arc } PQ} = \frac{QR}{\text{arc } PQ}.$$

But in the limit,

$\dfrac{\text{arc } PQ}{\text{chord } PQ} = 1$, and $\angle PQR = x.$

$$\therefore \frac{dy}{dx} = lt\frac{QR}{PQ} = \cos PQR = \cos x.$$

A similar proof can be given for the other circular functions.

50. Tan x and cot x.

Let $y = \tan x = \dfrac{\sin x}{\cos x}$

Then $\dfrac{dy}{dx} = \dfrac{\cos x \cdot \cos x - (-\sin x)\sin x}{\cos^2 x}$, (rule for quotients)

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

Similarly, if $y = \cot x$, $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x.$

Otherwise, as in Art. 48.

Hence, $\dfrac{d}{dx}\tan x = \sec^2 x$; $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x.$

51. Sec x and cosec x.

Let $y = \sec x = \dfrac{1}{\cos x}.$

Then $\dfrac{dy}{dx} = -\dfrac{1}{\cos^2 x}\cdot\dfrac{d}{dx}\cos x = \dfrac{\sin x}{\cos^2 x} = \sec x \tan x.$

Similarly, if $y = \operatorname{cosec} x$, $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$.

Hence $\dfrac{d}{dx}\sec x = \sec x \tan x$; $\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$.

NOTE.—All of the ratios might have been differentiated from first principles, as in the case of the sine.

Differentiation of Inverse Circular Functions.

52. $\sin^{-1} x$ and $\cos^{-1} x$.

Let $y = \sin^{-1} x$.

Then $x = \sin y$, and we shall use Art. 38.

$$\therefore \frac{dx}{dy} = \cos y = \sqrt{1 - x^2};$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Again, if $y = \cos^{-1} x = \dfrac{\pi}{2} - \sin^{-1} x$,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

Hence $\dfrac{d}{dx}\sin^{-1} x = +\dfrac{1}{\sqrt{1 - x^2}}$, $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1 - x^2}}$.

53. $\tan^{-1} x$ and $\cot^{-1} x$.

Let $y = \tan^{-1} x$.

Then $x = \tan y$, $\dfrac{dx}{dy} = \sec^2 y = 1 + x^2$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + x^2}.$$

Again, if $y = \cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x$, $\dfrac{dy}{dx} = -\dfrac{1}{1 + x^2}$.

Hence $\dfrac{d}{dx}\tan^{-1} x = +\dfrac{1}{1 + x^2}$; $\dfrac{d}{dx}\cot^{-1} x = -\dfrac{1}{1 + x^2}$.

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54. $\operatorname{Sec}^{-1} x$ and $\operatorname{cosec}^{-1} x$.

Let $y = \sec^{-1} x$.

$$\therefore x = \sec y;$$

$$\frac{dx}{dy} = \sec y \tan y = x\sqrt{x^2 - 1};$$

$$\therefore \frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}.$$

Again, if $y = \operatorname{cosec}^{-1} x = \frac{\pi}{2} - \sec^{-1} x$; $\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2 - 1}}$.

Hence $\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$; $\frac{d}{dx}\operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$.

Or, $\quad y = \sec^{-1} x = \cos^{-1}\frac{1}{x} = \cos^{-1} u$, where $u = \frac{1}{x}$; $\frac{du}{dx} = -\frac{1}{x^2}$;

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - u^2}}\frac{du}{dx} = -\frac{1}{\sqrt{1 - 1/x^2}}\left(-\frac{1}{x^2}\right) = \frac{1}{x\sqrt{x^2 - 1}}.$$

55. $\operatorname{Vers}^{-1} x$ and $\operatorname{covers}^{-1} x$.

Let $y = \operatorname{vers}^{-1} x$.

Then $\quad x = \operatorname{vers} y = 1 - \cos y$,

$$\frac{dx}{dy} = \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - (1 - x)^2} = \sqrt{2x - x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{2x - x^2}}.$$

Again, if $\quad y = \operatorname{covers}^{-1} x = \frac{\pi}{2} - \operatorname{vers}^{-1} x$; $\frac{dy}{dx} = -\frac{1}{\sqrt{2x - x^2}}$.

Hence $\quad \frac{d}{dx}\operatorname{vers}^{-1} x = \frac{1}{\sqrt{2x - x^2}}$; $\frac{d}{dx}\operatorname{covers}^{-1} x = -\frac{1}{\sqrt{2x - x^2}}$.

56. All the inverse functions may be differentiated by direct methods.

Thus, if $y = \sin^{-1} x$,

$$\frac{k}{h} = \frac{\sin^{-1}(x + h) - \sin^{-1} x}{h} = \frac{\sin^{-1}\{(x + h)\sqrt{1 - x^2} - x\sqrt{1 - (x + h)^2}\}}{h}.$$

Now, when h is small, the numerator is an angle whose sine is small (as may be seen by putting $h = 0$). And since $\sin\theta$ and θ approach equality when θ is diminished indefinitely, we may replace either by the other. [See, however, Art. 15.]

$$\text{Hence}\quad lt\,\frac{k}{h} = \frac{dy}{dx} = \underset{h=0}{lt}\,\frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}}{h}$$

$$= (\text{rationalizing the numerator})\,\underset{h=0}{lt}\,\frac{(x+h)^2(1-x^2) - x^2\{1-(x+h)^2\}}{h\{(x+h)\sqrt{1-x^2} + x\sqrt{1-(x+h)^2}\}}$$

$$= \underset{h=0}{lt}\,\frac{x^2(1-x^2) + 2xh(1-x^2) + h^2(1-x^2) - x^2\{(1-x^2) - 2xh - h^2\}}{h\{(x+h)\sqrt{1-x^2} + x\sqrt{1-(x+h)^2}\}}$$

which easily reduces to $\dfrac{1}{\sqrt{1-x^2}}$.

To differentiate $\cos^{-1}x$ in the same way, we must put

$$k = \cos^{-1}(x+h) - \cos^{-1}x$$
$$= \sin^{-1}[x\sqrt{1-(x+h)^2} - (x+h)\sqrt{1-x^2}];\ \text{etc.}$$

57. The expression $\dfrac{\ }{\sqrt{1-x^2}}$ double-signed. It should be noted, however, that *we have taken both* $\sin^{-1}x$ *and* $\cos^{-1}x$ *to be angles in the first quadrant.* In this case $\sin^{-1}(x+h)$ is greater than $\sin^{-1}x$; *i.e.* $\dfrac{\sin^{-1}(x+h) - \sin x}{h}$, and therefore $\dfrac{d}{dx}\sin^{-1}x$, is $+^{ve}$. Similarly, $\cos^{-1}(x+h)$ is *less* than $\cos^{-1}x$; whence $\dfrac{d}{dx}\cos^{-1}x$ is In reality, $\sin^{-1}x$ and $\cos^{-1}x$ are many-valued functions, and *in the case of both of them the d.c. is ambiguous in sign.*

Thus, suppose x to be $+^{ve}$; then $\sin^{-1}x$ represents an angle in either the first or second quadrant. In the former case we have seen that the d.c. is $+^{ve}$; in the latter case $\sin^{-1}(x+h)$ is *less* than $\sin^{-1}x$ (or the angle diminishes while the sine increases, as may be seen from a figure), whence the d.c. is $-^{ve}$.

If x be $-^{ve}$, then $\sin^{-1}x$ represents an angle in either the third or fourth quadrant, and we can show that the d.c. is $-^{ve}$ or $+^{ve}$ accordingly.

Similarly, for $\cos^{-1}x$; the other inverse functions may also be discussed in the same way.

58. Table of Fundamental Forms.

y	$\dfrac{dy}{dx}$	y	$\dfrac{dy}{dx}$
x^n	nx^{n-1}		
a^x	$a^x \log a$	$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
e^x	e^x		
$\log_a x$	$\dfrac{1}{x \log a}$	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\log x$	$\dfrac{1}{x}$	$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sin x$	$\cos x$	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\cos x$	$-\sin x$		
$\tan x$	$\sec^2 x$	$\sec^{-1} x$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\cot x$	$-\operatorname{cosec}^2 x$		
$\sec x$	$\sec x \tan x$	$\operatorname{cosec}^{-1} x$	$-\dfrac{1}{x\sqrt{x^2-1}}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot$		

This table should be committed to memory.

59. Examples.

Ex. 1. $y = \sin^3 x.$

$$\frac{dy}{dx} = 3\sin^2 x \,.\, \frac{d \sin x}{dx} = 3 \sin^2 x \cos x.$$

Ex. 2. $y = \sec^2 x \operatorname{cosec} x.$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec^2 x) \,.\, \operatorname{cosec} x + \sec^2 x \,.\, \frac{d}{dx}(\operatorname{cosec} x)$$

$$= (2 \sec x \,.\, \sec x \tan x) \operatorname{cosec} x + \sec^2 x(-\operatorname{cosec} x \cot x) \text{ [Art. 35]}$$

$$= 2\sec^3 x - \sec x \operatorname{cosec}^2 x = \sec x (2 \sec^2 x - \operatorname{cosec}^2 x).$$

Ex. 3. $y = \cot\left(\dfrac{5x^2-3}{x+1}\right) = \cot u$ say, where $u = \dfrac{5x^2-3}{x+1}.$

$$\therefore \frac{du}{dx} = \frac{10x(x+1) - (5x^2-3)}{(x+1)^2} = \frac{5x^2+10x+3}{(x+1)^2}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\operatorname{cosec}^2 u \frac{du}{dx} = -\frac{5x^2+10x+3}{(x+1)^2} \operatorname{cosec}^2\left(\frac{5x^2-3}{x+1}\right).$$

Ex. 4. $y = \dfrac{\cos x}{\sqrt{1+\cos^2 x}} = \dfrac{u}{\sqrt{1+u^2}}$ say, where $u = \cos x$.

$$\therefore \frac{du}{dx} = -\sin x.$$

To find dy/du, we have—

$$\log y = \log u - \tfrac{1}{2}\log(1+u^2).$$

$$\therefore \frac{1}{y}\frac{dy}{du} = \frac{1}{u} - \frac{1}{2}\cdot\frac{1}{1+u^2}\cdot\frac{d}{du}(1+u^2) \text{ [Art. 44, Cor.]}$$

$$= \frac{1}{u} - \frac{u}{1+u^2} = \frac{1}{u(1+u^2)}.$$

$$\therefore \frac{dy}{du} = \frac{y}{u(1+u^2)} = \frac{1}{(1+u^2)^{\frac{3}{2}}};$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{\sin x}{(1+\cos^2 x)^{\frac{3}{2}}}.$$

Ex. 5. $y = \sec^{-1}\frac{1}{2}\left(x+\dfrac{1}{x}\right) = \cos^{-1}\dfrac{2x}{1+x^2}\left(\text{since } \sec^{-1} a = \cos^{-1}\dfrac{1}{a}\right).$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-\dfrac{4x^2}{(1+x^2)^2}}}\cdot\frac{d}{dx}\frac{2x}{1+x^2} \text{ [Art. 35]}$$

$$= -\frac{1+x^2}{\sqrt{(1+x^2)^2-4x^2}}\cdot 2\frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= -2\frac{1+x^2}{1-x^2}\cdot\frac{1-x^2}{(1+x^2)^2} = -\frac{2}{1+x^2}.$$

Otherwise, put $x = \tan\theta$;

$$\therefore y = \cos^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \cos^{-1}\sin 2\theta = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x.$$

$$\therefore \frac{dy}{dx} = -2\cdot\frac{1}{1+x^2} = -\frac{2}{1+x^2}.$$

For a list of substitutions, see Art. 337.

EXAMPLES XII.

1. *Write down* the d.c. of—

(1) $\sin(x+1)$, $\cos 2x$, $\tan(1-x)$, $\cot(3-5x)$, $\sec bx$, $\operatorname{cosec}(a-bx)$.

(2) $\sin x^2$, $\sin^2 x$, $\cot\sqrt{x}$, $\sqrt{\cot x}$, $\operatorname{cosec}(1-\sqrt{x})$.

(3) $\tan\dfrac{1}{x}$, $\sin x\cos x$, $\sec\left(x-\dfrac{1}{x}\right)$, $\cot^3 x$, $\dfrac{1}{\cos x^3}$.

2. Find the d.c. of—

(1) $\sin x \cos^3 x$. (2) $\sec x \operatorname{cosec} x$. (3) $\sec x + \tan x$.

(4) $\operatorname{cosec} x + \cot x$. (5) $\sec^2 x + \tan^2 x$. (6) $\sec^n x$.

(7) $\cot \sqrt{ax^2 + b}$. (8) $\sec \dfrac{a}{\sqrt{a - x}}$. (9) $\tan \dfrac{2x - 3}{3x + 2}$.

(10) $\dfrac{1}{[illegible]}$. (11) $\dfrac{1 + \tan x}{1 - \tan x}$. (12) $\dfrac{\sec x + \tan x}{\sec x - \tan x}$.

3. Find the d.c. of—

(1) $\sin^{-1} \dfrac{x}{a}$. (2) $\dfrac{1}{a} \tan^{-1} \dfrac{x}{a}$. (3) $\dfrac{1}{a} \sec^{-1} \dfrac{x}{a}$.

(4) [illegible] (5) $\sec^{-1} \sqrt{x}$. (6) $\tan^{-1} \sqrt{x - 1}$.

(7) $\tan^{-1}(1 - 2x)$. (8) $\operatorname{cosec}^{-1} \sqrt{2 - x}$. (9) $\sec^{-1} \dfrac{1}{1 - x}$.

(10) $\sin^{-1} \sqrt{x} + \sin^{-1} \sqrt{1 - x}$. (11) $\tan^{-1} x + \tan^{-1} \dfrac{1}{x}$.

(12) $\sec^{-1} x + \sec^{-1} \dfrac{x}{\sqrt{x^2 - 1}}$. (13) $\cos^{-1} \sqrt{2 - x} + \cos^{-1} \sqrt{x - 1}$.

(14) $\operatorname{vers}^{-1} \dfrac{x}{a}$.

4. Find, by two methods, the d.c. of—

(1) $\sin^{-1} \sqrt{1 - x^2}$. (2) $\sin^{-1} 2x \sqrt{1 - x^2}$. (3) $\tan^{-1} \dfrac{x}{\sqrt{1 - x^2}}$.

(4) $\tan^{-1} \sqrt{x^2 - 1}$. (5) $\cot^{-1} \dfrac{x^2 - [illegible]}{[illegible]}$. (6) $\cos^{-1} \dfrac{\sqrt{2ax - [illegible]}}{a}$.

(7) $\operatorname{cosec}^{-1} \dfrac{x^2 + a^2}{2ax}$. (8) $2 \tan^{-1} \sqrt{\dfrac{1 - x}{1 + x}}$. (9) $\tan^{-1} \dfrac{ax}{ax}$

5. Differentiate both sides of the following identities, verifying the results:

(1) $\cos x = \dfrac{1}{\sec x}$. (2) $\sin 2x = 2 \sin x \cos x$.

(3) $\sin 3x = 3 \sin x - 4 \sin^3 x$.

(4) $\sin ax + \sin bx = 2 \sin \dfrac{a + b}{2} x \,.\, \cos \dfrac{a - b}{2} x$.

(5) $\tan^{-1} x = \sin^{-1} \dfrac{x}{\sqrt{1 + x^2}}$. (6) $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$.

(7) $\tan^{-1} x + \tan^{-1} (1 - x) = \cot^{-1} (1 - x + x^2)$.

(8) $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$.

(9) $$\sqrt{\frac{1 - \sin x}{1 + \sin x}} = \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}} = \tan \left(\frac{\pi}{4} - \frac{x}{2}\right).$$

6. Differentiate from first principles: (1) $\tan x$, (2) $\sec x$, (3) $\tan^{-1} x$.

7. Differentiate by a geometrical method: (1) $\cos x$, (2) $\tan x$, (3) $\sec x$.

8. Trace the graphs of $\sin x$, $\cos x$, etc.; also those of $\sin^{-1} x$, $\cos^{-1} x$, etc.

9. Illustrate geometrically by means of the graphs in the last question the fact that in the case of $\tan^{-1} x$ and $\cot^{-1} x$, dy/dx is single valued for any given value of x; while it has equal and opposite values for the other inverse functions.

ANSWERS.

1. (1) $\cos (x + 1)$, $-2 \sin 2x$, $-\sec^2 (1 - x)$, $5 \operatorname{cosec}^2 (3 - 5x)$,
$b \sec bx \tan bx$, $b \operatorname{cosec} (a - bx) \cot (a - bx)$.

(2) $2x \cos x^2$, $2 \sin x \cos x$, $-\dfrac{1}{2\sqrt{x}} \operatorname{cosec}^2 \sqrt{x}$, $-\dfrac{\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$,
$\dfrac{1}{2\sqrt{x}} \operatorname{cosec} (1 - \sqrt{x}) \cdot \cot (1 - \sqrt{x})$.

(3) $-\dfrac{\sec^2 1/x}{x^2}$, $\cos^2 x - \sin^2 x$, $\left(1 + \dfrac{1}{x^2}\right) \sec \left(x - \dfrac{1}{x}\right) \tan \left(x - \dfrac{1}{x}\right)$,
$-3 \cot^2 x \operatorname{cosec}^2 x$, $3x^2 \sec x^3 \tan x^3$.

2. (1) $\cos^2 x(\cos^2 x - 3 \sin^2 x)$. (2) $\sec x \operatorname{cosec} x (\tan x - \cot x)$

(3) $\sec x (\sec x + \tan x)$. (4) $-\operatorname{cosec} x (\operatorname{cosec} x + \cot x)$.

(5) $4 \sec^2 x \tan x$. (6) $n \sec^n x \tan x$.

(7) $\dfrac{-ax}{\sqrt{ax^2 + b}} \operatorname{cosec}^2 \sqrt{ax^2 + b}$. (8) $\dfrac{a}{2(a-x)^{\frac{3}{2}}} \sec \dfrac{a}{\sqrt{a-x}} \cdot \tan \dfrac{a}{\sqrt{a-x}}$.

(9) $\dfrac{13}{(3x + 2)^2} \sec^2 \dfrac{2x - 3}{3x + 2}$. (10) $\dfrac{\sec^2 x}{(1 - \tan x)^2}$.

(11) $\dfrac{2 \sec^2 x}{(1 - \tan x)^2}$. (12) $2 \sec x (\sec x + \tan x)^2$.

3. (1) $\dfrac{1}{\sqrt{a^2 - x^2}}$. (2) $\dfrac{1}{a^2 + x^2}$. (3) $\dfrac{1}{x\sqrt{x^2 - a^2}}$.

(4) $\dfrac{1}{2\sqrt{x}\sqrt{1 - x}}$. (5) $\dfrac{1}{2x\sqrt{x - 1}}$. (6) $\dfrac{1}{2x\sqrt{x - 1}}$.

(7) $-\dfrac{1}{1 - 2x + 2x^2}$. (8) $\dfrac{1}{2(2 - x)\sqrt{1 - x}}$. (9) $\dfrac{1}{\sqrt{2x - x^2}}$.

(10) 0. (11) 0. (12) 0. (13) 0. (14) $\dfrac{1}{\sqrt{2ax - x^2}}$.

4. (1) $-\dfrac{1}{\sqrt{1 - x^2}}$ [put $x = \cos\theta$]. (2) $\dfrac{2}{\sqrt{1 - x^2}}$ [$x = \sin\theta$].

(3) $\dfrac{1}{\sqrt{1 - x^2}}$ [$x = \sin\theta$]. (4) $\dfrac{1}{x\sqrt{x^2 - 1}}$ [$x = \sec\theta$].

(5) $-\dfrac{2}{1 + x^2}$ [$x = \cot\theta$]. (6) $-\dfrac{1}{\sqrt{2ax - x^2}}$ [$x = a$ covers θ].

(7) $\dfrac{2a}{a^2 + x^2}$ [$x = a\tan\theta$]. (8) $-\dfrac{1}{\sqrt{1 - x^2}}$ [$x = \cos\theta$].

(9) $-\dfrac{ab}{b^2 + a^2x^2}$ [put $x = \dfrac{b}{a}\cot\theta$; $\therefore y = \dfrac{\pi}{4} + \theta$].

CHAPTER VI.

MISCELLANEOUS EXAMPLES.

60. WE shall now work a few examples on all the preceding rules, and involving combinations of all the functions considered. Hyperbolic functions will be treated in the next chapter.

61. Ex. 1. *Differentiate* $y = \sin\{\sin(\sin x)\}$.

Here $\quad y = \sin u$, where $u = \sin(\sin x)$

$\qquad\qquad\qquad\qquad\qquad = \sin v$ say, where $v = \sin x$.

Also $\quad dy/du = \cos u,\ du/dv = \cos v,\ dv/dx = \cos x$,

$$\therefore\ dy/dx = \cos u \cos v \cos x = \cos\{\sin(\sin x)\} \,.\, \cos(\sin x) \,.\, \cos x.$$

Or,

$$dy/dx = \cos\{\sin(\sin x)\} \,.\, \frac{d}{dx}\sin(\sin x)$$

$$= \cos\{\sin(\sin x)\} \,.\, \cos(\sin x) \,.\, \frac{d}{dx}\sin x = \text{etc.}$$

In fact, the answer can be written down in one line with very little practice.

Ex. 2. $y = \sin^{-1}(\log x)$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\log x)^2}} \,.\, \frac{d}{dx}(\log x) = \frac{1}{x\sqrt{1-(\log x)^2}}.$$

Ex. 3. $y = (x + \sqrt{1-x^2})\, e^{\sin^{-1}x} = uv$ say,

where $\quad u = x + \sqrt{1-x^2},\ \therefore\ \dfrac{du}{dx} = 1 - \dfrac{x}{\sqrt{1-x^2}} = \dfrac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}}$;

and $\quad v = e^{\sin^{-1}x}\quad \therefore\ \dfrac{dv}{dx} = e^{\sin^{-1}x} \,.\, \dfrac{1}{\sqrt{1-x^2}}$,

$$\therefore\ \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x+\sqrt{1-x^2}) \,.\, \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} + e^{\sin^{-1}x} \,.\, \frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,(x + \sqrt{1-x^2} + \sqrt{1-x^2} - x) = 2e^{\sin^{-1}x}.$$

Or, put $x = \sin\theta$; $\therefore \dfrac{dx}{d\theta} = \cos\theta$.

$$\therefore y = (\sin\theta + \cos\theta)e^{\theta}.$$

and $$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = \frac{e^{\theta}(\cos\theta - \sin\theta) + e^{\theta}(\sin\theta + \cos\theta)}{\cos\theta}$$
$$= 2e^{\theta} = 2e^{\sin^{-1}x}.$$

Ex. 4. $y = (\tan x)^{\sin^{-1}x}$.

$$\log y = \sin^{-1}x \,.\, \log\tan x.$$

$$\frac{1}{y}\frac{dy}{dx} = \sin^{-1}x\frac{d}{dx}\log\tan x + \log\tan x \cdot \frac{d}{dx}\sin^{-1}x$$
$$= \sin^{-1}x \,.\, \frac{\sec^2 x}{\tan x} + \log\tan x \,.\, \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{\sin^{-1}x}{\sin x\cos x} + \frac{\log\tan x}{\sqrt{1 - x^2}}$$
$$\therefore \frac{dy}{dx} = (\tan x)^{\sin^{-1}x} \,.\, \left\{\frac{\sin^{-1}x}{\sin x\cos x} + \frac{\log\tan x}{\sqrt{1 - x^2}}\right\}.$$

Or, by Art. 47, Ex. 7 (Rule),

$$\frac{dy}{dx} = (\tan x)^{\sin^{-1}x}.\log\tan x \,.\, \frac{d}{dx}\sin^{-1}x + \sin^{-1}x\,(\tan x)^{\sin^{-1}x-1} \,.\, \frac{d}{dx}\tan x$$
$$= (\tan x)^{\sin^{-1}x}.\left\{\frac{\log\tan x}{\sqrt{1 - x^2}} + \frac{\sec^2 x\sin^{-1}x}{\tan x}\right\} = \text{etc.}$$

Note.—The second method should be adopted as well as the first. In fact, the second method is the preferable one for the more experienced student.

Ex. 5. *If* $x^3 + y^3 = 3axy$ (1); *find* dy/dx, *and determine for what values of* x *and* y, dy/dx *is zero or infinity. Give the geometrical interpretation.*

Here y is an *implicit* function of x. If we were to actually solve the equation, and substitute for y its equivalent expression in x, the equation would become an identity. We may therefore suppose this done, and regard y as the symbol which stands for this expression.

Differentiating both sides, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay$$
$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

We leave the answer in this form, as is usually done when the original equation is troublesome to solve.

Now (i) $dy/dx = 0$, if $ay = x^2$. Substituting in (1)

$$a^3x^3 + x^6 = 3a^3x^3, \therefore x = 0 \text{ or } a\sqrt[3]{2}.$$

To find y, substitute for x in (1);

we have $$y = 0, \text{ or } y^3 - 3a^2\sqrt[3]{2}\,.\,y + 2a^3 = 0,$$

which is satisfied by $y = a\sqrt[3]{4}$. Or, we may put $x^2 = ay$ in (1), giving a quadratic in $y^{\frac{3}{2}}$.

The values are $x = 0$, $y = 0$; and $x = 2^{\frac{1}{3}}a$, $y = 2^{\frac{2}{3}}a$.

(ii) $dy/dx = \infty$ if $ax = y^2$, and the method is similar to the above.

The values of x and y are evidently interchanged.

The geometrical interpretation is that at the points $(0, 0)$, $(2^{\frac{1}{3}}a, 2^{\frac{2}{3}}a)$ the tangent to the curve (1) is horizontal; and at the points $(0, 0)$, $(2^{\frac{2}{3}}a, 2^{\frac{1}{3}}a)$ it is vertical.

Ex. 6. *A ladder, of length* l, *slides in a vertical plane between a vertical wall and a horizontal plane. Compare the velocities of its extremities.*

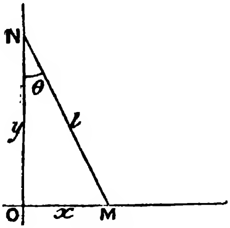

FIG. 9.

Let u and v be the velocities of M and N respectively; and let

$$OM = x,\ ON = y,\ \angle MNO = \theta;$$

$$\therefore dx/dt = u,\ dy/dt = v.$$

Now $x^2 + y^2 = l^2$, x and y being variable while l is constant. Hence y may be regarded as an implicit function of x. Differentiating in x,

$$2x + 2y\frac{dy}{dx} = 0; \therefore \frac{dy}{dx} = -\frac{x}{y} = -\tan\theta.$$

But $$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{v}{u} \text{ [Art. 24]}$$

$$\therefore v/u = -\tan\theta.$$

Since the ratio is —ve, it follows that as x increases, y diminishes; and *vice versâ.* This can be seen from the figure.

Ex. 7. *Differentiate* f(x) *with respect to* ϕ(x).

If $u = f(x)$, $v = \phi(x)$;

$$\text{then } \frac{du}{dv} = \frac{du}{dx}\Big/\frac{dv}{dx} = \frac{f'(x)}{\phi'(x)}.$$

EXAMPLES XIII.—MISCELLANEOUS.

Differentiate with respect to x—

1. $x\sin x + \cos x$. **2.** $x\log x - x$. **3.** $e^x(x-1)$.

4. $e^x \sin x$. **5.** $e^x(\sin x + \cos x)$. **6.** $\sqrt{\log x}$.

7. $\log\sin x$. **8.** $\log\tan x$. **9.** $\log(\tan x + \sec x)$.

10. $\sin x \,.\, \log\sin x$. **11.** $x\sin\log x$.

12. $x(\sin\log x + \cos\log x)$. **13.** $x\sin(\pi/4 + \log x)$.

14. $a^{\sin^{-1}x}$. **15.** $e^{\tan x}$. **16.** $e^{\log\tan x}$.

17. $(1+x^2)\,e^{\tan^{-1}x}$. **18.** $\tan^{-1}\sqrt{\cos x}$. **19.** $\log\tan(\pi/4 + x/2)$.

20. $\tan\tan x$. **21.** $\tan\tan\tan x$. **22.** $\log\sec\tan^{-1}x$.

23. $a^{\sqrt{x\cos x}}$. **24.** $\sin^{-1}x \,.\, \log\sin^{-1}x$. **25.** $f(x)\,.\,\log f(x)$.

26. $\sec^{-1}\dfrac{a-x}{a-2x}$. **27.** $\dfrac{\sqrt{a}-\sqrt{x}}{\sqrt{a-x}}$.

28. $\sqrt{2ax-x^2}\cos^{-1}\dfrac{a-x}{a}$. **29.** $\sqrt{2ax-x^2}$ vers

30. $(1+x)^{\frac{1}{x}}$. **31.** $(1+1/x)^x$. **32.** $(\log x)^{\frac{1}{x}}$.

33. $(\cos x)^{\sin x}$. **34.** e^{x^x}. **35.** e^{e^x}.

36. x^{e^x}. **37.** x^{x^x}. **38.** $x^{\log x}$.

39. $(\sqrt{x}+1)^{\log(\sqrt{x}+1)}$. **40.** $\dfrac{\log\sin x}{\log\cos x}$.

* *Most of the following examples, as far as* 70, *may be omitted on a first reading.*

41. $\log_x a$. **42.** $\log_x \sin x$.

43. $\log_v u$, u and v being functions of x.

44. $\dfrac{\sqrt{x^2-1}}{x^2} + \operatorname{cosec}^{-1}x$. **45.** $\log(\sqrt{x-a} + \sqrt{x-b})$.

46. $\log\sqrt{\dfrac{x-2}{x+2}} - \tan^{-1}\dfrac{x}{2}$. **47.** $\sec^{-1}\left(\sin\dfrac{a}{x} + \cos\dfrac{a}{x}\right)$.

48. $\sin^{-1}(x - \sqrt{x^2-1})$. **49.** $\dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$.

50. $\dfrac{x\sqrt{a^2+x^2}}{2} + \dfrac{a^2}{2}\log\{x+\sqrt{a^2+x^2}\}$.

51. $\log\sqrt{\dfrac{\tan x - 1}{\tan x + 1}} - x$

52. $\log\tan\frac{1}{2}(\tan^{-1}x + \tan^{-1}\frac{3}{4})$.

53. $\tan^{-1}\left(\sqrt{\dfrac{m-n}{m+n}}\tan\dfrac{x}{2}\right)$.

54. $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}} - \sec^{-1}\dfrac{1}{\sqrt{1-x^2}}$.

55. $\operatorname{cosec}^{-1}\left(2\cos^2\dfrac{x}{2}\right)$.

56. $\log\dfrac{a\cos x + b\sin x}{a\cos x - b\sin x}$.

57. $(\operatorname{cosec}^2 x + 2)\cos x + \log\tan^3\dfrac{x}{2}$.

58. $e^{ax^2}\sec^2 bx$.

59. $\dfrac{(a+x)\,e^{a\tan^{-1}x}}{(1+a^2)(1+x^2)^{\frac{1}{2}}}$.

60. $\dfrac{1}{\sqrt{b^2-a^2}}\log\dfrac{\sqrt{b+a}+\sqrt{b-a}\tan\dfrac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\dfrac{x}{2}}$.

61. $\log\dfrac{e^x+\sqrt{e^{2x}-a^2}}{e^x-\sqrt{e^{2x}-a^2}}$.

62. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$.

63. $x\sin^{-1}x\log\dfrac{ae}{x} + \sqrt{1-x^2}\log\dfrac{a}{x} + \log\dfrac{x}{1+\sqrt{1-x^2}}$.

64 $\dfrac{1}{6}\log\dfrac{(x^4-1)^2}{x^8+x^4+1} + \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x^4+1}{\sqrt{3}}$.

65. Differentiate x with respect to $\sin x$.

66.	,,	x	,,	$\sqrt{1-x^2}$.
67.	,,	$\sin x$	,,	$\cos x$.
68.	,,	$\log\cos x$	,,	$\sec x$.
69.	,,	$\sqrt{1-x^2}$	,,	$\sin^{-1}x$.
70.	,,	$e^{\sin^{-1}x}$	,,	$e^{-\cos^{-1}x}$.

Find at what points of the following curves the tangent is equally inclined to the *positive* directions of the axes of co-ordinates :—

71. $y = \sin x$. **72.** $y^2 = 4ax$. **73.** $y = \dfrac{b}{a}\sqrt{a^2-x^2}$.

74. $y = \dfrac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$. **75.** $x^3 - y^3 + 3axy = 0$.

76. Q is a point on the line $y = 1$, and the ordinate QPN cuts the curve $y = \log x$ in P. Show that the tangent to the curve at P is parallel to OQ, O being the origin.

77. The axis of a parabola is vertical, the vertex downwards. At what point is the curve twice as steep as at an extremity of the latus rectum?

78. A and B are two points facing each other on opposite sides of a river 30 feet wide. A person is walking on the bank, on the same side as A, and 40 feet from it at the point P. Use differentiation to show that the rate at which PB diminishes is $\frac{4}{5}$ths of the rate at which PA diminishes. [For the method, see answer.]

79. When $x = 0{\cdot}6$, find the increase in the function $x\sqrt{1 - x^2}$ produced by an increase of $0{\cdot}001$ in the value of x, approximately.

80. Show by differentiation that when $x = \dfrac{1}{2}$, the function $x^3\sqrt{\dfrac{1-x}{1+x}}$ is unaltered by a small change in the value of x.

81. If x be the length of the diameter of a sphere, and also that of a diagonal of the square base of a prism of height h; compare the rates of increase of the volumes of these solids as x increases. Also show that for the value of x at which the volumes are increasing at the same rate, the areas of the surfaces (not including the square ends of the prism) are increasing in the ratio of $\sqrt{2} : 1$.

82. Draw the graph of the function $\sqrt{\dfrac{x-2}{x-1}}$, and show that at no point is the direction of the curve parallel to the axis of x.

Find the inclination of the tangent, at the point where $x = \frac{9}{4}$, to the axis of x.

83. From the identity

$$\log(1 - x) + \log(1 + x) + \log(1 + x^2) + \dots + \log(1 + x^{2^n}) = \log(1 - x^{2^{n+1}}),$$

establish by differentiation an algebraical identity, and verify the latter.

Answers.

1. $x\cos x$. **2.** $\log x$. **3.** xe^x. **4.** $e^x(\sin x + \cos x)$.

5. $2e^x\cos x$. **6.** $\dfrac{1}{2x\sqrt{\log x}}$. **7.** $\cot x$. **8.** $2\operatorname{cosec} 2x$.

9 $\sec x$. **10.** $\cos x(1 + \log\sin x)$. **11.** $\sin\log x + \cos\log x$.

12. $2\cos\log x$. **13.** $\sin(\pi/4 + \log x) + \cos(\pi/4 + \log x) = \sqrt{2}\cos\log x$.

14. $\dfrac{a^{\sin^{-1}x}\log a}{\sqrt{1-x^2}}$. **15.** $\sec^2 x\, e^{\tan x}$. **16.** $\sec^2 x$. **17.** $e^{\tan^{-1}x}(2x+1)$.

18. $\dfrac{-\sin x}{2\sqrt{\cos x}(1+\cos x)} = -\dfrac{1}{2}\tan\dfrac{x}{2}\cdot\sqrt{\sec x}$. **19.** $\sec x$.

20. $\sec^2 x\,.\,\sec^2(\tan x)$. **21.** $\sec^2 x\,.\,\sec^2(\tan x)\,.\,\sec^2(\tan\tan x)$.

22. $\dfrac{x}{(1+x^2)}$. **23.** $\dfrac{a^{\sqrt{x\cos x}}\log a(\cos x - x\sin x)}{2\sqrt{x\cos x}}$. **24.** $\dfrac{(\log\sin^{-1}x+1)}{\sqrt{1-x^2}}$.

25. $f'(x)\{\log f(x)+1\}$. **26.** $\dfrac{a}{(a-x)\sqrt{x(2a-3x)}}$.

27. $\dfrac{-\frac{1}{2}\sqrt{a}(\sqrt{a}-\sqrt{x})}{\sqrt{x}(a-x)^{\frac{3}{2}}} = \dfrac{-\sqrt{a}}{2\sqrt{x}(\sqrt{a}+\sqrt{x})\sqrt{a-x}}$.

28. $\dfrac{a-x}{\sqrt{2ax-x^2}}\,.\,\cos^{-1}\dfrac{a-x}{a}+1$. **29.** $\dfrac{a-x}{\sqrt{2ax-x^2}}\,.\,\text{vers}^{-1}\dfrac{x}{a}+1$.

30. $\dfrac{(1+x)^{\frac{1}{x}-1}\{x-(1+x)\log(1+x)\}}{x^2}$.

31. $\dfrac{1}{x}\left(1+\dfrac{1}{x}\right)^{x-1}\left\{(1+x)\log\left(1+\dfrac{1}{x}\right)-1\right\}$.

32. $\dfrac{1}{x^2}(\log x)^{\frac{1}{x}-1}\{1-\log x\,.\,\log\log x\}$.

33. $(\cos x)^{\sin x-1}\,.\,(\cos^2 x\log\cos x - \sin^2 x)$. **34.** $e^{x^x}\,.\,x^x(\log x+1)$.

35. $e^{e^x}\,.\,e^x = e^{e^x+x}$. **36.** $e^x\,.\,x^{e^x-1}(1+x\log x)$.

37. $x^{x^x+x-1}\,.\,\{1+x\log x(\log x+1)\}$. **38.** $2\log x\,.\,x^{\log x-1}$.

39. $\dfrac{1}{\sqrt{x}}\log(\sqrt{x}+1)\,.\,(\sqrt{x}+1)^{\log(\sqrt{x}+1)-1}$.

40. $\dfrac{(\cot x\,.\,\log\cos x+\tan x\,.\,\log\sin x)}{(\log\cos x)^2}$.

41. $\dfrac{-1}{(\log_e x)^2\,.\,x\log_e a} = \dfrac{-\log a}{x(\log x)^2}$. **42.** $\dfrac{(x\cot x\,.\,\log x - \log\sin x)}{x(\log x)^2}$.

43. $\dfrac{u'v\log v - v'u\log u}{uv(\log v)^2}$. **44.** $\dfrac{-2\sqrt{x^2-1}}{x^3}$. **45.** $\dfrac{}{2\sqrt{(x-a)(x-b)}}$.

46. $\dfrac{16}{x^4-16}$. **47.** $\dfrac{a\left(\sin\dfrac{a}{x}-\cos\dfrac{a}{x}\right)}{x^2\left(\sin\dfrac{a}{x}+\cos\dfrac{a}{x}\right)\sqrt{\sin\dfrac{2a}{x}}} = \dfrac{\dfrac{a}{x^2}\left(1-\sin\dfrac{2a}{x}\right)}{\cos\dfrac{2a}{x}\sqrt{\sin\dfrac{2a}{x}}}$.

48. $\dfrac{-\sqrt{x}-\sqrt{x^2-1}}{\sqrt{2}(x^2-1)^{\frac{3}{4}}} = \dfrac{-(\sqrt{x+1}-\sqrt{x-1})}{2(x^2-1)^{\frac{3}{4}}}.$ 49. $\sqrt{a^2-x^2}.$

50. $\sqrt{a^2+x^2}.$ 51. $2/(\tan^2 x - 1).$

52. $1/(1+x^2)\sin(\tan^{-1}x + \tan^{-1}\frac{3}{4}) = 5/(4x+3)\sqrt{1+x^2}.$

53. $\dfrac{\sqrt{m^2-n^2}}{2(m+n\cos x)}$ 54. 0. 55. $\tan\dfrac{x}{2}\Big/\sqrt{4\cos^4\dfrac{x}{2}-1}.$

56. $2ab/(a^2\cos^2 x - b^2\sin^2 x).$ 57. $-2\cos x\cot^3 x.$

[illegible] $\sec^2 bx\,(ax + b\tan bx).$ 59. $\dfrac{e^{a\tan^{-1}x}}{(1+x^2)^{\frac{1}{2}}}$ [illegible] $\dfrac{[illegible]}{a+b\cos x}$

[illegible] $\dfrac{2e^{x}}{\sqrt{e^{2x}-a^2}}.$ 62. $\dfrac{3a}{a^2+x^2}.$ 63. $\sin^{-1}x\,.\log\dfrac{a}{x} + \dfrac{2x}{\sqrt{1-x^2}}.$

[illegible] $\dfrac{[illegible]x^7}{x^{12}-1}.$ [illegible] $\sec x.$ 66. $-\dfrac{\sqrt{1-x^2}}{x}.$ 67. $-\cot x.$

68. $-\cos x.$ 69. $-x.$ 70. $e^{\sin^{-1}x+\cos^{-1}x} = e^{\pi/2}.$

71. $dy/dx = 1$ at $(0,0)$. 72. $(a, 2a)$. 73. $\left(\mp\dfrac{a^2}{\sqrt{a^2+b^2}}, \pm\dfrac{b^2}{\sqrt{a^2+b^2}}\right).$

74. $\{c\log(1+\sqrt{2}), c\sqrt{2}\}.$ 75. $\left(\dfrac{3a}{2}, -\dfrac{3a}{2}\right)$ and (∞, ∞).

77. For the parabola $x^2 = 4ay$, the point is $(4a, 4a)$.

78. If $PA = x$, $PB = y$; $y^2 = x^2 + 30^2$; $\therefore dy/dx = x/y$. *Now* put $x = 40$, $\therefore y = 50$, etc.

[illegible] 0·00053. [illegible] $\pi x : 2h.$ 82. $\tan^{-1}\left(\dfrac{8}{5\sqrt{5}}\right).$

[illegible] $\dfrac{1}{[illegible]\,1} + \dfrac{1}{x+1} + \dfrac{[illegible]x}{x^2+1} + \dfrac{4x^3}{x^4+1} + \dots + \dfrac{2^n.x^{2^n-1}}{x^{2^n}+1} = \dfrac{2^{n+1}.x^{2^{n+1}-1}}{x^{2^{n+1}}-1}.$

* CHAPTER VII.

HYPERBOLIC FUNCTIONS AND THEIR DIFFERENTIATION.

62. Direct Functions.—The hyperbolic functions *cosh x*, *sinh x* (pronounced *cosh*, *shĭn*) are provisionally defined as follows:—

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = e^x - e^{-x}$$

and are termed respectively the *hyperbolic cosine*, and *hyperbolic sine* of x.

For an explanation of the terms, see Art. 68.

The other hyperbolic functions are *tanh x*, *coth x*, *sech x*, *cosech x* (pronounced *than*, *coth*, *shec*, *coshec*), and are thus defined:—

$$\tanh x = \frac{\sinh x}{\cosh x}; \quad \coth x = \frac{1}{\tanh x}; \quad \operatorname{sech} x = \frac{1}{\cosh x};$$

$$\operatorname{cosech} x = \frac{1}{\sinh x};$$

the relations being quite similar to those connecting the corresponding circular functions.

63. The following relations can be easily verified from the above definitions, and should, if possible, be committed to memory. We place them side by side with their analogues in the case of the circular functions.

Hyperbolic Functions.	Circular Functions.
$\cosh^2 x - \sinh^2 x = 1$	$\cos^2 x + \sin^2 x = 1$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$1 + \tan^2 x = \sec^2 x$
$\coth^2 x - 1 = \operatorname{cosech}^2 x$	$\cot^2 x + 1 = \operatorname{cosec}^2 x$
$\sinh(-x) = -\sinh x$	$\sin(-x) = -\sin x$
$\cosh(-x) = \cosh x$	$\cos(-x) = \cos x$
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$	$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cosh^2 x - 1$	$= 2\cos^2 x - 1$
$= 1 + 2\sinh^2 x$	$= 1 - 2\sin^2 x$
$\sinh 2x = 2 \sinh x \cosh x$	$\sin 2x = 2 \sin x \cos x$
Note also that $\cosh x = \frac{1}{2}(e^x + e^{-x})$	$\cos x = \frac{1}{2}(e^{xi} + e^{-xi})$
$\sinh x = \frac{1}{2}(e^x - e^{-x})$	$\sin x = \frac{1}{2i}(e^{xi} - e^{-xi})$

64. Inverse Functions.

Let $y = \sinh^{-1} x$; then $x = \sinh y = \dfrac{e^y - e^{-y}}{2} = \dfrac{e^{2y} - 1}{2e^y}$;

$$\therefore\ e^{2y} - 2xe^y - 1 = 0, \text{ or } e^y = x \pm \sqrt{x^2+1}.$$

$$\therefore\ y, \text{ or } \sinh^{-1} x, = \log\left(x \pm \sqrt{x^2+1}\right).$$

Since x is numerically $< \sqrt{x^2+1}$, it follows that *we cannot take the lower sign for real values of y* (the logarithm of a negative quantity being imaginary).

Hence $$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right),$$
where x may have any value between $-\infty$ and $+\infty$.

65. If $y = \cosh^{-1} x$, we get $e^y = x \pm \sqrt{x^2-1}$;

$\therefore\ y$, or $\cosh^{-1} x$, $= \log\left(x \pm \sqrt{x^2-1}\right)$;

or, since $x - \sqrt{x^2-1} = \dfrac{1}{x + \sqrt{x^2-1}}$,

$$\cosh^{-1} x = \pm \log\left(x + \sqrt{x^2-1}\right),$$

the $+^{\text{ve}}$ value of the root being taken.

Hence, *$\cosh^{-1} x$ is double-signed;* also, *for real values of $\cosh^{-1} x$, x must be $+^{\text{ve}}$ and > 1*, since (1) x is numerically $> \sqrt{x^2-1}$, and therefore if x were $-^{\text{ve}}$, $x + \sqrt{x^2-1}$ would also be $-^{\text{ve}}$; and (2) $x^2 - 1$ must be $+^{\text{ve}}$.

66. If $y = \tanh^{-1} x$; then $x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}} = \dfrac{e^{2y} - 1}{e^{2y} + 1}$;

$$\therefore \text{(comp}^{\text{o}}\text{. and div}^{\text{o}}\text{.)}, \ e^{2y} = \frac{1+x}{1-x};$$

$$\therefore y, \text{ or } \tanh^{-1} x, = \frac{1}{2} \log \frac{1+x}{1-x},$$

where x must lie between $+1$ and -1 for real values of $\tanh^{-1} x$.

67. Similarly, $\coth^{-1} x = \dfrac{1}{2} \log \dfrac{x+1}{x-1}$,

where x cannot lie between $+1$ and -1 for real values of $\coth^{-1} x$.

Again, $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x} = \pm \log \dfrac{1 + \sqrt{1 - x^2}}{x}$,

where x must lie between 0 and $+1$ for real values of $\operatorname{sech}^{-1} x$.

NOTE.—$\operatorname{Sech}^{-1} x$, like $\cosh^{-1} x$, is double-signed.

$$\text{Finally, } \operatorname{cosech}^{-1} x = \sinh^{-1} \frac{1}{x} = \log \left(\frac{1}{x} \pm \sqrt{\frac{1}{x^2} + 1} \right)$$
$$= \log \frac{1 \pm \sqrt{1 + x^2}}{x}.$$

And $\log \dfrac{1 + \sqrt{1 + x^2}}{x}$ will be real only if x is $+^{\text{ve}}$, while $\log \dfrac{1 - \sqrt{1 + x^2}}{x}$ will be real only if x is $-^{\text{ve}}$. Hence $\operatorname{cosech}^{-1} x$ will have one real value for *any* value of x between $-\infty$ and $+\infty$.

68. Reason for the Term "Hyperbolic."

Let a point $P(x, y)$ move on the circle $x^2 + y^2 = a^2$ . . . . (1)

Then, since $x = a \cos \phi$, $y = a \sin \phi$ satisfy (1) for all values of ϕ, we may regard the latter expressions as the co-ordinates of a point on the circle.

Similarly, if P (x, y) move on the rectangular hyperbola

$$x^2 - y^2 = a^2; \qquad \ldots \ldots \ldots \qquad (2)$$

then since $x = a\cosh u$, $y = a\sinh u$ satisfy (2) for all values of u, we may regard the latter expressions as the co-ordinates of a point on a rectangular hyperbola.

But while we know that ϕ is the angle AOP in the circle, we do not as yet know what u represents in the hyperbola.

It will be shown, however, that whereas the area of the circular sector AOP is $\frac{1}{2}a^2\theta$, that of the hyperbolic sector AOP is $\frac{1}{2}a^2u$ [Art. 439].

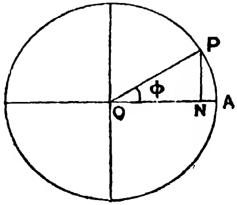

FIG. 10.

Now let $\frac{1}{2}a^2\theta = \Theta$, $\frac{1}{2}a^2u = U$; or $\theta = 2\Theta/a^2$, $u = 2U/a^2$; then for the circle we have $x = a\cos\dfrac{2\Theta}{a^2}$, $y = a\sin\dfrac{2\Theta}{a^2}$, and for the rectangular hyperbola, $x = a\cosh\dfrac{2U}{a^2}$, $y = a\sinh\dfrac{2U}{a^2}$. The analogy is therefore more apparent if we express, in each case, the co-ordinates of P as functions of the sector bounded by OA and OP.

For a geometrical treatment, see *Levett and Davison's Trigonometry.*

69. The Gudermannian.

In Fig. 11 draw NR to touch the auxiliary circle.

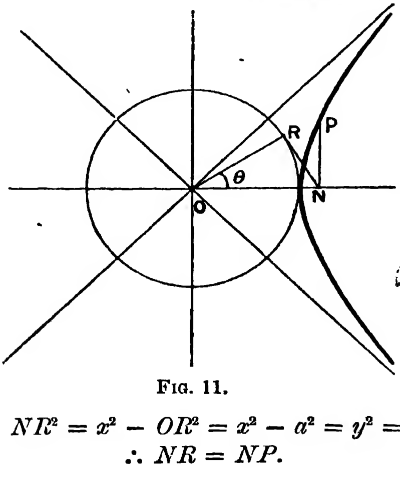

FIG. 11.

Then $$NR^2 = x^2 - OR^2 = x^2 - a^2 = y^2 = NP^2$$
$$\therefore NR = NP.$$

If $\angle RON = \theta$, then $x = a\sec\theta$, $y = NR = a\tan\theta$, and these latter expressions satisfy (2) for all values of θ. This angle θ is called the *gudermannian* of u, and is written $\theta = \operatorname{gd} u$.

Thus, if $\theta = \operatorname{gd} u$, this means that $\sec\theta = \cosh u$.

NOTE.—θ is many valued, but the ***principal*** (or smallest $+^{\text{ve}}$) value is here taken.

70. Inverse Gudermannian.

If $\theta = \operatorname{gd} u$ then we may write $u = \operatorname{gd}^{-1}\theta$.

$$\begin{aligned} \text{And } u = \cosh^{-1}\sec\theta &= \pm\log(\sec\theta + \sqrt{\sec^2\theta - 1}) \text{ [by Art 65]},\\ &= \pm\log(\sec\theta + \tan\theta),\\ \text{or} \quad &= \pm\log\tan(\pi/4 + \theta/2). \quad [\textit{Misc. Theorems.}] \end{aligned}$$

NOTE.—u is double-signed for any given value of θ, but hereafter we shall take the upper sign only.

EXAMPLES XIV.

Prove the following identities:—

1. $\sinh x = \dfrac{\sin xi}{i} = i\sin\dfrac{x}{i}$

$\sin x = \dfrac{\sinh xi}{i} = i\sinh\dfrac{x}{i}$

2. $\cosh x = \cos xi = \cos\dfrac{x}{i}$

$\cos x = \cosh xi = \cosh\dfrac{x}{i}$

3. $\tanh x = \dfrac{\tan xi}{i} = i\tan\dfrac{x}{i}$

$\tan x = \dfrac{\tanh xi}{i} = i\tanh\dfrac{x}{i}$

4. $x = \log(\cosh x + \sinh x)$

$= -\log(\cosh x - \sinh x)$.

5. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

6. $\sinh x = \dfrac{\tanh x}{\sqrt{1 - \tanh^2 x}} = \dfrac{\sqrt{1 - \operatorname{sech}^2 x}}{\operatorname{sech} x}$.

7. $x = \dfrac{1}{2}\log\dfrac{1 + \tanh x}{1 - \tanh x} = \log\dfrac{1 + \tanh x}{\operatorname{sech} x} = \log\dfrac{\operatorname{sech} x}{1 - \tanh x}$.

8. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x\tanh y}$.

9. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$.

10. $\sinh 2x = \dfrac{2\tanh x}{1 - \tanh^2 x}$.

11. $\dfrac{\cosh 2x - 1}{\cosh 2x + 1} = \tanh^2 x.$ 12. $e^{2x} = \dfrac{1 + \tanh x}{1 - \tanh x}.$

13. $\left(\dfrac{1 + \tanh x}{1 - \tanh x}\right)^2 = \dfrac{1 + \tanh 2x}{1 - \tanh 2x}.$

14. $\dfrac{\cosh 2x - 1}{\sinh 2x} = \dfrac{\sinh 2x}{\cosh 2x + 1} = \tanh x.$

15. $\coth x + \operatorname{cosech} x = \coth \dfrac{x}{2}.$ 16. $\coth x - \operatorname{cosech} x = \tanh \dfrac{x}{2}.$

17. $\sinh^{-1} \dfrac{x}{a} = \log(x + \sqrt{x^2 + a^2}) - \log a.$

18. $\cosh^{-1} \dfrac{x}{a} = \log(x + \sqrt{x^2 - a^2}) - \log a.$

19. $\tanh^{-1} \dfrac{x}{a} = \dfrac{1}{2} \log \dfrac{a + x}{a - x}.$ 20. $\tanh^{-1} \dfrac{a - b}{2x - a - b} = \dfrac{1}{2} \log \dfrac{x - b}{x - a}.$

If $\theta = \operatorname{gd} u$, prove that :—

21. $\left.\begin{array}{l}\sin\theta = \tanh u\\ \tan\theta = \sinh u\end{array}\right\}$; $\left.\begin{array}{l}\cos\theta = \operatorname{sech} u\\ \sec\theta = \cosh u\end{array}\right\}$; $\left.\begin{array}{l}\cot\theta = \operatorname{cosech} u\\ \operatorname{cosec}\theta = \coth u\end{array}\right\}$

22. $e^u = \tan\theta + \sec\theta.$ 23. $\theta = 2\tan^{-1} e^u - \pi/2.$

24. $\tan \dfrac{\theta}{2} = \tanh \dfrac{u}{2}.$ 25. $\sinh u = \operatorname{cosech} \log \cot \dfrac{v}{2}.$

Prove that :—

26. $\log x = \tanh^{-1} \dfrac{x^2 - 1}{x^2 + 1}.$ 27. $\log \cot x = \tanh^{-1} \cos 2x.$

28. $\sec x = \cosh \log(\tan x + \sec x).$

29. $\sin x = \operatorname{sech} \log(\operatorname{cosec} x + \cot x).$

30. $\operatorname{gd}^{-1}\theta = \tanh^{-1}(\sin\theta) = \dfrac{1}{2} \log \dfrac{1 + \sin\theta}{1 - \sin\theta}.$

31. $\operatorname{gd}^{-1}\theta = \sinh^{-1}(\tan\theta) = \log(\tan\theta + \sec\theta).$

32. $\operatorname{gd}^{-1} 2\theta = \log \dfrac{1 + \tan\theta}{1 - \tan\theta} = 2\tanh^{-1}\tan\theta = \tanh^{-1}\sin 2\theta.$

33. $\sin^{-1}\tanh x = \tan^{-1}\sinh x.$ 34. $\sinh^{-1}\tan x = \tanh^{-1}\sin x.$

35. $e^{\tanh^{-1} x} = \sqrt{\dfrac{1 + x}{1 - x}}.$ 36. $e^{\sinh^{-1} x} = x + \sqrt{1 + x^2}.$

71. Differentiation of Hyperbolic Functions. The hyperbolic functions, $\sinh x$ and $\cosh x$, can be differentiated by methods similar to those adopted for the circular functions. But though this would still further bring out the analogy between the two classes of functions, yet the following method is obviously simpler. For the rest the methods are the same as for the corresponding circular functions.

72. Direct Functions.

If $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$; $\dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{2} = \cosh x$.

Similarly, if $y = \cosh x$, $dy/dx = \sinh x$.

73. If $y = \tanh x = \dfrac{\sinh x}{\cosh x}$; $\dfrac{dy}{dx} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$ (by rule for quotients) $= \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$.

Similarly, if $y = \coth x$, $dy/dx = -\operatorname{cosech}^2 x$.

74. If $y = \operatorname{sech} x = \dfrac{1}{\cosh x}$; $\dfrac{dy}{dx} = -\dfrac{\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$

Similarly, if $y = \operatorname{cosech} x$, $dy/dx = -\coth x \operatorname{cosech} x$.

75. Inverse Functions.

If $y = \sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$; then $x = \sinh y$.

$$\therefore \frac{dy}{dx} = 1\bigg/\frac{dx}{dy} = \frac{1}{\cosh y} = \frac{1}{\sqrt{x^2+1}}.$$

Similarly, if $y = \cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$, $[\infty > x > 1]$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}.$$

76. If $y = \tanh^{-1} x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$, $[1 > x > -1]$;

then
$$x = \tanh y.$$

$$\therefore \frac{dy}{dx} = 1\bigg/\frac{dx}{dy} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-x^2}.$$

Similarly, if $y = \coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1}, [x > 1 \text{ or } < -1]$:

$$\frac{dy}{dx} = -\frac{1}{x^2-1}.$$

77. If $y = \operatorname{sech}^{-1} x = \log\frac{1+\sqrt{1-x^2}}{x}, [1 > x > 0]$;

then $x = \operatorname{sech} y$.

$$\therefore \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = -\frac{1}{\operatorname{sech} y \tanh y} = -\frac{1}{x\sqrt{1-x^2}}.$$

Similarly, if $y = \operatorname{cosech}^{-1} x = \log\frac{1 \pm \sqrt{1+x^2}}{x}$ (according as x is $+^{ve}$ or $-^{ve}$),

then $\dfrac{dy}{dx} = \mp\dfrac{1}{x\sqrt{1+x^2}}$ (according as x is $+^{ve}$ or $-^{ve}$).

78. **gd** x.

Let $y = \operatorname{gd} x$; then $\sec y = \cosh x$;

$$\therefore \sec y \tan y \frac{dy}{dx} = \sinh x; \quad \frac{dy}{dx} = \frac{\sinh x}{\sec y \tan y}$$

$$= \frac{\sinh x}{\cosh x \sinh x} \text{ (Exs. XIV. No. 21) } = \operatorname{sech} x.$$

79. **gd**$^{-1}$ x.

Let $y = \operatorname{gd}^{-1} x = \log(\tan x + \sec x) = \log\tan(\pi/4 + x/2)$.

Then $x = \operatorname{gd} y$; $\therefore \dfrac{dy}{dx} = 1\Big/\dfrac{dx}{dy} = \dfrac{1}{\operatorname{sech} y} = \cosh y = \sec x$.

80. We append a table, subject, however, to the remarks made above and in Art. 57.

COMPARATIVE TABLE OF CIRCULAR AND HYPERBOLIC FUNCTIONS.

Circular Functions.	Differential Coefficient.	Hyperbolic Functions.	Differential Coefficient.
$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$\sec^2 x$	$\tanh x$	$\operatorname{sech}^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$	$\coth x$	$-\operatorname{cosech}^2 x$
$\sec x$	$\sec x \tan x$	$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sinh^{-1} x = \log(x+\sqrt{x^2+1})$	$\dfrac{1}{\sqrt{x^2+1}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$	$\cosh^{-1} x = \log(x+\sqrt{x^2-1})$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\tanh^{-1} x = \frac{1}{2}\log\dfrac{1+x}{1-x}$	$\dfrac{1}{1-x^2}$
$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$	$\coth^{-1} x = \frac{1}{2}\log\dfrac{x+1}{x-1}$	$-\dfrac{1}{x^2-1}$
$\sec^{-1} x$	$\dfrac{1}{x\sqrt{x^2-1}}$	$\operatorname{sech}^{-1} x = \log\dfrac{1+\sqrt{1-x^2}}{x}$	$-\dfrac{1}{x\sqrt{1-x^2}}$
$\operatorname{cosec}^{-1} x$	$-\dfrac{1}{x\sqrt{x^2-1}}$	$\operatorname{cosech}^{-1} x = \log\dfrac{1+\sqrt{1+x^2}}{x}$	$-\dfrac{1}{x\sqrt{1+x^2}}$
		$\operatorname{gd} x$	$\operatorname{sech} x$
		$\operatorname{gd}^{-1} x$	$\sec x$

EXAMPLES XV.

Differentiate :—

1. $\sinh x + \cosh x$.
2. $x \cosh x - \sinh x$.
3. $\log \sinh x$.
4. $\log(\sinh x + \cosh x)$.
5. $\log \tanh \dfrac{x}{2}$.
6. $\log(\coth x + \operatorname{cosech} x)$.
7. $\log\{\sinh x + \sqrt{\sinh^2 x + 1}\}$
8. $\log \dfrac{\cosh \frac{x}{2} - 1}{\cosh \frac{x}{2} + 1}$.
9. $\log \dfrac{1+\tanh x}{1-\tanh x}$.

10. $\dfrac{x\sqrt{x^2+a^2}}{2}+\dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a}.$ **11.** $x\sinh^{-1}x-\sqrt{1+x^2}.$

12. $\sinh^{-1}\tan x.$ **13.** $\tan^{-1}\sinh x.$ **14.** $\tanh^{-1}\sin x.$

15. $\sin^{-1}\tanh x.$ **16.** $\cosh^{-1}\sec x.$ **17.** $\sec^{-1}\cosh x.$

18. $\tan^{-1}\dfrac{x}{a}+\tanh^{-1}\dfrac{x}{a}.$ **19.** $(\sqrt{1+x^2}-x)\,e^{\sinh^{-1}x}.$

20. $\sinh^{-1}2x$ **21.** $\tanh^{-1}\dfrac{x}{\sqrt{1+x^2}}.$ **22.** $\tanh^{-1}\dfrac{2x}{1+x^2}.$

23. $\tanh^{-1}\dfrac{a-x}{a}-\tanh^{-1}\dfrac{b-x}{b}.$ **24.** $\tanh^{-1}\dfrac{a-b}{2x-a-b}.$

ANSWERS.

1. $\cosh x+\sinh x.$ **2.** $x\sinh x.$ **3.** $\coth x.$ **4.** 1. **5.** $\operatorname{cosech} x.$

6. $-\operatorname{cosech} x.$ **7.** 1. **8.** $\operatorname{cosech} x.$ **9.** 2. **10.** $\sqrt{x^2+a^2}.$

11. $\sinh^{-1}x.$ **12.** $\sec x.$ **13.** $\operatorname{sech} x.$ **14.** $\sec x.$ **15.** $\operatorname{sech} x.$

16. $\sec x.$ **17.** $\operatorname{sech} x.$ **18.** $\dfrac{2a^3}{a^4-x^4}.$ **19.** 0. **20.** $\dfrac{2}{\sqrt{1+x^2}}.$

21. $\dfrac{1}{\sqrt{1+x^2}}.$ **22.** $\dfrac{2}{1-x^2}.$ **23.** $\dfrac{a-b}{(2a-x)(2b-x)}.$ **24.** $\dfrac{b-a}{2(x-a)(x-b)}.$

CHAPTER VIII.

SMALL QUANTITIES—INFINITESIMALS—ERRORS.

81. Large and Small are Purely Relative Terms.—The magnitude, or size, of a quantity, although in itself an absolute thing, cannot be estimated by us except by comparison with another like quantity. When we say that a quantity is large or small, we only mean that it is larger or smaller than a second quantity which we have in our minds, and which we are unconsciously taking as the unit, or standard of comparison.

Thus a foot is regarded as a small quantity when we are considering the distance between one town and another, but we call the same quantity enormous when considering the lengths of microscopical objects. Hence *the same quantity may be regarded as large at one time and small at another.*

Suppose now that we have chosen our standard of comparison; then the relative value of the quantity under consideration will be expressed by an abstract number. According as this is greater or less than 1, will the quantity tend towards "large" or "small."

82. Errors.—An error in measuring a given length is only properly estimated by comparing it with the true length. In practice, of course, the error is always considerably less than the true length, so that the fraction $\dfrac{\text{error}}{\text{true length}}$ is less than 1, and therefore tends towards "small."

But yet we sometimes speak of an error as being "large," the reason being that we are comparing this with some smaller error which we take as the standard, and the latter depends either on

the amount of accuracy desired or on the amount actually obtainable. In practical everyday life we only desire a moderate amount of accuracy, but in work such as is performed in observatories we desire as much as possible, the limit being only set by the imperfections of our instruments.

83. Approximation.—We have seen that in differentiation from first principles, as in Art. 6, we usually obtain a series of terms involving h and its powers; and in the final stage when h is indefinitely diminished (whether we have divided down by h or not) all terms beyond the first are rejected. So long as h is finite, however small, the final result is only an approximation; but by including more and more of the terms which were otherwise rejected, we can (without further reducing h) approximate as close as we please to the true value.

For example, the error in the computed volume of a cube, due to a small error, h, in measuring the length of a side of (correct) length x,

$$= (x + h)^3 - x^3 = 3x^2h + 3xh^2 + h^3.$$

Now suppose $x = 3$ feet, $h = 0{\cdot}001$ feet; the error

$$= 27 \times 0{\cdot}001 + 9 \times 0{\cdot}000001 + 0{\cdot}000000001$$
$$= 0{\cdot}027 + 0{\cdot}000009 + 0{\cdot}000000001 = 0{\cdot}027009001.$$

Hence, to a first approximation, the error = $3x^2h = 0{\cdot}027$.

" second " " $3x^2h + 3xh^2 = 0{\cdot}027009$.

" third " " $3x^2h + 3xh^2 + h^3 = 0{\cdot}027009001$.

84. In practice, this error 0·001 would itself be only roughly accurate; hence there is no value to be set upon the second and third approximations, since the first term 0·027 is only roughly true, apart from the other terms being omitted.

85. Small Quantities.

Def.—A magnitude, whose ratio to a standard magnitude (chosen at will) is finite, but smaller than a given ratio (previously agreed upon and taken as the standard), is termed a *small quantity*.

Two quantities are said to be *comparable* with or to each other

when the ratio of the smaller to the larger is not so small as to come within the meaning of the term "small quantity."

Taking unity as the standard, if x and y be small quantities, and $y = kx$, where k is comparable with unity (so that x and y are comparable with each other), then y is said to be a small quantity of the *same order* as x, which we may regard as of the *first order*.

Again, a quantity whose ratio to a small quantity of the first order is smaller than the standard ratio, is said to be of the *second order*. Thus, if x and y are the same as above, then x^2, xy, and y^2 are each of the second order. We may similarly define the third and higher orders of small quantities.

NOTE.—Precisely similar remarks apply to the term *large quantity*.

86. It should be observed that (given their ratios to each other) the smaller $x, y, z, \ldots$ are compared with unity, the more distinctly do the different higher orders stand out from each other.

We shall illustrate this by a simple example.

Ex. Let $x = 10y$.

(1) Then if $x = 0{\cdot}01$, $y = 0{\cdot}001$, the standard ratio being $1:100$ say,

$$x^3 = 1000y^3 = y^2, \text{ since } y = 0{\cdot}001.$$

Hence the second and third orders are not distinct, but are confused with each other.

(2) Again, let $x = 1/10^{12}$, $y = 1/10^{13}$, the standard ratio being $1:10^{12}$ say.

Then $x^3 = 1000y^3$ as before; but 1000 is not large enough in this case to alter the order of the term y^3; hence the 2nd and 3rd orders are quite distinct from each other.

But $x^{13} = y^{12}$; hence the 12th and 13th orders are confused with each other.

(3) Generally, let $x = 1/10^n$, $y = 1/10^{n+1}$, the standard ratio being $1:10^n$ say.

Then $x^{n+1} = y^n$, and the nth and $(n+1)$th orders are confused with each other.

When n is *very* large, then x and y become very small, and it is only the *very* high order, *viz.* the nth, which becomes confused with the next higher order, the lower orders standing out distinctly from each other.

87. Infinitesimals.

Def.—A magnitude whose ratio to unity is indefinitely small is termed an *infinitesimal quantity*, or *infinitesimal*.

By "indefinitely small" we mean "smaller than any quantity we can conceive, without being absolutely nothing."

If x and y be two such quantities, and $y = kx$, where k is *finite*, then y is an infinitesimal of the *same order* as x, which we may regard as of the *first order*.

Again, if x and y be of the first order, then the quantities x^2, xy, and y^2 are each of the *second order*, as also are kx^2, kxy, and ky^2; k being finite. And similarly for higher orders.

Ex. Let θ be infinitesimal. Then

$$\theta - \sin\theta = \theta - \{\theta - \theta^3/3! + \theta^5/5! - \ldots\} = \theta^3/3! - \theta^5/5! \ldots,$$

which is of the 3rd order of infinitesimals, the higher powers of θ being quite negligible in comparison with $\theta^3/3!$. [See Art. 90.]

NOTE 1.—If θ be small, the error in writing θ for $\sin\theta$ is $\theta - \sin\theta$, which is of the 3rd order of small quantities, *i.e. of the 2nd order as compared with θ.*

NOTE 2.—The confusion of different orders in the case of small finite quantities (Art. 86), does not occur in the case of infinitesimals.

NOTE 3—We shall hereafter, for brevity, speak of *infinitesimals* only, though what follows will generally apply to small quantities if we substitute "comparable with unity" for "finite."

88. Infinitesimals in Geometry.

Let ABC be an isosceles triangle whose vertical angle, 2θ, is indefinitely small, and draw AD $\perp^r$ to BC, DE $\perp^r$ to AB, EF $\perp^r$ to BD.

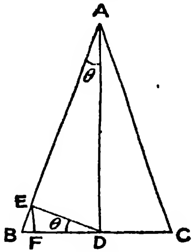

FIG. 12.

Let $AB = AC = a$, which is finite. Then—

(1) $AD = a\cos\theta$ and is obviously *finite*.

(2) $BD = a\sin\theta = a(\theta - \theta^3/3! + \ldots) =$ *1st* order of infinitesimals.

(3) $DE = BD\cos\theta = a(\theta - \ldots)(1 - \theta^2/2! + \ldots) = a(\theta + \ldots) =$ *1st* order.

(4) $BE = BD\sin\theta = a\sin^2\theta = a(\theta - \ldots)^2 = a(\theta^2 + \ldots) =$ *2nd* order. Otherwise, $BE/BD = BD/BA$; $\therefore BE = BD^2/a =$ 2nd order, since BD is of the 1st order.

(5) $EF = BE \cos\theta = $ *2nd* order, since $\cos\theta$ is finite.

(6) $BF = BE \sin\theta = a \sin^3\theta = a(\theta^3 + \ldots) = $ *3rd* order.

(7) $AB - AD = a(1 - \cos\theta) = a\{1 - (1 - \theta^2/2! + \ldots)\} = a(\theta^2/2! + \ldots) = $ *2nd* order.

(8) Similarly $AD - AE = $ *2nd* order.

(9) $BD - ED = $ 2nd order compared with BD or $ED = $ *3rd* order.

(10) Similarly $EB - EF = $ *4th* order.

89. Infinitesimal Arc of a Circle, etc.

Let PQ be an infinitesimal arc of a circle of radius r; and

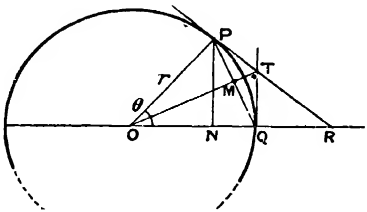

FIG. 13

let θ be the angle at the centre of the circle subtended by PQ. Then—

(11) Arc PQ − chord $PQ = r\theta - 2r\sin\frac{1}{2}\theta$

$$= r\left[\theta - 2\left\{\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \ldots\right\}\right] = \frac{r\theta^3}{24} + \ldots = \textit{3rd} \text{ order.}$$

(12) Sum of tangents PT, QT − arc PQ

$$= 2r\tan\frac{\theta}{2} - r\theta = r\left[2\left\{\frac{\theta}{2} - \frac{1}{3!}\cdot\frac{\theta^3}{8}\ldots\right\}\left\{1 - \frac{1}{2!}\cdot\frac{\theta^2}{4}\ldots\right\}^{-1} - \theta\right]$$

$$= r\left\{\left(\theta - \frac{\theta^3}{24}\ldots\right)\left(1 + \frac{\theta^2}{8}\ldots\right) - \theta\right\} = \frac{r\theta^3}{12}\ldots = \textit{3rd} \text{ order.}$$

(13) $RQ = OR - OQ = r(\sec\theta - 1)$

$$= r\,.\,2\sin^2\frac{\theta}{2}/\cos\theta = 2r\,.\left\{\left(\frac{\theta}{2}\right)^2 - \ldots\right\}/(1 - \ldots) = \textit{2nd} \text{ order.}$$

(14) $NQ = r(1 - \cos\theta) = \textit{2nd}$ order.

(15) $RQ - QN = r\{(\sec\theta - 1) - (1 - \cos\theta)\} = r\dfrac{(1-\cos\theta)^2}{\cos\theta}$

$= \textit{4th}$ order.

From (11) and (12) it follows that the error in substituting the chord PQ for either the infinitesimal arc PQ, or the sum of the tangents PT and TQ, is of the *2nd* order *compared with either of the three.*

This statement, which is important, can be shown to be, in general, true for any curve. See *Williamson's Diff. Calc.*, art. 37, and footnote.

90. The sum of the series $a_1\theta + a_2\theta^2 + a_3\theta^3 + \ldots$ (a_1, a_2, etc. being finite, and θ infinitesimal) can never be finite; for suppose a_n to be the greatest coefficient, then the series is

$$< a_n(\theta + \theta^2 + \theta^3 + \ldots \text{ ad inf.})$$

$$\textit{i.e.} < a_n \frac{\theta}{1-\theta};$$

and since θ is infinitesimal, the denominator is finite; therefore the fraction is infinitesimal.

From this it follows that the sum of the series $a_2\theta^2 + a_3\theta^3 + \ldots$ is an infinitesimal of the second order; and so on.

In the series $\underset{n=\infty}{lt}\left(\dfrac{1}{n} + \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{2n}\right)$ there are n terms, but *all of the same order*, and in this example the sum of the series is finite and equal to $\log_e 2$ in the limit. [See Ex. 1, Art. 415].

91. General Statement—Method of Infinitesimals.

Prop.—*If* f(x,y,z,...) *be a function of several variables* x,y,z..., *and if each of the latter be increased by a quantity which is infinitesimal compared with it, then the function itself will, in general, be increased by an infinitesimal of the first order.*

This follows easily in the case of one variable, for if h be infinitesimal, $f(x+h) - f(x) = hf'(x)$ to the first order; and we

have seen that for all ordinary functions $f'(x)$ is finite, *except for special values of x.* It can also be shown to be true in the general case. (See chapter on "Partial Differentiation.")

92. The following is a short proof for algebraical functions :—

Let $x' = x + \alpha$, $y = y' + \beta$; α and β being infinitesimals of the first order.

Then we must show that

$$\begin{array}{lll} (1)\ x' + y' = x + y + \text{infinitesimal of 1st order,} \\ (2)\ x' - y' = x - y + \quad " \quad " \\ (3)\ x'y' \quad = xy \quad + \quad " \quad " \\ (4)\ x' \div y' = x \div y + \quad " \quad " \\ (5)\ (x')^n \quad = x^n \quad + \quad " \quad " \end{array}$$

for all values of n.

(1) (2) and (3) follow readily. (4) is thus proved:—

$$\frac{x'}{y'} = \frac{x+\alpha}{y+\beta} = \frac{x}{y}\left(1+\frac{\alpha}{x}\right)\left(1+\frac{\beta}{y}\right)^{-1} = \frac{x}{y}\left(1+\frac{\alpha}{x}\right)\left(1-\frac{\beta}{y}+\dots\right)$$
$$= \frac{x}{y}\left(1+\frac{\alpha}{x}-\frac{\beta}{y}+\dots\right)$$
$$= \frac{x}{y}+\left(\frac{\alpha}{x}-\frac{\beta}{y}\right)\frac{x}{y}+\text{higher orders}$$
$$= \frac{x}{y}+\text{inf}^{\text{l}}.\text{ of 1st order.}$$

(5) follows similarly from the Binomial Theorem.

Since any algebraical function is but a combination of the above operations, the proposition follows at once, for algebraical functions.

93. The truth of this proposition justifies the use of the *method of infinitesimals,* which is due to Leibnitz. In using this method, if x', y', z', etc., are any quantities which differ from x, y, z, etc., by infinitesimals of the first order *compared with each respectively;* we may, in reasoning with these quantities, replace x' by x, y' by y, and so on. If one of the quantities (x, say) is already an infinitesimal of the first order, then x' must differ from it by an infinitesimal of a *higher* order than the first.

94. Example on the Use of Infinitesimals.

A straight line of constant length slides between two fixed straight lines Ax, Ay. *If* BC *be a given position, and* DE *a consecutive position making an infinitesimal angle with* BC *and cutting it in* F; *prove that* FB = GC,

where G *is the foot of the perpendicular from* A *on* BC, *the error being infinitesimal.*

(1) *Formal Proof.*—We shall use the symbols θ_1, θ_2, etc., to denote respectively *some* infinitesimal of the 1st, 2nd, etc., orders; so that if repeated during the work *they do not necessarily denote the same infinitesimal quantity.*

Draw BH, EK perpendiculars to DE and BC respectively.

Then, since $DE = BC$, or $DF + FE = BF + FC$,

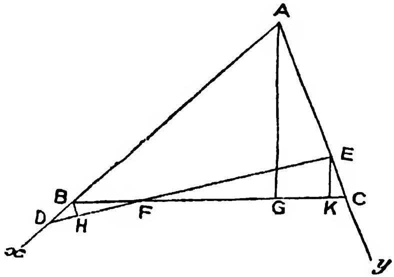

Fig. 14.

$$\therefore DF - BF = FC - FE,$$

$$\text{or } DH + \theta_2 = KC + \theta_2; \quad \text{[Art. 88, (7).]}$$

$$\therefore DH = KC + \theta_2;$$

$$\therefore BH \cot D = KE \cot C + \theta_2;$$

$$\therefore BH \cot (B - \theta_1) = KE \cot C + \theta_2;$$

$$\therefore BH (\cot B + \theta_1) = KE \cot C + \theta_2;$$

$$\therefore BH \cot B + \theta_2 = KE \cot C.$$

$$\therefore \text{dividing by } KE \cot B,\ \frac{BH}{KE} + \theta_1\dagger = \cot C \tan B;$$

$$\therefore \frac{BF \sin F}{KF \tan F} + \theta_1 = \cot C \tan B;$$

$$\frac{BF}{KF} + \theta_1 = \cot C \tan B;$$

$$\therefore \frac{BF}{FC} = \cot C \tan B + \theta_1$$

$$= \frac{GC}{GA}\frac{GA}{BG} = \frac{GC}{BG}, \text{ omitting the error.}$$

$$\therefore \text{(componendo) } \frac{BF}{BC} = \frac{GC}{BC}, \text{ or } FB = GC.$$

† Since KE is of 1st order.

(2) *Practical Method.*—Adopting the method of infinitesimals prope., we omit errors which are of the 1st order *compared with the quantities in error respectively.* Thus:—

$$DE = BC, \text{ or } DH + HF + FE = BF + FK + KC;$$
$$\therefore DH = KC.$$

$\therefore BH \cot B = EK \cot C$ (for the error in B being infinitesimal, the error in $\cot B$ is the same, by Art. 91).

$\therefore \cot C \tan B = BH/EK = BF/EF$ (since BH and EK may be regarded as arcs of circles) $= BF/FC$; etc., as above.

EXAMPLES XVI.

1. If x be an infinitesimal of the first order, state to what order the following expressions belong:—

(1) $\cos x$. (2) $\sin x$. (3) $\tan x$.
(4) $1 - \cos x$. (5) $\sec x - 1$. (6) $\operatorname{cosec} x - \cot x$.
(7) $x \operatorname{cosec} x - 1$. (8) $\tan x - \sin x$. (9) $x - 2 \sin \frac{x}{2}$.
(10) $x^2 - \sin^2 x$. (11) $x^2 + x \sin x - 4 \operatorname{vers} x$.
(12) $e^x - x - 1$. (13) $\log(1 + x) - \frac{x}{2}(2 - x)$.

2. ABC is a triangle, B and C being infinitesimals of the first order; AD is drawn perpendicular to BC. State to what order the following belong:—

(1) $AB \sim AC$. (2) AD. (3) $BD - DC$. (4) $AB + AC - BC$.

3. ABC is a triangle, B being infinitesimal; AD is perpendicular to BC. State to what order the following belong:—

(1) AD. (2) DC. (3) $BC \sim BA$.
(4) $AC - AD$. (5) $AC - DC$. (6) $AB + AC - BC$.

4. In the preceding question, show that the area of the triangle BAD is of the first order, and CAD of the second order.

5. If $x' = x + \alpha$; $y' = y + \beta$; α and β being infinitesimals of the first order, prove, by formulæ (1) to (5) in Art. 92, that an error of the first order is made in writing x and y for x' and y' in the following expressions:—

(1) $\sqrt{x' + y'}$. (2) $\sqrt{x' + \sqrt{1 + x'}}$. (3) $\dfrac{1}{\sqrt{1 + x'}\sqrt{1 + y'}}$.

6. If x' is as in Ex. 5, prove that, neglecting orders beyond the first—

$$(1)\ \sin x' = \sin x + \alpha \cos x.$$
$$(2)\ \tan x' = \tan x + \alpha \sec^2 x.$$

7. P and Q are two points indefinitely near to each other on a curve. PM and QN are their ordinates, and PR is drawn parallel to Ox. Prove that the error in taking the rectangle PN for the figure $PMNQ$ is of the first order of infinitesimals compared with either.

8. If $a - b$ is an infinitesimal of the first order, show that $\sqrt{a} - \sqrt{b}$ is also of the first order, as also is $a^n - b^n$, n being any quantity except zero.

9. In the preceding question, show that the A. mean of a and b is greater than the G. mean by an infinitesimal of the second order.

10. OAB is a sector of a circle, centre O, and radius r, the angle AOB ($= \theta$) being infinitesimal; AT and BT are the tangents at A and B.

Prove that the area of the triangle $ABT = \frac{1}{8} r^2\theta^3$ nearly.

Hence show that the error in taking the area of the triangle OAB for that of the sector, or that of the figure $OABT$, is of the third order *at least* (*i.e.* the order may be *higher* than the third).

11. OA, OB are any two straight lines drawn from O. MN, $M'N'$ are two straight lines intercepted between OA and OB, and inclined at an infinitesimal angle to each other; they also cut off equal areas, MON, $M'ON'$. Show that their point of intersection ultimately bisects either of the intercepted lines (see question 4).

Deduce from this that the portion of the tangent to a hyperbola, intercepted between the asymptotes, is bisected at the point of contact.

12. In question 10, if OT meet the circle in C, and AB in F; and if the tangent at C meet AT in D, and BT in E; prove that ultimately

$$(1)\ AT = DE = 2AD; \qquad (2)\ TC = CF;$$
$$(3)\ \triangle ABT = 4\triangle DTE = 2\triangle CAB.$$

13. PQ is an infinitesimal arc of a circle, centre E, and radius ρ. O is any origin. Prove that $OQ \sim OP = \rho\alpha \cos\phi$, where α is the angle PEQ, and ϕ the inclination of OP to the tangent at P.

Answers.

1. (1) finite. (2) 1st. (3) 1st. (4) 2nd. (5) 2nd. (6) 1st. (7) 2nd. (8) 3rd. (9) 3rd. (10) 4th. (11) 6th. (12) 2nd. (13) 3rd.

2. (1) finite generally. (2) 1st. (3) finite. (4) 2nd.

3. (1) 1st. (2) 1st. (3) 1st. (4) 1st. (5) 1st. (6) 1st.

CHAPTER IX.

SUCCESSIVE DIFFERENTIATION.

95. Notation.—When a function has been differentiated once, the result may be again differentiated, and the process repeated indefinitely. This is called *successive differentiation.*

The *second d.c. of* y *with respect to* x is the d.c. of the d.c. of y with respect to x, i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$; and is written $\frac{d^2y}{dx^2}$.

Similarly the 3rd, 4th, etc., are written d^3y/dx^3, d^4y/dx^4, etc.

Other notations are $y_1, y_2, y_3 \ldots y_n$

$y', y'', y''' \ldots y^{(n)}$

$f'(x), f''(x), f'''(x) \ldots f^{(n)}(x)$ or $f^n(x)$.

$Dy, D^2y, D^3y \ldots D^ny$.

Each has its advantages, and the notation used may be varied according to circumstances.

It is sometimes convenient [*vide* Leibnitz's Theorem] to put y_0 instead of y, to make the notation uniform.

Further, the notation $(dy/dx)_0$, $(d^2y/dx^2)_0$, ... $(d^ny/dx^n)_0$, or $(y_1)_0$, $(y_2)_0$ ... $(y_n)_0$, is used when x is to be put equal to 0 *after* one, two, ... n, differentiations; while $(y)_0$ or $(y_0)_0$ denotes that x is put equal to 0 in y itself. These may be also written $f(0), f'(0)$, $f''(0)$, etc.

96. Except in a very few cases, the higher d.c.'s of y become more and more complicated, x^n, e^x, and log x being the only functions for which y_n can be found directly; for one or two forms, however, a compact expression can be obtained by means

of special artifices, such as the use of subsidiary angles. Some of these are given below.

97. x^n.

Let $y = x^n$; $\therefore y_1 = nx^{n-1}$; $y_2 = n(n-1)x^{n-2}$; etc.

$$\therefore y_r = n(n-1)\dots(n-r+1)x^{n-r} = \frac{n!}{(n-r)!}x^{n-r}, \text{ or } nP_rx^{n-r}.$$

$\therefore$ also $y_n = n!$, and $y_{n+1} = 0$.
Hence the $(n+1)$th d.c. of x^n vanishes.

98. e^x and e^{ax}.

Let $y = e^x$; $\therefore y_1 = e^x$; $y_2 = e^x$; etc. $\therefore y_n = e^x$.
Similarly, if $y = e^{ax}$, $y_n = a^ne^{ax}$.
Hence, *differentiating* e^{ax} n *times is equivalent to multiplying it by* a^n. This may also be written $D^n.e^{ax} = a^n.e^{ax}$.

99. $\log x$ and $\log_a x$.

Let $y = \log x$; $\therefore y_1 = \frac{1}{x}$; $y_2 = -\frac{1}{x^2}$; $y_3 = (-1)(-2)\frac{1}{x^3}$;

$y_4 = (-1)(-2)(-3)\frac{1}{x^4}$; etc.

$$\therefore y_n = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

Similarly, if $y = \log_a x$, $y_n = (-1)^{n-1}\frac{(n-1)!}{x^n \log a}$.

100. Sin $(mx + \alpha)$ and cos $(mx + \alpha)$.

Let $y = \sin(mx + \alpha)$.
$\therefore y_1 = m\cos(mx + \alpha)$, which may with advantage be written

$$= m\sin\left(mx + \alpha + \frac{\pi}{2}\right) = m\sin(mx + \alpha_1), \text{ where } \alpha_1 = \alpha + \frac{\pi}{2}.$$

Hence, *differentiating* $\sin(mx + \alpha)$ *is equivalent to multiplying it by* m, *and altering the* constant α *into* α_1, *without altering the nature of the function.*

We can therefore write down y_2, y_3, etc., at once; † thus—

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† To show this more clearly; since α is *any* constant, suppose we write α_1 instead of it. Then $y = \sin(mx + \alpha_1)$; hence, by the above reasoning, $_1 = m\sin\left(mx + \alpha_1 + \frac{\pi}{2}\right)$. If, therefore, $y_1 = \sin(mx + \alpha_1)$, then $y_2 =$

$\therefore y_2 = m^2 \sin(mx + \alpha_2)$, where $\alpha_2 = \alpha_1 + \frac{\pi}{2} = \alpha + 2 . \frac{\pi}{2}$;

$y_3 = m^3 \sin(mx + \alpha_3)$, where $\alpha_3 = \alpha + 3 . \frac{\pi}{2}$;

$y_n = m^n \sin(mx + \alpha_n)$ or $m^n \sin\left(mx + \alpha + n . \frac{\pi}{2}\right)$.

This may be written, $D^n \sin(mx + \alpha) = m^n \sin\left(mx + \alpha + \frac{n\pi}{2}\right)$.

Cor.—If $y = \sin x$, $y_1 = \sin\left(x + \frac{\pi}{2}\right)$; $y_n = \sin\left(x + \frac{n\pi}{2}\right)$.

Similarly, if $y = \cos(mx + \alpha)$, $y_1 = m\cos\left(mx + \alpha + \frac{\pi}{2}\right)$;

$$y_n = m^n \cos\left(mx + \alpha + \frac{n\pi}{2}\right);$$

and if $\qquad y = \cos x$, $y_1 = \cos\left(x + \frac{\pi}{2}\right)$; $y_n = \cos\left(x + \frac{n\pi}{2}\right)$.

101. $e^{ax} \sin(mx + \alpha)$ and $e^{ax} \cos(mx + \alpha)$.

Let $y = e^{ax} \sin(mx + \alpha)$.

$$\begin{aligned} \therefore y_1 &= ae^{ax} \sin(mx + \alpha) + me^{ax} \cos(mx + \alpha) \\ &= e^{ax} \{a \sin(mx + \alpha) + m\cos(mx + \alpha)\}. \end{aligned}$$

Put $\left.\begin{aligned} a &= r\cos\phi \\ m &= r\sin\phi \end{aligned}\right\}$ so that $r = (a^2 + m^2)^{\frac{1}{2}}$; $\phi = \tan^{-1}\frac{m}{a}$.

$$\begin{aligned} \therefore y_1 &= re^{ax} \{\cos\phi \sin(mx + \alpha) + \sin\phi \cos(mx + \alpha)\} \\ &= re^{ax} \sin(mx + \alpha + \phi) \\ &= re^{ax} \sin(mx + \alpha_1), \text{ where } \alpha_1 = \alpha + \phi. \end{aligned}$$

Hence, *differentiating* $e^{ax} \sin(mx + \alpha)$ *is equivalent to multiplying it by the* constant r *and altering the* constant α *into* α_1, *without altering the nature of the function.*

We can therefore write down y_2, y_3, etc., at once; † thus

$$\begin{aligned} y_2 &= r^2 e^{ax} \sin(mx + \alpha_1 + \phi) = r^2 e^{ax} \sin(mx + \alpha_2), \text{ where } \alpha_2 = \alpha + 2\phi. \\ y_3 &= r^3 e^{ax} \sin(mx + \alpha_2 + \phi) = r^3 e^{ax} \sin(mx + \alpha_3), \text{ where } \alpha_3 = \alpha + 3\phi, \end{aligned}$$

and $y_n = r^n e^{ax} \sin(mx + \alpha_n)$, where $\alpha_n = \alpha + n\phi$.

i.e. $y_n = r^n e^{ax} \sin(mx + \alpha + n\phi)$

$$= (a^2 + m^2)^{n/2} e^{ax} \sin\left(mx + \alpha + n \tan^{-1}\frac{m}{a}\right).$$

$m \sin\left(mx + \alpha_1 + \frac{\pi}{2}\right)$; etc. The beginner may best satisfy himself by going through the original reasoning a second time, for y_2.

† The beginner should, however, go through the same reasoning for y_2 as for y_1. *Note that a is not involved in the reasoning*

Cor.—If $a = 0$, $y = \sin(mx + \alpha)$, $r = m$, $\phi = \tan^{-1}\infty = \pi/2$,

and $$y_n = m^n \sin\left(mx + \alpha + \frac{n\pi}{2}\right) \text{ as before.}$$

Similarly, if $y = e^{ax}\cos(mx + \alpha)$, $y_n = r^n e^{ax}\cos(mx + \alpha + n\phi)$.

102. $\tan^{-1}x$.

Let $y = \tan^{-1}x$.

$$\therefore y_1 = \frac{1}{1+x^2} = \frac{1}{2i}\left\{\frac{1}{x-i} - \frac{1}{x+i}\right\};$$

$$\therefore y_n = \frac{(-1)^{n-1}}{2i}\left\{\frac{(n-1)!}{(x-i)^n} - \frac{(n-1)!}{(x+i)^n}\right\}$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left\{\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right\}.$$

Put $\left.\begin{aligned} x &= r\cos\theta \\ 1 &= r\sin\theta \end{aligned}\right\}$ so that $r^2 = 1 + x^2$; $x = \cot\theta$.

$$\therefore y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left\{\frac{1}{(\cos\theta - i\sin\theta)^n} - \frac{1}{(\cos\theta + i\sin\theta)^n}\right\}$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left\{\frac{1}{\cos n\theta - i\sin n\theta} - \frac{1}{\cos n\theta + i\sin n\theta}\right\}$$

[by Demoivre's Theorem],

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta, \text{ or } \frac{(-1)^{n-1}(n-1)!}{(1+x^2)^{n/2}}\sin\left(n\tan^{-1}\frac{1}{x}\right),$$

$$\text{or } (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta,$$

$$\because r = \operatorname{cosec}\theta.$$

We can prove this also by induction, for assuming

$y_n = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, we have

$$y_{n+1} = \frac{d}{d\theta}(y_n)\cdot\frac{d\theta}{dx} = \frac{dy_n}{d\theta}\bigg/\frac{dx}{d\theta}$$

$$= (-1)^{n-1}(n-1)![n\sin^{n-1}\theta\cos\theta\sin n\theta + n\sin^n\theta\cos n\theta]$$

$$\div(-\operatorname{cosec}^2\theta), \because x = \cot\theta,$$

$$= (-1)^n n!\sin^{n+1}\theta(\sin n\theta\cos\theta + \cos n\theta\sin\theta)$$

$$= (-1)^n n!\sin^{n+1}\theta\sin(n+1)\theta,$$

which is of the same form as before, but with $n + 1$ in place of n.

103. In Ch. XXIV. (*q.v.*) we shall show that a rational algebraical function of the form $f(x)/\phi(x)$ can be split up into partial fractions, whose denominators are of the form $x - a$ or $(x - a)^r$, a being real or imaginary.

See the notes at end of Arts. 368 and 373. The case in which a is real presents no difficulty as regards finding the nth d.c., but when it is imaginary the work is rather involved. Exs. 1 and 4 of the next Article will throw some light on the subject; see also the preceding Article.

104. Examples on Successive Differentiation.

Ex. 1.

$$y = \frac{1}{x^2 - 4x + 3} = \frac{1}{(x-3)(x-1)} = \frac{\text{some quantity}}{x-3} + \frac{\text{some quantity}}{x-1}.$$

By rough trials we can soon obtain the numerators; thus

$$y = \frac{1}{2}\left(\frac{1}{x-3} - \frac{1}{x-1}\right) = \frac{1}{2}\frac{1}{x-3} - \frac{1}{2}\frac{1}{x-1};$$

$$\therefore y_1 = \tfrac{1}{2}(-1)\frac{1}{(x-3)^2} - \tfrac{1}{2}(-1)\frac{1}{(x-1)^2}$$

$$= \tfrac{1}{2}(-1)\left\{\frac{1}{(x-3)^2} - \frac{1}{(x-1)^2}\right\}.$$

So
$$y_2 = \tfrac{1}{2}(-1)(-2)\left\{\frac{1}{(x-3)^3} - \frac{1}{(x-1)^3}\right\}$$

$$= \frac{(-1)^2 2!}{2}\left\{\frac{1}{(x-3)^3} - \frac{1}{(x-1)^3}\right\};$$

$$y_3 = \tfrac{1}{2}(-1)(-2)(-3)\left\{\frac{1}{(x-3)^4} - \frac{1}{(x-1)^4}\right\}$$

$$= \frac{(-1)^3 3!}{2}\left\{\frac{1}{(x-3)^4} - \frac{1}{(x-1)^4}\right\}.$$

Hence
$$y_n = \frac{(-1)^n n!}{2}\left\{\frac{1}{(x-3)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right\}.$$

Ex. 2. *If $y = \sec x$, find $(y_1)_0$, etc., to $(y_6)_0$ inclusive.*

The work will be shortened if we use s and t for $\sec x$ and $\tan x$ respectively. Moreover, as we approach y_6, we shall be able to omit from our work the higher powers of t which will occur, since t vanishes with x.

We have	$y = s$	$\therefore (y)_0 = 1.$
	$y_1 = st$	$(y_1)_0 = 0.$
	$y_2 = st^2 + s^3$	$(y_2)_0 = 1.$
(Omit t^4 terms)†	$y_3 = st^3 + 2s^3t + 3s^3t = st^3 + 5s^3t$	$(y_3)_0 = 0.$
(„ t^3 „)	$y_4 = \dots + 3s^3t^2 + 15s^3t^2 + 5s^5$	
	$= \dots + 18s^3t^2 + 5s^5$	$(y_4)_0 = 5.$
(„ t^2 „)	$y_5 = \dots\dots + 36s^5t + 25s^5t = \dots + 61s^5t$	$(y_5)_0 = 0.$
(„ t „)	$y_6 = \dots\dots + 61s^7$	$(y_6)_0 = 61.$

† To find what powers of t are to be omitted, reckon backwards from y_6; y_3, however, has *no* t^4 terms.

Ex. 3. *If $y = \sin^3 x$, find y_n.*

Here $y = \frac{1}{4}(3\sin x - \sin 3x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

$$\therefore y_n = \frac{3}{4}\sin\left(x + \frac{n\pi}{2}\right) - \frac{3^n}{4}\sin\left(3x + \frac{n\pi}{2}\right).$$

Ex. 4. *Find* $\dfrac{d^n}{dx^n}\left(\dfrac{1}{x^2 - 2x + 2}\right)$.

We have $y = \dfrac{1}{x^2 - 2x + 2} = \dfrac{1}{(x-1)^2 + 1} = \dfrac{1}{2i}\left\{\dfrac{1}{x - 1 - i} - \dfrac{1}{x - 1 + i}\right\}$.

$$\therefore y_n = \frac{1}{2i}(-1)^n n!\left\{\frac{1}{(x - 1 - i)^{n+1}} - \frac{1}{(x - 1 + i)^{n+1}}\right\}.$$

Let $\left.\begin{aligned} x - 1 &= r\cos\theta \\ 1 &= r\sin\theta \end{aligned}\right\}$, $\therefore \tan\theta = \dfrac{1}{x-1}$, and $r^2 = x^2 - 2x + 2$.

$$\therefore y_n = \frac{1}{2i}(-1)^n n!\left\{\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right\}$$

$$= \frac{1}{2i}\frac{(-1)^n n!}{r^{n+1}}\left\{\frac{1}{(e^{-\theta i})^{n+1}} - \frac{1}{(e^{\theta i})^{n+1}}\right\}$$

$$= \frac{(-1)^n n!}{r^{n+1}} \cdot \frac{e^{(n+1)\theta i} - e^{-(n+1)\theta i}}{2i}$$

$$= \frac{(-1)^n n!}{r^{n+1}}\sin(n+1)\theta = \frac{(-1)^n n! \sin\left\{(n+1)\tan^{-1}\dfrac{1}{x-1}\right\}}{(x^2 - 2x + 2)^{\frac{n+1}{2}}}$$

or $= (-1)^n n! \sin^{n+1}\theta \,.\, \sin(n+1)\theta$, since $r = \operatorname{cosec}\theta$.

EXAMPLES XVII.

1. If $y = x^7$, find d^6y/dx^6.
2. If $y = 3x^5 - 4x^4 + 3x^3 + x^2 - x - 6$, find y_4.
3. If $y = (ax^3 + b)^3$, find y_9.
4. If $y = x\sin x$, find y_5.
5. If $y = x^6\log x$, find y_7.
6. If $y = e^{ax^2}$, show that $y_3 = 4a^2x(3 + 2ax^2)y$.
7. If $y = \sqrt{1 + x^2}$, find y_4.

Find y_n in the following cases:—

8. $y = 1/x$. **9.** $y = x^p$, $p > n$. **10.** $y = e^{ax+b}$.

11 $y = a^m$ **12.** $y = \dfrac{1}{x - a}$. **13.** $y = \dfrac{1}{2x - 5}$.

14. $y = \dfrac{1}{a - bx}$. **15.** $y = \dfrac{a - x}{a + x}$. **16.** $y = \dfrac{x^2}{1 - x}$.

17. $y = \log(a + x)$. **18.** $y = \log(2 - 3x)$. **19.** $y = \log\dfrac{a - x}{a + x}$.

20. $y = x^{n-1}\log x$. **21.** $y = \sin ax$. **22.** $y = \sin x \cos x$.

23. $y = \cos^2 x$. **24.** $y = \cos^3 x$. **25.** $y = \cos 3x \sin x$.

26. $y = \sin(x + a)\sin(x + b)$. **27.** $y = \dfrac{1}{1 - x^2}$.

28. $y = \dfrac{1}{x^2 - 5x + 6}$. **29.** $y = \dfrac{1}{(x - a)(x - b)}$.

30. If $y = (x - 1)^n$, prove that $y + y_1 + \dfrac{y_2}{2!} + \dfrac{y_3}{3!} + \dots + \dfrac{y_n}{n!} = x^n$.

31. If $y_0 = \sin x$, show that $y_0 = y_4 = y_8 =$ etc., and that $y_n = y_{n+4}$.

32. Prove that $a\cos x + b\sin x = (a^2 + b^2)^{\frac{1}{2}}\sin\left(x + \tan^{-1}\dfrac{a}{b}\right)$.

Hence show that if $y = a\cos x + b\sin x$,

$$y_n = (a^2 + b^2)^{\frac{1}{2}}\sin\left(x + \tan^{-1}\frac{a}{b} + \frac{n\pi}{2}\right).$$

33. If $y_0 = e^x\sin x$, prove that $y_1 = \sqrt{2}\,e^x\sin\left(x + \dfrac{\pi}{4}\right)$,

$$\text{and that } y_n = 2^{n/2}e^x\sin\left(x + \frac{n\pi}{4}\right).$$

Prove also that $y_0 = -\dfrac{1}{4}y_4 = \dfrac{1}{4^2}y_8 = \dots = \left(-\dfrac{1}{4}\right)^n y_{4n}$.

34. If $y_0 = e^x(\sin x + \cos x)$, prove that $y_n = 2^{\frac{n+1}{2}}e^x\sin\left\{x + \dfrac{(n + 1)\pi}{4}\right\}$,
and that $y_0 = -\dfrac{1}{4}y_4 = \dfrac{1}{4^2}y_8 = \dots = \left(-\dfrac{1}{4}\right)^n y_{4n}$.

35. If $y_0 = e^x(a\cos x + b\sin x)$, prove that $y_n = 2^{n/2}re^x\sin\left(x + \phi + \dfrac{n\pi}{4}\right)$
where $r^2 = a^2 + b^2$, $\phi = \tan^{-1}\dfrac{a}{b}$, and that

$$y_0 = -\frac{1}{4}y_4 = \frac{1}{4^2}y_8 = \dots = \left(-\frac{1}{4}\right)^n y_{4n}.$$

36. If $y = \cos(mx + \alpha)$, find y_n.

37. If $y = e^{ax}\cos bx$, find y_n.

38. If $y = e^{ax}\cos^2 bx$, find y_n.

39. If $y = e^{x\cos\alpha}\sin(x\sin\alpha)$, prove that $y_n = e^{x\cos\alpha}\sin(x\sin\alpha + n\alpha)$.

40. If $y = a^x\cos 3x$, prove that $y_n = r^n a^x\cos(3x + n\phi)$, where
$r^2 = (\log a)^2 + 9$; $\tan\phi = 3/\log a$.

41. If $y = e^x\cos x\cos 2x$, find y_n.

42. If $y = \log\sin x$, find $(y_5)_0$.

43. If $y = \log\sec x$, find $(y_7)_0$.

44. If $y = \tan^2 x$, find $(y_1)_0$, $(y_2)_0$, etc., to $(y_7)_0$ inclusive.

45. If $y = e^x\sec x$, prove that $(y_3)_0 = 4$.

46. If $y = e^{\sin x}$, prove that $y_4 = (s^4 + 6s^3 + 5s^2 - 5s - 3)c^s$.

47. If $y = e^u$, prove that $y_3 = (u_3 + 3u_1u_2 + u_1^3)e^u$,
and that $y_4 = (u_4 + 4u_1u_3 + 6u_1^2u_2 + u_1^4 + 3u_2^2)e^u$,
u being any function of x, and u_1, u_2, ... the derived functions [Note 1. Art. 5].

48. If $y = \dfrac{1}{x^2 + a^2}$, show that

$$y_n = \frac{(-1)^n n!\sin(n+1)\theta}{a(x^2+a^2)^{\frac{n+1}{2}}}, \text{ or } \frac{(-1)^n n!\sin(n+1)\theta.\sin^{n+1}\theta}{a^{n+2}};$$

where $x = a\cot\theta$.

49. If $y = \dfrac{1}{a^2 - x^2}$, prove that

$$y_{2n} = \frac{(2n)!}{a(a^2-x^2)^{\frac{2n+1}{2}}}\cdot\cosh\left\{(2n+1)\tanh^{-1}\frac{x}{a}\right\},$$

or
$$\frac{(2n)!}{a^{2n+2}}\cosh(2n+1)u.\cosh^{2n+1}u, \text{ where } u = \tanh^{-1}\frac{x}{a}.$$

50. If $y = \dfrac{x}{a^2 + x^2}$, prove that $a^{n+1}y_n = (-1)^n n!\cos(n+1)\theta.\sin^{n+1}\theta$,
where $x = a\cot\theta$.

51. If $y = \tan^{-1}\dfrac{x\sin\alpha}{1 - x\cos\alpha}$, prove that

$$y_n = \frac{(n-1)!}{(1 - 2x\cos\alpha + x^2)^{n/2}}\sin n(y+\alpha),$$

and that $\quad (y_n)_0 = (n-1)!\sin n\alpha.$

52. If $y = \tanh^{-1}\dfrac{a-b}{2x-a-b}$;

$$y_n = \frac{(-1)^n(n-1)!}{[(x-a)(x-b)]^{n/2}}\cdot\sinh\left\{n\tanh^{-1}\frac{a-b}{2x-a-b}\right\}$$
$$= \frac{2^n(-1)^n(n-1)!}{(a-b)^n}\sinh ny.\sinh^n y.$$

ANSWERS.

1. $7!\,x$. **2.** $360x - 96$. **3.** $9!\,a^3$. **4.** $x\cos x + 5\sin x$. **5.** $6!/x$.

7. $\dfrac{3(4x^2-1)}{(1+x^2)^{7/2}}$. **8.** $(-1)^n\dfrac{n!}{x^{n+1}}$. **9.** $p(p-1)\ldots(p-n+1)x^{p-n}$.

10. $a^n . e^{ax+b}$. **11.** $(m\log a)^n a^{mx+n}$. **12.** $(-1)^n\dfrac{n!}{(x-a)^{n+1}}$. **13.** $\dfrac{(-2)^n n!}{(2x-5)^{n+1}}$

14. $\dfrac{b^n n!}{(a-bx)^{n+1}}$. **15.** $\dfrac{(-1)^n 2a\,n!}{(a+x)^{n+1}}$ **16.** $\dfrac{n!}{(1-x)^{n+1}}$. **17.** $\dfrac{(-1)^{n-1}(n-1)!}{(a+x)^n}$.

18. $-\dfrac{3^n(n-1)!}{(2-3x)^n}$. **19.** $(n-1)!\left\{\dfrac{(-1)^n}{(a+x)^n} - \dfrac{1}{(a-x)^n}\right\}$. **20.** $\dfrac{(n-1)!}{x}$.

21. $a^n\sin\left(ax + \dfrac{n\pi}{2}\right)$. **22.** $2^{n-1}\sin\left(2x + \dfrac{n\pi}{2}\right)$. **23.** $2^{n-1}\cos\left(2x + \dfrac{n\pi}{2}\right)$

24. $\dfrac{3}{4}\left\{3^{n-1}\cos\left(3x + \dfrac{n\pi}{2}\right) + \cos\left(x + \dfrac{n\pi}{2}\right)\right\}$.

25. $2^{n-1}\left\{2^n\sin\left(4x + \dfrac{n\pi}{2}\right) - \sin\left(2x + \dfrac{n\pi}{2}\right)\right\}$.

26. $-2^{n-1}\cos\left(2x + a + b + \dfrac{n\pi}{2}\right)$. **27.** $\dfrac{1}{2}n!\left\{\dfrac{1}{(1-x)^{n+1}} + \dfrac{(-1)^n}{(1+x)^{n+1}}\right\}$.

28. $(-1)^n n!\left\{\dfrac{1}{(x-3)^{n+1}} - \dfrac{1}{(x-2)^{n+1}}\right\}$.

29. $\dfrac{(-1)^n n!}{a-b}\left\{\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x-b)^{n+1}}\right\}$. **36.** $m^n\cos\left(mx + a + \dfrac{n\pi}{2}\right)$.

37. $r^n e^{ax}\cos(bx + n\phi)$, if $r^2 = a^2 + b^2$, $\phi = \tan^{-1}\dfrac{b}{a}$.

38. $\dfrac{1}{2}e^{ax}\left\{a^n + (a^2+4b^2)^{n/2}\cos\left(2bx + n\tan^{-1}\dfrac{2b}{a}\right)\right\}$.

41. $\dfrac{1}{2}e^x\left\{10^{n/2}\cos(3x + n\tan^{-1}3) + 2^{n/2}\cos\left(x + \dfrac{n\pi}{4}\right)\right\}$.

42. ∞. **43.** 0. **44.** 0, 2, 0, 16, 0, 272, 0.

105. Leibnitz's Theorem.—When we know the nth d.c. of each of two functions of x, we can obtain by means of this theorem the nth d.c. of their product. The statement is as follows :—

Prop.—*If* $y_0 = u_0v_0$, u_0 *and* v_0 *being any functions of* x, *then* $y_n = u_nv_0 + {}_nC_1u_{n-1}v_1 + \dots + {}_nC_{r-1}u_{n-r+1}v_{r-1} + {}_nC_ru_{n-r}v_r + \dots + u_0v_n$.

We can easily verify that

$$y_1 = u_1v_0 + u_0v_1,$$
$$y_2 = u_2v_0 + 2u_1v_1 + u_0v_2,$$
$$y_3 = u_3v_0 + 3u_2v_1 + 3u_1v_2 + u_0v_3,$$

which shows that the theorem holds good when $u = 1, 2$, or 3. Hence, adopting the proof by induction, assume the theorem to be true for y_n, and differentiate both sides of the equation with respect to x. That is, assume,

$$y_n = u_nv_0 + {}_nC_1u_{n-1}v_1 + \dots + {}_nC_{r-1}u_{n-r+1}v_{r-1} + {}_nC_ru_{n-r}v_r + \dots + u_0v_n.$$

$$\therefore\ y_{n+1} = (u_{n+1}v_0 + u_nv_1) + {}_nC_1(u_nv_1 + u_{n-1}v_2) + \dots$$
$$+ {}_nC_{r-1}(u_{n-r+2}v_{r-1} + u_{n-r+1}v_r) + {}_nC_r(u_{n-r+1}v_r + u_{n-r}v_{r+1})$$
$$+ \dots + (u_1v_n + u_0v_{n+1}).$$

The general term is $({}_nC_{r-1} + {}_nC_r)u_{n-r+1}v_r$, *i.e.* ${}_{n+1}C_r \cdot u_{n+1-r}v_r$ [*Misc. Theorems* (1)]; while $u_nv_1 + {}_nC_1u_nv_1 = (n+1)u_nv_1 = {}_{n+1}C_1u_nv_1$;

$$\therefore\ y_{n+1} = u_{n+1}v_0 + {}_{n+1}C_1u_nv_1 + \dots + {}_{n+1}C_ru_{n+1-r}v_r + \dots + u_0v_{n+1},$$

which is what we should have obtained by merely changing n into $n + 1$ in the given equation. Hence, if the law is true for y_n it is true for y_{n+1}; but it is true for y_1, y_2, and y_3; therefore it is true for y_4, and therefore again for y_5, and so on. Hence it is always true.

106. Comparison with Binomial Theorem.

If we differentiate u_mv_n, we get $u_{m+1}v_n + u_mv_{n+1}$.

If we multiply u^mv^n by $u + v$, we get $u^{m+1}v^n + u^mv^{n+1}$.

Hence, by changing suffixes into indices, and the operation of differentiation into that of multiplication by $u + v$, we shall see the analogy between Leibnitz's Theorem and the Binomial Theorem, as shown in the table below. The indices are inserted in the case of u^0, u^1, v^0, v^1, to make the analogy complete.

COMPARATIVE TABLE.

Differentiation of u_0v_0	Multiplication of u^0v^0 by $u+v$ or y
$y_0 = u_0v_0$	$y^0 = u^0v^0$
$y_1 = u_1v_0 + u_0v_1$	$y^1 = u^1v^0 + u^0v^1$
$y_2 = (u_2v_0 + u_1v_1) + (u_1v_1 + u_0v_2)$	$y^2 = (u^2v^0 + u^1v^1) + (u^1v^1 + u^0v^2)$
$\quad = u_2v_0 + 2u_1v_1 + u_0v_2$	$\quad = u^2v^0 + 2u^1v^1 + u^0v^2$
$y_3 = u_3v_0 + 2u_2v_1 + u_1v_2$	$y^3 = u^3v^0 + 2u^2v^1 + u^1v^2$
$\quad + u_2v_1 + 2u_1v_2 + u_0v_3$	$\quad + u^2v^1 + 2u^1v^2 + u^0v^3$
$\quad = u_3v_0 + 3u_2v_1 + 3u_1v_2 + u_0v_3$	$\quad = u^3v^0 + 3u^2v^1 + 3u^1v^2 + u^0v^3$
etc.	etc.

Similarly for the general case.

Cor.—If $y = xu$; $y_n = xu_n + nu_{n-1}$, since the higher d.c.'s of x vanish.

Again, if $y = x^2u$; $y_n = x^2u_n + 2nxu_{n-1} + n(n-1)u_{n-2}$;
and so on, the number of terms in y_n being one more than the index of x in y.

107. Examples.

Ex. 1. *If $y = x^3e^x$, find y_n,*

If $e^x = u$, $x^3 = v$, then, since $u_n = e^x$,

$$\therefore\ y_n = e^x . x^3 + ne^x . \frac{d}{dx}(x^3) + \frac{n(n-1)}{1.2}e^x . \frac{d^2}{dx^2}(x^3) \dots$$

$$= e^x x^3 + ne^x . 3x^2 + \frac{n(n-1)}{1.2}e^x . 6x + \frac{n(n-1)(n-2)}{1.2.3}e^x . 6$$ (the remaining terms vanishing)

$$= e^x\{x^3 + 3nx^2 + 3n(n-1)x + n(n-1)(n-2)\}.$$

Ex. 2. *If $y = \log x/x$, find y_5 and y_n.*

Let $u = 1/x$; $v = \log x$.

Then $u_n = (-1)^n\dfrac{n!}{x^{n+1}}$; and therefore u_5, u_4, etc., may be found from this by putting $u = 5$, 4, etc. We may differentiate $\log x$ successively as we come to each term.

$$(1)\ y_5 = u_5v_0 + \frac{5}{1}u_4v_1 + \frac{5.4}{1.2}u_3v_2 + \frac{5.4}{1.2}u_2v_3 + \frac{5}{1}u_1v_4 + u_0v_5$$

$$= -\frac{5!}{x^6}\log x + \frac{5}{1}\cdot\frac{4!}{x^5}\cdot\frac{1}{x} - \frac{5.4}{1.2}\cdot\frac{3!}{x^4}\left(-\frac{1}{x^2}\right) + \frac{5.4}{1.2}\cdot\frac{2!}{x^3}\left(\frac{2}{x^3}\right)$$
$$-\frac{5}{1}\cdot\frac{1}{x^2}\left(\frac{-6}{x^4}\right) + \frac{1}{x}\left(\frac{24}{x^5}\right)$$

$$= -\frac{5!}{x^6}\log x + \frac{5!}{x^6} + \frac{5!}{2x^6} + \frac{5!}{3x^6} + \frac{5!}{4x^6} + \frac{5!}{5x^6}$$

$$= \frac{5!}{x^6}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \log x\right).$$

$$(2)\ y_n = u_n v_0 + \frac{n}{1} u_{n-1} v_1 + \frac{n(n-1)}{1.2} u_{n-2} v_2 + \dots + u_0 v_n$$

$$= (-1)^n \frac{n!}{x^{n+1}} \log x + (-1)^{n-1} \frac{n}{1} \frac{(n-1)!}{x^n} \cdot \frac{1}{x} + (-1)^{n-2} \frac{n(n-1)}{1.2} \cdot \frac{(n-2)!}{x^{n-1}} \left(-\frac{1}{x^2}\right)$$
$$+ (-1)^{n-3} \frac{n(n-1)(n-2)}{1.2.3} \cdot \frac{(n-3)!}{x^{n-2}} \left(\frac{2}{x^3}\right) + \dots + \frac{1}{x} (-1)^{n-1} \cdot \frac{(n-1)!}{x^n}$$
$$= (-1)^n \frac{n!}{x^{n+1}} \left\{\log x - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)\right\},$$

for all the terms after the second will have the same sign, since each term is made up of two factors, each of which is alternately $+^{ve}$ and $-^{ve}$.

Ex. 3. If $y = e^x \sin x$.

Making $e^x = u$, $\sin x = v$, we have

$$y_n = D^n e^x . \sin x + {}_nC_1 D^{n-1} e^x . D \sin x + {}_nC_2 D^{n-2} e^x . D^2 \sin x + \dots$$
$$= e^x \left\{\sin x + {}_nC_1 \sin\left(x + \frac{\pi}{2}\right) + {}_nC_2 \sin\left(x + \frac{2\pi}{2}\right) + \dots + \sin\left(x + \frac{n\pi}{2}\right)\right\}.$$

Cor.—On referring to Ex. 33 of the previous set,

$$y_n = 2^{n/2} e^x \sin\left(x + \frac{n\pi}{4}\right).$$

Hence, dividing each expression for y_n by e^x, we have the identity

$$\sin x + {}_nC_1 \sin\left(x + \frac{\pi}{2}\right) + {}_nC_2 \sin\left(x + \frac{2\pi}{2}\right) + \dots + \sin\left(x + \frac{n\pi}{2}\right)$$
$$= 2^{n/2} \sin\left(x + \frac{n\pi}{4}\right).$$

This result can be written

$$\sin x + {}_nC_1 D \sin x + {}_nC_2 D^2 \sin x + \dots + D^n \sin x = 2^{n/2} \sin\left(x + \frac{n\pi}{4}\right),$$

or, treating the *operators* D, D^2, etc., as if they were algebraical quantities connected with $\sin x$ by multiplication only, we have the following *symbolical notation* :—

$$(1 + D)^n \sin x = 2^{n/2} \sin\left(x + \frac{n\pi}{4}\right).$$

This is a particular case of a more general theorem given as an example below. [See Ex. XVIII. No. 29.]

108. Formation of Differential Equations—Elimination of Functions.

If $y = f(x)$ . . . . (1)

then an equation which involves any of the quantities x, y, y_1, y_2, etc., or their powers is called an *ordinary differential equation.*

If we differentiate (1) one or more times, we shall obtain equations involving y_1, y_2, etc., and from these and (1) we may or may not eliminate any quantity we please; the resulting equation, however obtained, will be a differential equation. The formation of differential equations is useful in expansions of functions (see next chapter), and we shall see that in cases where an involved function occurs it is usual, if possible, to eliminate the function by preference.

109. Examples.

Ex. 1. If $y = \sin x$; then $y_1 = \cos x$;

$$\therefore\ y^2 + y_1{}^2 = 1, \text{ or } y^2 + (dy/dx)^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Differentiating (a), we have $2yy_1 + 2y_1y_2 = 0$,

$\therefore$ dividing by $2y_1$ (which is not necessarily 0), $y + y_2 = 0$.

Or, since $y_1 = \cos x$, $\therefore\ y_2 = -\sin x = -y$; $\therefore\ y + y_2 = 0$.

Ex. 2. If $y = \sin^{-1} x$;

then
$$y_1 = \frac{1}{\sqrt{1 - x^2}}, \quad \therefore\ \sqrt{1 - x^2}\,.\,y_1 = 1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating both sides of (1)

$$\sqrt{1 - x^2}\,.\,y_2 - \frac{x}{\sqrt{1 - x^2}}\,.\,y_1 = 0$$

$$\therefore\ (1 - x^2)\,y_2 - xy_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Or, since $y_1 = \dfrac{1}{\sqrt{1 - x^2}}$, $\therefore\ y_2 = \dfrac{x}{(1 - x^2)^{\frac{3}{2}}} = \dfrac{x}{1 - x^2}\,.\,\dfrac{1}{\sqrt{1 - x^2}} = \dfrac{xy_1}{1 - x^2}$;

$$\therefore\ (1 - x^2)y_2 = xy_1 \text{ as before.}$$

Now differentiate (2) n times by Leibnitz's Theorem,

$$\therefore\ (1 - x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1.2}(-2)y_n$$
$$- xy_{n+1} \qquad - ny_n \qquad = 0,$$

the higher d.c.'s of $1 - x^2$ and x being zero.

$$\therefore\ (1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0.$$

Again, if A_r be the value of y_r when x is put equal to 0 (*after* differentiation), we have

$$A_{n+2} - n^2A_n = 0.$$

NOTE.—Squaring (1), we have $(1 - x^2)y_1{}^2 = 1$; by differentiating this

equation, and dividing out by $2y_1$ [which does not vanish, see (1)], we shall obtain (2) as before.

EXAMPLES XVIII.

Use Leibnitz's Theorem in the following examples:—

1. If $y = e^x \sin x$, find y_4.

2. If $y = xe^x$, find y_5, y_n.

3. If $y = x^3 e^{ax}$, find y_3, y_n.

4. If $y = x \log x$, find y_5, y_n.

5. If $y = x^3 \log x$, find y_4, y_n.

6. If $y = \dfrac{\log x}{x^2}$, find y_4.

7. If $y = \dfrac{\sin x}{x}$, find y_5.

8. If $y = \dfrac{x}{\sqrt{1+x}}$, find y_n.

9. If $y = a \sin mx + b \cos mx$, prove that $y_2 + m^2 y = 0$.

10. If $y = ae^{mx} + be^{-mx}$, prove that $y_2 = m^2 y$.

11. If $y = \cos^{-1} ax$, then $(1 - a^2x^2)y_2 = a^2 x y_1$.

12. If $y = \dfrac{1}{a} \sec^{-1} \dfrac{x}{a}$, then $x(x^2 - a^2)y_2 + (2x^2 - a^2)y_1 = 0$.

13. If $y = \dfrac{1}{\sqrt{1 - x^2}}$, prove that

(1) $(1 - x^2)y_1 = xy$.
(2) $(1 - x^2)y_4 - 7xy_3 - 9y_2 = 0$.
(3) $(1 - x^2)y_{n+1} - (2n + 1)xy_n - n^2 y_{n-1} = 0$.

14. If $y = \dfrac{3x + 4}{5x + 6}$, prove that $2y_2 y_3 = y_1 y_4$.

15. If $y = \sinh^{-1} x \equiv \log(x + \sqrt{1 + x^2})$, then $(1 + x^2)y_2 + xy_1 = 0$.

16. If $y = \sin(m \sinh^{-1} x)$, then $(1 + x^2)y_2 + xy_1 + m^2 y = 0$.

17. If $y = (1 - x^2)^{-\frac{1}{2}} \sin^{-1} x$, then $(1 - x^2)y_1 - xy - 1 = 0$.

18. If $y = x \log \dfrac{x}{a + bx}$, then $x^3 y_2 = (y - xy_1)^2$.

19. If $y = \sin(m \tan^{-1} x)$, and $A_r \equiv (y_r)_0$, prove that

$$A_{n+2} + (2n^2 + m^2) A_n + n(n - 1)^2 (n - 2) A_{n-2} = 0.$$

20. If $y = \sin(m \sin^{-1} x)$, show that $(1 - x^2)y_2 - xy_1 + m^2 y = 0$, and that $A_{n+2} = (n^2 - m^2) A_n$.

21. If $y = e^{x^2} \sin x$, prove that $y_2 - 4xy_1 + (4x^2 - 1)y = 0$, and that $A_{n+2} - (4n + 1) A_n + 4n(n - 1) A_{n-2} = 0$.

22. If $y = a\sin(m\log x + \alpha)$, show that $x^2y_2 + xy_1 + m^2y = 0$.

23. If $y = x^{n-1}\log x$, find y_n, and show that the result $= \dfrac{(n-1)!}{x}$, as obtained by the direct method.

24. If $y = \sqrt{1-x^2}$, prove that

(1) $yy_3 + 3y_1y_2 = 0$.

(2) $yy_5 + 5y_1y_4 + 10y_2y_3 = 0$.

(3) $yy_6 + 6y_1y_5 + 15y_2y_4 + 10y_3^2 = 0$.

25. If $y = e^{a\tan^{-1}x}$, then $A_n = aA_{n-1} - (n-1)(n-2)A_{n-2}$.

26. If $y = x^ne^x$, $y_n = e^x\left\{x^n + n^2x^{n-1} + \dfrac{n^2(n-1)^2}{2!}x^{n-2} + \dots + n!\right\}$, the $(r+1)^{\text{th}}$ term being $e^x\left(\dfrac{n!}{(n-r)!}\right)^2\dfrac{x^{n-r}}{r!}$.

27. If $y = xe^x\sin x$, find y_n. [See Ex. 33 of the previous set.]

28. If $y = x^2e^{x\sqrt{3}}\sin x$, show that

$$y_n = 2^ne^{x\sqrt{3}}\left\{x^2\sin\left(x + \frac{n\pi}{6}\right) + nx\sin\left(x + \frac{(n-1)\pi}{6}\right)\right.$$
$$\left. + \frac{n(n-1)}{4}\sin\left(x + \frac{(n-2)\pi}{6}\right)\right\}.$$

29. If $y = e^{ax}u$, where u is any function of x, prove that

$D^ny \doteq e^{ax}(a+D)^n.u$, the notation being symbolical.

30. If $\psi(D)$ be a rational integral algebraical function of D, prove that

(1) $\psi(D).e^{ax} = \psi(a)e^{ax}$.

(2) $\psi(D)(e^{ax}.u) = e^{ax}\psi(a+D)u$.

31. From the identity $\cos ax\cos bx = \frac{1}{2}\{\cos(a+b)x - \cos(a-b)x\}$, prove, by differentiating n times, that

$$a^n\cos\left(ax + n\frac{\pi}{2}\right) + {}_nC_1a^{n-1}b\cos\left(ax + (n-1)\frac{\pi}{2}\right)\cos\left(bx + \frac{\pi}{2}\right) + \dots$$
$$+ {}_nC_ra^{n-r}b^r\cos\left(ax + (n-r)\frac{\pi}{2}\right)\cos\left(bx + \frac{r\pi}{2}\right) + \dots$$
$$= \tfrac{1}{2}(a+b)^n\cos\left\{(a+b)x + n\frac{\pi}{2}\right\} + \tfrac{1}{2}(a-b)^n\cos\left\{(a-b)x + n\frac{\pi}{2}\right\}.$$

32. Using two methods for finding y_n, where $y = e^{ax}\sin mx$, prove the identity

$$a^n\sin mx + {}_nC_1a^{n-1}m\sin\left(mx + \frac{\pi}{2}\right) + {}_nC_2a^{n-2}m^2\sin\left(mx + 2\frac{\pi}{2}\right) + \dots$$
$$= (a^2+m^2)^{n/2}\sin\left(mx + n\tan^{-1}\frac{m}{a}\right).$$

ANSWERS.

1. $-4e^x \sin x$.

2. $e^x(5+x)$; $e^x(n+x)$.

3. $e^{ax}(a^3x^3 + 9a^2x^2 + 18ax + 6)$; $a^{n-3}e^{ax}\{a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)\}$.

4. $\dfrac{4!}{x^5}$; $\dfrac{(-1)^n(n-2)!}{x^{n-1}}$.

5. $\dfrac{6}{x}$; $\dfrac{6(-1)^n(n-4)!}{x^{n-3}}$.

6. $\dfrac{1}{x^6}\{120\log x - 154\}$.

7. $\cos x\left(\dfrac{120}{x^5} - \dfrac{20}{x^3} + \dfrac{1}{x}\right) - 5\sin x\left(\dfrac{24}{x^6} - \dfrac{12}{x^4} + \dfrac{1}{x^2}\right)$.

8. $\dfrac{(-1)^{n-1}}{2^n}.1.3.5\ldots(2n-3)(x+2n)/(1+x)^{\frac{2n+1}{2}}$.

27. $e^x\left\{2^{n/2}.x\sin\left(x+\dfrac{n\pi}{4}\right) + n.2^{\frac{n-1}{2}}\sin\left(x+\dfrac{(n-1).\pi}{4}\right)\right\}$.

CHAPTER X.

INTERPRETATION OF SIGNS.

110. Continuous Functions.

Let $y = f(x)$; then if (1) for every value of x between two given values a and b, there is a corresponding finite value of y, and (2) for an indefinitely small increment of x there is a corresponding indefinitely small increment of y, then y is said to be *finite and continuous* between $x = a$ and $x = b$.

Similarly, if for every value of x between a and b there is a corresponding finite value of dy/dx, and for an indefinitely small increment of x there is a corresponding indefinitely small increment of dy/dx, then dy/dx is said to be finite and continuous between $x = a$ and $x = b$.

And so for the higher derivatives.

In the adjoining figure, y satisfies condition (1) between the points A and G, between G and H, and between K and L. But

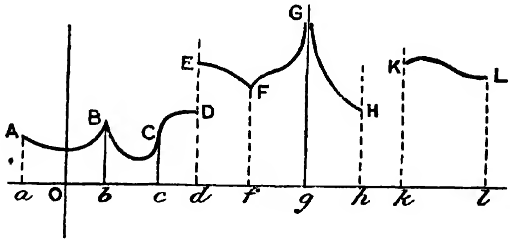

FIG. 15.

at G it is infinite, and does not satisfy (1), neither does it in the space between $x = Oh$ and $x = Ok$.

Again, at D it suddenly changes by the amount ED, hence condition (2) is not satisfied. The function y is therefore said to be discontinuous at the points D, E, G, H, and K.

Again, dy/dx is infinite at B, C, D, and G, while it suddenly changes at F. It is therefore discontinuous at the points B, C, D, F, G; and obviously at H and K.

111. We may remark, that even when these *singular values*, as they are called, do occur, they are usually *very limited in number, and are separated by an infinite number of ordinary values.* Hence, it is usual to deal with functions on the supposition that they are finite and continuous, as well as their d.c.'s; and to take the exceptions singly.

If y have more than one value for a given value of x, we may fix on one value at a time, ignoring the others. This is equivalent to considering one branch of a curve at a time.

112. Example in Electricity.

A simple example of discontinuity occurs in the case of the potential of point in the vicinity of an electrically charged sphere. If q be the charge,

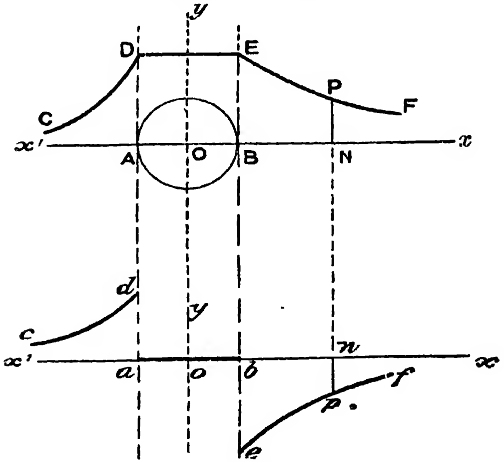

Fig. 16.

and r the distance of a point N from the centre O of the sphere, then the potential of the point N is known to be q/r. In the first place, however, the potential is always $+^{ve}$, and in the second place it is constant for all points within the sphere, or on its surface.

The variation of potential is exhibited in the upper curve of the above figure; thus the ordinate PN denotes the potential of the point N. This curve consists of three distinct parts, namely, CD, DE, and EF. Taking O as the origin, the equations of the curves CD and EF are respectively $y = \mp \frac{q}{x}$; and, if a be the radius of the sphere, the equation of DE is $y = q/a$.

The lower curve represents the force on a unit charge at different points along $x'ox$; thus pn denotes the force at N† in magnitude and direction, and is the value of dy/dx at the point P.

Hence the three portions cd, ab, ef, have for their respective equations

$$y = +\frac{q}{x^2};\ y = 0;\ \text{and}\ y = -\frac{q}{x^2}.$$

The points at which the discontinuities occur are of course D, E in the upper curve, and d, e in the lower curve.

Sign of dy/dx and d^2y/dx^2—Maxima and Minima.

113. Let y be a finite and continuous function of x, and let us suppose x to be always increasing, so that dx is always $+^{ve}$. Then if dy/dx be $+^{ve}$ for a given value of x (a say), dy is $+^{ve}$, or y is increasing as x increases. Suppose that for a higher value of x (b say), dy/dx is $-^{ve}$; then dy is $-^{ve}$, or y is diminishing as x increases.

But if y is increasing when $x = a$, and diminishing when $x = b$ it must have reached its greatest or *maximum* value at some intermediate point. And if dy/dx is $+^{ve}$ when $x = a$, and $-^{ve}$ when $x = b$, it must have passed through zero at some intermediate point. In fact, when dy/dx is zero, y is constant for the moment, and is neither increasing nor diminishing; it has reached its maximum value.

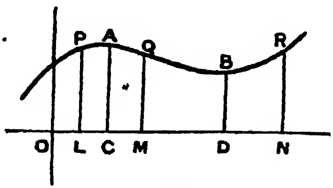

FIG. 17.

In the above figure, dy/dx is $+^{ve}$ at P, and $-^{ve}$ at Q, an

† For convenience, the charge q has been taken as a *negative* charge

at the intermediate point, A, it is zero, while y is a maximum at that point.

Similarly, if dy/dx is $-^{ve}$ when $x = a$, as at Q, and $+^{ve}$ when $x = b$, as at R; then y is first diminishing and then increasing. Hence at some intermediate point y must have its *minimum* value, as at B, while dy/dx is zero at that point.

It should be noted that in both cases dy/dx changes sign in passing through zero.

Hence the condition for a maximum or minimum value of y is that $dy/dx = 0$, *and changes sign.*

114. Distinction between Maxima and Minima.

We have seen that when dy/dx is $+^{ve}$, y is increasing with x.

Similarly, when $\frac{d^2y}{dx^2}$, i.e. $d.\frac{dy}{dx}/dx$, is $+^{ve}$, $\frac{dy}{dx}$ is increasing with x, *i.e.* $\tan\psi$ (and therefore ψ) is increasing; hence the curve is getting either more uphill or less downhill as x increases; as between Q and R.

The curve is also convex to Ox, between these points.

Therefore, *at a minimum value of* y, d^2y/dx^2 is $+^{ve}$.

Similarly, when d^2y/dx^2 is $-^{ve}$, dy/dx is diminishing as x increases, the curve is getting either less uphill or more downhill; as between P and Q. The curve is also concave to Ox between these points.

Therefore, *at a maximum value of* y, d^2y/dx^2 is $-^{ve}$.

Ex. If $y = 1 - x + x^2$; $dy/dx = -1 + 2x = 0$ for a max. or min.
$\therefore x = \frac{1}{2}$.
$d^2y/dx^2 = 2$, and is $+^{ve}$; hence y is a minimum when $x = \frac{1}{2}$.

Note.—Both of the terms maximum and minimum are included in either of the single expressions, *stationary-*, or *turning-value.*

115. Points of Inflexion.

When d^2y/dx^2 is first $+^{ve}$ for a given value of x, and subsequently passes through zero and becomes $-^{ve}$, then, by the above reasoning, dy/dx (or $\tan\psi$) is at first increasing, becomes

stationary, and then diminishes. At the point where $d^2y/dx^2 = 0$, *provided it is changing sign*, we have a *point of inflexion* as at C in Fig. 18; for as we move from P to Q the curve gets steeper until we arrive at C, the point of maximum steepness, after which it gets less steep.

Hence, at C, dy/dx and ψ are each a maximum.

Similarly, when d^2y/dx^2 is first $-^{ve}$, and passes through zero and becomes $+^{ve}$, then dy/dx is first diminishing, becomes

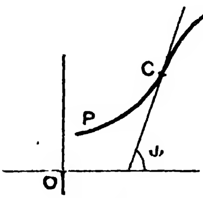

FIG. 18.

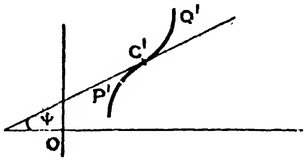

FIG. 19.

stationary, and afterwards increases; and when $d^2y/dx^2 = 0$, we have a *point of inflexion*, which is also a point of minimum steepness, as at C' in Fig. 19.

The condition that dy/dx should be a maximum (d^2y/dx^2 being zero), is that d^3y/dx^3 is $-^{ve}$; and for a minimum, that d^3y/dx^3 is $+^{ve}$.

Ex. $y = x - x^2 + x^3$.

$y_1 = 1 - 2x + 3x^2$.

$y_2 = -2 + 6x = 0$ for turning values of dy/dx; $\therefore x = \frac{1}{3}$.

$y_3 = 6$, which is $+^{ve}$. Hence y_1 is a minimum.

Also where $x = \frac{1}{3}$, $y = \frac{7}{27}$; $y_1 = \frac{2}{3}$.

Hence there is a point of inflexion of the second kind at the point $(\frac{1}{3}, \frac{7}{27})$; the value of ψ being $\tan^{-1}\frac{2}{3}$.

116. Example on the Signs of y, y_1, y_2.

We give in a tabulated form the signs of y, y_1, y_2, for every point of the accompanying curve, in each of the four quadrants; these should be carefully verified by the student. In passing from 3 to 1, or from 2 to 4 there is a point of inflexion, and y_2 will be seen to change sign in both cases.

Note that in 3, y is $-^{\text{ve}}$, but increasing algebraically as we pass from left to right; while in 4, y is $-^{\text{ve}}$ and diminishing algebraically.

drant	y	y_1	y_2
1	+	+	+
2	+	−	+
3	−	+	−
4	−	−	−

Fig. 20.

117. Velocities and Accelerations.

If time be the independent, and space the dependent, variable, we have seen that ds/dt is the velocity of a moving point at any instant. And given the function which s is of t, we can find the velocity at any time t.

Also the rate of change of velocity, *i.e.* the acceleration at any instant, is represented by dv/dt, i.e. d^2s/dt^2.

If v or ds/dt is $-^{\text{ve}}$, the body is moving in the $-^{\text{ve}}$ direction, since s is diminishing as t increases.

If dv/dt or d^2s/dt^2 is $-^{\text{ve}}$, the velocity is diminishing.

118. Simple Harmonic Motion.

Let P, Q, R, S be four positions of a point moving *uniformly* round a circle, and K, L, M, N their projections on AB. Then the latter will be four positions of a point moving in simple harmonic motion along AB.

Suppose t to be reckoned from the instant that P is at B, and let ω be the uniform angular velocity of P about O.

$\therefore \angle POB = \omega t$, and if $OK = s$, $OP = a$, we have

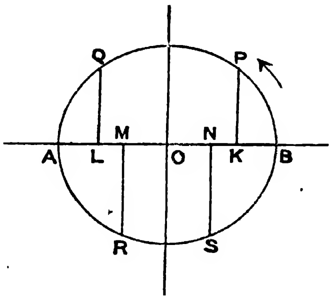

Fig. 21.

$$s = a\cos\omega t \quad . \quad . \qquad (1)$$
$$\therefore \dot{s} \text{ or } ds/dt = -a\omega\sin\omega t \qquad (2)$$
$$\ddot{s} \text{ or } d^2s/dt^2 = -a\omega^2\cos\omega t\,. \qquad (3)$$

From (2) and (3) we can obtain the sign of $\dot{s}$ and $\ddot{s}$ in each of the four quadrants, these depending on the signs of $\sin\theta$ and $\cos\theta$ for the different values of θ between 0 and 2π. We give these in tabular form.

Position	s	$\dot{s}$	
K	$+$	$-$	
L	$-$	$-$	$+$
M	$-$	$+$	$+$
N	$+$	$+$	

The student should try to interpret these signs. Thus, for N, $\dot{s}$ is $+^{\text{ve}}$, $\ddot{s}$ is $-^{\text{ve}}$, which shows that the point is moving to the right, but is stopping.

EXAMPLES XIX.

1\. Find the maximum and minimum values of :—

(1) $x^2 - 3x + 1$. (2) $x + \dfrac{1}{x}$. (3) $x(x+1)(x+2)$.

(4) $x \log x$. (5) $\dfrac{x^2 + x + 2}{x - 1}$. (6) $x(2\cos x + x\sin x)$.

2\. Find the points of inflexion in the following curves, stating whether ψ is a max. or min. :—

(1) $y = x^3$. (2) $y = x^2(x-3)$. (3) $y = \sin x$.
(4) $y = x(x+1)(x+2)$. (5) $y = x^2 \log x$ (take $e = 2{\cdot}7$).

3\. If y is the number of gallons in a leaking tank of water, x the number of hours since the tank was full, what does dy/dx represent, and is it $+^{\text{ve}}$ or $-^{\text{ve}}$? If d^2y/dx^2 were $+^{\text{ve}}$, what information would this convey?

4\. If x denote the population of a country, and y the number of years that have elapsed since a fixed date, what is the meaning of dx/dy and d^2x/dy^2 respectively?

5\. Let x denote the annual expenditure, and y the annual receipts of a trading company. If dy/dx be $+^{\text{ve}}$, and d^2y/dx^2 be $-^{\text{ve}}$ for a given value of x, what inference would you draw? What additional statement could you make if you also knew whether dy/dx were greater or less than unity?

6\. Give, in a tabular form, the signs of y, y_1, and y_2 in the case of the circle $x^2 + y^2 = a^2$, for each quadrant, as ascertained by actual differentiation; and verify your results geometrically.

7\. Draw a curve at every point of which

(1) x is $-^{\text{ve}}$, y $+^{\text{ve}}$, y_1 $-^{\text{ve}}$, y_2 $-^{\text{ve}}$
(2) x is $+^{\text{ve}}$, y $-^{\text{ve}}$, y_1 $-^{\text{ve}}$, y_2 $+^{\text{ve}}$
(3) x is $-^{\text{ve}}$, y $-^{\text{ve}}$, y_1 $+^{\text{ve}}$, y_2 $-^{\text{ve}}$.

8. A particle moves along the axis of x, so that $x = t^2(t - 3)$, t being the time in seconds reckoned from a given instant.

(1) Find its velocity and acceleration at any given time.

(2) What is happening when $t = -1, 0, 1, 2, 3$ respectively?

(3) When is the velocity a minimum?

(4) When is the velocity constant for an instant?

(5) Describe its motion from $t = -10$ to $+10$.

(6) Representing times by abscissæ on the axis of y, and distances by abscissæ on the axis of x, draw the curve representing its motion.

Answers.

1. (1) $-\frac{5}{4}$, min. (2) $+2$, min.; -2, max.

(3) $-\dfrac{2}{3\sqrt{3}}$, min.; $+\dfrac{2}{3\sqrt{3}}$, max. (4) $-\dfrac{1}{e}$, min.

(5) 7, min.; -1, max. (6) $\pi^2/4$, max.

2. (1) (0, 0), min. (2) $(1, -2)$, min. (3) (0, 0), max.

(4) $(-1, 0)$, min. (5) $(\cdot 22, -\cdot 076)$, min.

3. The rate of increase in gals. per hour; $-^{ve}$; that the rate of leaking was diminishing.

4. The rate of increase (at any moment) in units per year; the rate at which this rate is increasing.

5. That an increase in the annual expenditure brings with it an increase in the annual receipts; but that the latter increase tends to become smaller as the expenditure becomes greater. If $dy/dx > 1$, a profit is made.

8. (1) $3t(t - 2)$; $6(t - 1)$. (3) and (4) When $t = 1$.

CHAPTER XI.

EXPANSION OF FUNCTIONS.

119. Definition.—When a function of any quantity x is expressed in the form of the series $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, where the a's are independent of x, it is said to be *expanded in positive integral ascending powers of* x.

Thus the expansion of $(1 + x)^n$ is $1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots$

It is not every function that can be expanded in this way; $\log x$ cannot, for instance, as we shall see hereafter. But at present we shall, with one exception, deal only with those functions that can be so expanded, reserving to a later portion of the chapter the consideration of the possibility of such an expansion, as well as the conditions for convergency.

120. Odd and Even Functions.—A function of x is said to be *even* when it is not altered in magnitude or sign, on changing x into $-x$; it is said to be *odd* when it is altered in sign, but not in magnitude.

Thus, $\cos(-x) = \cos x$; $(-x)^2 = x^2$; hence $\cos x$ and x^2 are *even* functions.

But $\sin(-x) = -\sin x$; $(-x)^3 = -x^3$; hence $\sin x$ and x^3 are *odd* functions.

If $f(x)$ is a function capable of expansion in $+^{ve}$ ascending powers of x, we may write—

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \quad . \quad . \quad . \quad (1)$$

Now (i) suppose $f(x)$ an *even* function; then $f(x) = f(-x)$,

$$\therefore\ a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$
$$= a_0 + a_1(-x) + a_2(-x)^2 + a_3(-x)^3 + \dots$$
$$= a_0 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - a_5x^5 + \dots$$
$$\therefore \text{ transposing,} \qquad a_1x + a_3x^3 + a_5x^5 + \dots = 0.$$

Subtracting from (1), we have

$$f(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots \qquad (2)$$

Similarly, (ii) suppose $f(x)$ an *odd* function; then we can show that

$$f(x) = a_1x + a_3x^3 + a_5x^5 + \dots \qquad (3)$$

Hence an *even* function of x contains *even* powers of x when expanded; and an *odd* function, *odd* powers. This might have been expected, for if in (2) we change x into $-x$, the series is not altered because of the even powers of x, while in (3) all the terms will be altered because of the odd powers.

121. Algebraical Method.—By help of the known expansions of the functions $(1 \pm x)^n$, a^x, $\log(1+x)$, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, and any others that we choose to include, we can expand combinations of these functions, but only to a few terms. If, however, x is small, this will not matter, as the higher powers of x will be negligible. For the expansions, see *Miscellaneous Theorems*, above.

Ex. 1. *Expand* $\log \cos x$ *as far as the term containing* x^4.

We have $\log \cos x = \log\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} \dots\right) = \log\left\{1 - \left(\dfrac{x^2}{2} - \dfrac{x^4}{24}\dots\right)\right\}$

$$= -\left\{\left(\frac{x^2}{2} - \frac{x^4}{24}\dots\right) + \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{24}\dots\right)^2 + \dots\right\}$$
$$= -\left\{\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{8}\dots\right\} = -\frac{x^2}{2} - \frac{x^4}{12}\dots$$

Ex. 2. *Expand* $\dfrac{x}{e^x - 1}$ *as far as* x^4.

$$\frac{x}{e^x - 1} = \frac{x}{x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}\dots} = \frac{1}{1 + \dfrac{x}{2!} + \dfrac{x^2}{3!}\dots}$$
$$= \left\{1 + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \dots\right)\right\}^{-}$$

$$= 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} \cdots\right) + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} \cdots\right)^2$$

$$- \left(\frac{x}{2} + \cdots\right)^3 + \left(\frac{x}{2} \cdots\right)^4 -$$

$$= 1 - \tfrac{1}{2}x - \tfrac{1}{6}x^2 - \tfrac{1}{24}x^3 - \tfrac{1}{120}x^4 \cdots$$
$$+ \tfrac{1}{4}x^2 + \tfrac{1}{6}x^3 + (\tfrac{1}{36} + \tfrac{1}{24})x^4 \cdots$$
$$- \tfrac{1}{8}x^3 - \tfrac{1}{8}x^4 \cdots$$
$$+ \tfrac{1}{16}x^4 \cdots$$
$$= 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 \cdots$$

Ex. 3 *Expand* cosec x, *as far as* x^4.

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{x - \dfrac{x^3}{6} + \dfrac{x^5}{120} \cdots} = \frac{1}{x} \cdot \frac{1}{1 - \dfrac{x^2}{6} + \dfrac{x^4}{120} \cdots}$$

$$= \frac{1}{x}\left\{1 - \left(\frac{x^2}{6} - \frac{x^4}{120} \cdots\right)\right\}^{-1}$$

$$= \frac{1}{x}\left\{1 + \left(\frac{x^2}{6} - \frac{x^4}{120} \cdots\right) + \left(\frac{x^2}{6} \cdots\right)^2 \cdots\right\}$$

$$= \frac{1}{x}\left\{1 + \frac{x^2}{6} + \frac{7x^4}{360} \cdots\right\} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} \cdots$$

This case should be noted. If $x = 0$, $\operatorname{cosec} x = \infty$, which points to the fact that $\operatorname{cosec} x$ cannot be expanded in *positive* integral powers of x; for referring to equation (1), Art. 120, we see that $f(0) = a_0$, which is finite. The expansion of $\operatorname{cosec} x$, however, contains the term $1/x$, or x^{-1} (a negative power), which becomes infinite when $x = 0$.

122. Inverse Functions.—When we know the expansion of a direct function, $x = f(y)$, say, in $+^{\text{ve}}$ integral ascending powers of y, we can find that of the inverse function y in ascending powers of x, provided that the known expansion, *if an infinite series*, has no absolute term (or term without x).

In this case the resulting, or *reversed*, series will be found to have no absolute term, since by hypothesis the graph of $x = f(y)$, and therefore the graph of $y = f^{-1}(x)$, passes through the origin.

123. Ex. 1. *Expand* $\sin^{-1} x$ *as far as* x^5.

Since $\sin^{-1} x$ is an *odd* function of x, we may assume

$$y = \sin^{-1} x = a_1 x + a_3 x^3 + a_5 x^5 \cdots$$

The known series is $x = \sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots$

$$\therefore x = (a_1x + a_3x^3 + a_5x^5 \ldots) - \frac{1}{3!}(a_1x + a_3x^3 \ldots)^3 + \frac{1}{5!}(a_1x \ldots)^5 \ldots$$
$$= a_1x + (a_3 - \tfrac{1}{6}a_1^3)x^3 + (a_5 - \tfrac{1}{2}a_1^2a_3 + \tfrac{1}{120}a_1^5)x^5 \ldots$$

Now, if $a_1, a_3, \ldots$ were all known, this would be an identity in x, and therefore (conversely), assuming it is so, the coefficients of like powers of x should be the same on both sides.

Equating these coefficients, we have

(Coefficients of x)	$1 = a_1$,	$\therefore a_1 = 1$.
(„ x^3)	$0 = a_3 - \frac{1}{6}a_1^3$,	$\therefore a_3 = \frac{1}{6}$.
(„ x^5)	$0 = a_5 - \frac{1}{2}a_1^2a_3 + \frac{1}{120}a_1^5$,	$\therefore a_5 = \frac{3}{40}$.

Hence $\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 \ldots$

124. If the series is not infinite, then, if it contains an absolute term, the reversed series will also contain one. To find it we shall show that we have to solve the equation $f(a_0) = 0$, where $x = f(y)$, and y is assumed equal to $a_0 + a_1x + a_2x^2 \ldots$

For if $x = f(y) \equiv p_0 + p_1y + p_2y^2 + \ldots + p_ny^n$, this being the given series,

$$x = p_0 + p_1(a_0 + a_1x + \ldots) + p_2(a_0 + a_1x \ldots)^2 + \ldots + p_n(a_0 + a_1x \ldots)^n$$
$$= (p_0 + p_1a_0 + p_2a_0^2 + \ldots + p_na_0^n) + Kx + Lx^2 \ldots$$
$$= f(a_0) + Kx + Lx^2 \ldots,$$

and equating coefficients of x^0, we get $f(a_0) = 0$.

The other equations for finding a_1, a_2, etc., will be found to be all of the first degree only.

Ex. 2. *If* $x = (1 - y)(1 - 2y)$, *expand* y *in ascending powers of* x.

Assuming $y = a_0 + a_1x + a_2x^2 \ldots$, we have

$$x = \{(1 - a_0) - a_1x - a_2x^2 \ldots\}\{(1 - 2a_0) - 2a_1x - 2a_2x^2 \ldots\}.$$

Begin by equating coefficients of x^0; $\therefore 0 = (1 - a_0)(1 - 2a_0)$;

$$\therefore a_0 = 1 \text{ or } \tfrac{1}{2}.$$

(1) If $a_0 = 1$; $x = (-a_1x - a_2x^2 \ldots)(-1 - 2a_1x - 2a_2x^2 \ldots)$

$$= (a_1x + a_2x^2 \ldots)(1 + 2a_1x + 2a_2x^2 \ldots);$$

and equating coefficients,

$$1 = a_1,$$
$$0 = a_2 + 2a_1^2, \text{ etc.}$$
$$\therefore a_1 = 1,\ a_2 = -2.$$
$$\therefore y = 1 + x - 2x^2 \ldots$$

(2) If $a_0 = \frac{1}{2}$; $x = (\frac{1}{2} - a_1x - a_2x^2 \ldots)(-2a_1x - 2a_2x^2 \ldots)$;

and we shall get $\qquad y = \frac{1}{2} - x + 2x^2 \ldots$

The reason that there are two alternative expansions is that we have a quadratic in y, which has two roots.

Thus $2y^2 - 3y + (1 - x) = 0$.

$$\therefore y = \frac{3 \pm \sqrt{9 - 8(1 - x)}}{4} = \tfrac{1}{4}(3 + \sqrt{1 + 8x}), \text{ or } \tfrac{1}{4}(3 - \sqrt{1 + 8x}).$$

If we expand the surds, we shall get the same two series as before.

Ex. 3. $x = (1 + y)^3 = (1 + a_0 + a_1x + a_2x^2 \ldots)^3$ suppose;

$$\therefore (1 + a_0)^3 = 0, \quad \therefore x = (a_1x + a_2x^2 \ldots)^3.$$

But this gives, on equating coefficient of x, $1 = 0$; an absurdity.

In fact, y cannot be expanded in integral powers of x; as may be otherwise seen, for $1 + y = x^{\frac{1}{3}}$, i.e. $y = -1 + x^{\frac{1}{3}}$.

125. Implicit Functions.

The same method may be adopted in this case.

Ex. 4. *If* $x^3 + 2xy^2 - y^3 + x = 1$; . . . . (1)
expand y *in ascending powers of* x.

Assume $y = a_0 + a_1x + a_2x^2 \ldots$;

$\therefore$ in (1), $x^3 + 2x(a_0 + a_1x + a_2x^2 \ldots)^2 - (a_0 + a_1x + a_2x^2 \ldots)^3 + x = 1$.

or $x^3 + 2x(a_0{}^2 + 2a_0a_1x + \ldots) - \{a_0{}^3 - 3a_0{}^2a_1x + (3a_0a_1{}^2 + 3a_0{}^2a_2)x^2 \ldots\} + x = 1$.

$$\therefore -a_0{}^3 = 1, \text{ or } a_0 = -1;$$

$$2a_0{}^2 + 3a_0{}^2a_1 + 1 = 0, \quad \therefore 3a_1 + 3 = 0, \text{ or } a_1 = -1;$$

$$4a_0a_1 - (3a_0a_1{}^2 + 3a_0{}^2a_2) = 0, \quad \therefore 4 + 3 - 3a_2 = 0, \text{ or } a_2 = \tfrac{7}{3};$$

$$\therefore y = -1 - x + \tfrac{7}{3}x^2 \ldots$$

Since a_0 is the value of y when $x = 0$, we may obtain this at once by putting $x = 0$ in (1). This gives $(y)_0$, or a_0, $= -1$.

Practically the same remarks as those made in the last article apply in the case of implicit functions, when algebraical, rational and integral.

For a rigid discussion of the subject, see Chrystal's "Algebra," vol. ii. p. 349.

EXAMPLES XX.

1. Prove that

(1) $\log \dfrac{\sin x}{x} = -\dfrac{1}{6}x^2 - \dfrac{1}{180}x^4 - \dfrac{1}{2835}x^6 \ldots$

(2) $e^{\sin x} = 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{15}x^5 - \tfrac{1}{240}x^6 \ldots$

(3) $\sin(\sin x) = x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 \ldots$

(4) $\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 \ldots$

(5) $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 \ldots$

(6) $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 \ldots$

(7) $\operatorname{sech} x = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 \ldots$

(8) $\log\{1 + \log(1 + x)\} = x - x^2 + \frac{7}{6}x^3 - \frac{35}{24}x^4$

(9) $$\sqrt{\frac{1 + e^x}{2e^x}} = 1 - \frac{1}{4}x + \frac{3}{32}x^2 \ldots$$

(10) $\operatorname{gd}^{-1} x \equiv \log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} \cdots$

2. State whether the following functions are odd or even, or neither :-

(a) $\sin^{-1} x$; $\sec x$; $e^x + e^{-x}$; $e^x - e^{-x}$; $\cosh x$; $\sinh x$.

(b) $\tan^{-1} x$; $\log(x + \sqrt{1 + x^2})$; $\log \frac{1 + x}{1 - x}$; $e^{\cos^{-1}}$ $e^{\sin^{-1} x}$

(c) $\frac{\cos x - \sin x}{\cos x + \sin x}$; $\log \frac{\cos x - \sin x}{\cos x + \sin x}$; $\log(x \operatorname{cosec} x)$; $\log(x \cos x)$; $x\frac{e^x + 1}{e^x - 1}$; $\sin(a \tan^{-1} x)$; $\log \tan^{-1} x$.

3. Employ the method of reversion of series in the following examples :—

(1) Given $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \ldots$, expand $\tan x$ as far as x^5.

(2) Given $y = \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$, show that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} \cdots$

(3) From Ex. 1 (10), deduce that $\operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} \cdots$

(4) If $x^3 + y^3 + xy - 1 = 0$, prove that $y = 1 - \frac{1}{3}x - \frac{26}{81}x^3 \ldots$

(5) If $y^3 - (1 + x)y = 6$, expand y in ascending powers of $1/x$ as far as $1/x^3$.

(6) If $6x = (1 - y^2)(1 + 2y)$, expand y in ascending powers of x as far as x^2, giving three series.

Answers.

2. (a) Odd; even; even; odd; even; odd. (b) Odd; odd; odd; even; neither. (c) Neither; odd; even; neither; even; odd; neither.

3. (1) $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \ldots$ (5) $y = -\frac{6}{x} + \frac{6}{x^2} - \frac{6}{x^3} \cdots$

(6) $y = 1 - x - \frac{7}{6}x^2 \ldots$; $y = -1 - 3x + \frac{45}{2}x^2 \ldots$; or $y = -\frac{1}{2} + 4x - \frac{64}{3}x^2 \ldots$

Method of the Differential Calculus.—

126. Taylor's Theorem.

Prop.—*To prove that, if* f(x + h) *be a function of* x + h, *and capable of expansion into a convergent series of positive integral powers of* h, *then*

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots + \frac{h^n}{n!}f^{(n)}x + \dots$$

The proof below is given strictly on the assumption that $f(x + h)$ is capable of expansion in positive integral powers of h, which is by no means the case always; so that the above equation must not be taken as true unless the condition is satisfied at the outset. And even when the function is capable of such expansion, it is necessary to consider for what values of x and h the series is convergent.

A strict proof will be given below (Arts. 136, etc.).

We shall first establish the following—

127. Lemma.

If x *and* h *be independent of each other, then*

$$\frac{d}{dx}f(x+h) = \frac{d}{dh}f(x+h),$$

where h is regarded as constant in the first case, since it is not a function of x; and x as constant in the second case for a similar reason.

Let $y = f(x + h) = f(z)$, say, where $z = x + h$;

$$\therefore\ dz/dx = 1, \text{ and } dz/dh = 1.$$

Then
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz};$$

also
$$\frac{dy}{dh} = \frac{dy}{dz}\frac{dz}{dh} = \frac{dy}{dz};$$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{dh}, \text{ i.e. } \frac{d}{dx}f(x+h) = \frac{d}{dh}f(x+h).$$

Otherwise, let x increase by a small quantity a, which ultimately vanishes. Then

$$\frac{d}{dx}f(x+h) = \ \text{lt}\ \frac{f(x+h+a) - f(x+h)}{}$$

Again, let h increase by the same small quantity a (which is quite permissible, since a is merely *some* quantity which ultimately vanishes); then

$$\frac{d}{dh} f(x+h) = \underset{a=0}{lt} \frac{f(x+a+h) - f(x+h)}{a},$$

which is the same expression as the other; ∴ etc.

Ex. If $f(x+h) \equiv (x+h)^3$

$$\frac{df}{dx} = 3(x+h)^2 \frac{d}{dx}(x+h) = 3(x+h)^2$$

$$\frac{df}{dh} = 3(x+h)^2 \frac{d}{dh}(x+h) = 3(x+h)^2.$$

If we expand $(x+h)^3$, the theorem is not so obviously true; thus

if $f(x+h) \equiv x^3 + 3x^2h + 3xh^2 + h^3$

$$df/dx = 3x^2 + 6xh + 3h^2 = 3(x+h)^2$$

$$df/dh = 3x^2 + 6xh + 3h^2 = 3(x+h)^2.$$

128. Proof of Taylor's Theorem.

By hypothesis we may assume

$$f(x+h) = A_0 + A_1h + A_2h^2 + A_3h^3 \ldots \quad (1)$$

where A_0, A_1, etc., are functions of x, and *do not contain* h.

Then
$$\frac{df}{dx} = \frac{dA_0}{dx} + h.\frac{dA_1}{dx} + h^2.\frac{dA_2}{dx} + h^3.\frac{dA_3}{dx} \ldots \quad (2)$$

$$\frac{df}{dh} = A_1 + 2A_2h + 3A_3h^2 + 4A_4h^3 \ldots \quad (3)$$

Since (2) and (3) are identically equal, we have, comparing coefficients of like powers of h,

$$A_1 = \frac{dA_0}{dx} = f'(x)\text{; since, putting } h = 0 \text{ in (1), } A_0 = f(x).$$

$$2A_2 = \frac{dA_1}{dx} = f''(x)\text{; } \therefore A_2 = \frac{1}{2}f''(x).$$

$$3A_3 = \frac{dA_2}{dx} = \frac{1}{2}f'''(x)\text{; } \therefore A_3 = \frac{1}{3!}f'''(x)\text{; etc., etc.}$$

$$\therefore \text{ in (1) } f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

$$+ \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

129. Maclaurin's (or Stirling's) Theorem.

Prop.—*To prove that, if* f(x) *be a function of* x *capable of expansion into a convergent series of positive integral powers of* x, *then*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

Originally this was proved from first principles, the proof being similar to the above; but it can be at once deduced from Taylor's Theorem, thus:

Put $x = 0$ in the formula of Art. 126;

$$\therefore f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \dots$$

Since it is immaterial what symbol we use, put x for h;

$$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

where, in $f'(0), f''(0), \dots$, x is put $= 0$ *after* differentiation.

Other modes of writing the theorem are

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \dots + \frac{x^n}{n!}(y_n)_0 + \dots;$$

and sometimes

$$y = A_0 + A_1 x + A_2\frac{x^2}{2!} + A_3\frac{x^3}{3!} \dots + A_n\frac{x_n}{n!} + \dots .$$

These two theorems are of exceptional importance, since they embrace as particular cases all the well-known expansions.

130. Ex. 1. If $f(x+h) \equiv (x+h)^n$; then $f(x) = x^n$, $f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$, etc.

$\therefore$ by Taylor's Theorem, $(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{1.2}x^{n-2}h^2 \dots$.

Ex. 2. If $y = a^x$, then $y_1 = a^x \log a$, $y_2 = a^x(\log a)^2$, $y_3 = a^x(\log a)^3, \dots$

And $(y)_0 = 1$, $(y_1)_0 = \log a$, $(y_2)_0 = (\log a)^2$, $(y_3)_0 = (\log a)^3, \dots$

whence by Maclaurin's Theorem,

$$a^x = 1 + x\log a + \frac{x^2(\log a)^2}{2!} + \frac{x^3(\log a)^3}{3!} \dots .$$

Ex. 3. If $y = \log(1+x)$; $y_1 = \frac{1}{1+x}$, $y_2 = -\frac{1}{(1+x)^2}$, $y_3 = \frac{2}{(1+x)^3} \dots$

$$\therefore (y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = -1,\ (y_3)_0 = 2;$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Moreover, $y_n = (-1)^{n-1}\dfrac{(n-1)!}{(1+x)^n}$; $\therefore (y_n)_0 = (-1)^{n-1}(n-1)!$

Hence the general term $= \dfrac{x^n}{n!}(y_n)_0 = (-1)^{n-1}\dfrac{x^n}{n}$.

Ex. 4. If $y = \sin x$; $y_1 = \cos x$, $y_2 = -\sin x$, $y_3 = -\cos x$, $y_4 = \sin x$, etc. $(y)_0 = 0$, $(y_1)_0 = 1$, $(y_2)_0 = 0$, $(y_3)_0 = -1$, $(y_4)_0 = 0$, $(y_5)_0 = 1$, etc.

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Similarly for $\cos x$.

131. An objection might be raised that, in order to apply Taylor's Theorem to the expansion of $(x+h)^n$, we have to differentiate x^n; while the proof of the rule for differentiating x^n depends on the expansion of $(x+h)^n$. But there are methods of differentiating x^n without the use of the Binomial Theorem, and similar remarks apply to the other functions e^x, $\log x$, etc. [See Ex. VI., 2 (1); and Arts. 41, 45, 46.]

Hence it is quite possible for Taylor's Theorem to have been discovered before these other theorems (although such was not actually the case); and in this sense it may be regarded as the parent of all the latter theorems.

132. Further Examples.

Ex. 5. *Expand* $\log(1+\cos x)$ *as far as* $\mathbf{x}^4$.

We have $y = \log(1+\cos x) = \log\left(2\cos^2\dfrac{x}{2}\right) = \log_e 2 + 2\log\cos\dfrac{x}{2}$;

$$\therefore (y)_0 = \log_e 2.$$

$$y_1 = -\tan\frac{x}{2}; \qquad \therefore (y_1)_0 = 0.$$

$$y_2 = -\frac{1}{2}\sec^2\frac{x}{2}; \quad \therefore (y_2)_0 = -\tfrac{1}{2}.$$

$$y_3 = -\frac{1}{2}.2\sec\frac{x}{2}\cdot\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2} = -\tfrac{1}{2}\sec^2\frac{x}{2}\tan\frac{x}{2};$$

$$\therefore (y_3)_0 = 0.$$

$$y_4 = -\frac{1}{2}\left\{\left(2\sec\frac{x}{2}\cdot\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right)\tan\frac{x}{2} + \sec^2\frac{x}{2}\cdot\frac{1}{2}\sec^2\frac{x}{2}\right\}$$

$$\therefore (y_4)_0 = -\tfrac{1}{4}.$$

Hence $\log(1+\cos x) = \log_e 2 - \dfrac{1}{2}\dfrac{x^2}{2!} - \dfrac{1}{4}\dfrac{x^4}{4!}\cdots$

Ex. 6. *Expand* $y = e^{ax}\cos bx$.

We have $y_1 = re^{ax}\cos(bx+\phi)$, where $r=(a^2+b^2)^{\frac{1}{2}}$; $\tan\phi = \frac{b}{a}$. [Art. 101.]

$$y_2 = r^2e^{ax}\cos(bx+2\phi);\ \text{etc.}$$

$$\therefore\ (y)_0 = 1,\ (y_1)_0 = r\cos\phi,\ (y_2)_0 = r^2\cos 2\phi,\ \text{etc.}$$

$$\therefore\ e^{ax}\cos bx = 1 + xr\cos\phi + \frac{x^2r^2\cos 2\phi}{2!} + \dots + \frac{x^nr^n\cos n\phi}{n!}\dots$$

This can be also expanded algebraically, thus:—

$$\begin{aligned} e^{ax}\cos bx &= \tfrac{1}{2}e^{ax}(e^{bxi}+e^{-bxi}) = \tfrac{1}{2}(e^{(a+bi)x}+e^{(a-bi)x}) \\ &= \tfrac{1}{2}\{e^{re^{\phi i}x}+e^{re^{-\phi i}x}\}\dagger = \tfrac{1}{2}[(1+re^{\phi i}x+\dots)+(1+re^{-\phi i}x+\dots)] \\ &= 1 + r\cos\phi . x + \dots \end{aligned}$$

EXAMPLES XXI.

1. Use Taylor's Theorem to prove that—

(1) $e^{x+h} = e^x\left(1 + h + \frac{h^2}{2!} + \dots + \frac{h^n}{n!}\dots\right)$.

(2) $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots$

(3) $\frac{1}{x+h} = \frac{1}{x} - \frac{h}{x^2} + \frac{h^2}{x^3} - \frac{h^3}{x^4} + \dots$

(4) $\sin(\alpha+\beta) = \sin\alpha + \beta\cos\alpha - \frac{\beta^2}{2!}\sin\alpha - \frac{\beta^3}{3!}\cos\alpha + \dots$

(5) $\sqrt{m+n} = m^{\frac{1}{2}} + \frac{1}{2}\frac{n}{m^{1/2}} - \frac{1.1}{2.4}\frac{n^2}{m^{3/2}} + \frac{1.1.3}{2.4.6}\frac{n^3}{m^{5/2}} - \dots$

2. Use Maclaurin's Theorem to prove that—

(1) $\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4\dots$

(2) $\log(1+\tan x) = x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{7}{12}x^4\dots$

(3) $e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!}\dots$

(4) $\sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40}\dots$

(5) $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!}\dots$ [see Art. 104, Ex. 2].

(6) $\tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!}\dots$ [see Art. 104, Ex. 2].

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† Since $a = r\cos\phi$, $b = r\sin\phi$

3. Prove that—

(1) $f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) \dots$

(2) $f(1+x) = f(x) + f'(x) + \frac{f''(x)}{2!} + \dots + \frac{f^n(x)}{n!}$

$$= f(1) + xf'(1) + \frac{x^2}{2!}f''(1) + \dots$$

(3) $\tan^{-1}(x+h) = \tan^{-1}x + \frac{}{1+x^2} - \frac{xh^2}{(1+x^2)^2} \dots$

(4) $\tan^{-1}(1+x) = \tan^{-1}x + \frac{1}{1+x^2} - \frac{}{(1+x^2)^2} - \frac{1-3x^2}{3(1+x^2)^3}$

$$+ \frac{x-x^3}{(1+x^2)^4} \dots$$

(5) $\tan^{-1}(1+x) = \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} \dots$

(6) $\tan^{-1}(x+h) = \tan^{-1}x + h\sin^2\theta - \frac{1}{2}h^2\sin^2\theta\sin 2\theta$

$$+ \frac{1}{3}h^3\sin^3\theta\sin 3\theta - \dots,$$

where $\theta = \cot^{-1}x$. [See Art. 102.]

(7) $\tan^{-1}(1+\tan\theta) = \theta + \cos\theta.\cos\theta - \frac{1}{2}\sin 2\theta\cos^2\theta - \frac{1}{3}\cos 3\theta\cos^3\theta \dots$

(8) $e^h\sin(x+h) = \sin x + h.2^{\frac{1}{2}}\sin\left(x+\frac{\pi}{4}\right) + \frac{h^2}{2!}.2^{\frac{2}{2}}\sin\left(x+\frac{2\pi}{4}\right) + \dots$

$$+ \frac{h^n}{n!}.2^{n/2}\sin\left(x+\frac{n\pi}{4}\right) \dots$$

4. If $e^{3x}\cos 4x = y$, prove that $y_r = 5^r e^{3x}\cos(4x + r\phi)$, where $\phi = \tan^{-1}\frac{4}{3}$.

Hence prove that $e^{3x}\cos 4x = 1 + 5\cos\phi.x + \frac{5^2\cos 2\phi}{2!}.x^2 + \dots$

$$+ \frac{5^n\cos n\phi}{n!}.x^n \dots$$

$$= 1 + 3x - \frac{7}{2}x^2 - \frac{39}{2}x^3 \dots$$

133. Expansion by the Use of Differential Equations, and by Leibnitz's Theorem.

Ex. 1. Let $y = e^x$; then $y_1 = e^x$; and we have

$$y_1 = y, \quad \dots \dots \dots \quad (1)$$

which is the differential equation required.

Assume $\qquad y = a_0 + a_1x + a_2x^2 + a_3x^3 \dots$

Put $x = 0$; $\qquad \therefore a_0 = (y)_0 = e^0 = 1.$

Also $\qquad y_1 = a_1 + 2a_2x + 3a_3x^2 \dots$

$\therefore$ from (1) $\quad a_1 + 2a_2x + 3a_3x^2 \dots = a_0 + a_1x + a_2x^2 + \dots$

Comparing coefficients of like powers of x, since this is an identity in x, [see Ex. 1, Art. 123], we get

$$a_1 = a_0 = 1;$$
$$2a_2 = a_1, \quad \therefore a_2 = \tfrac{1}{2};$$
$$3a_3 = a_2, \quad \therefore a_3 = 1/3!; \text{ etc.}$$
$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

Ex. 2. Let $y = \tan^{-1} x$, then $y_1 = \dfrac{1}{1+x^2}$,

$$\therefore (1 + x^2)y_1 = 1, \text{ the differential equation required.}$$

Assume $y = a_1 x + a_3 x^3 + a_5 x^5 \dots$, *since* $\tan^{-1} x$ *is an odd function.*

$$\therefore y_1 = a_1 + 3a_3 x^2 + 5a_5 x^4 \dots$$
$$\therefore (1 + x^2)(a_1 + 3a_3 x^2 + 5a_5 x^4 \dots) = 1;$$

whence
$$a_1 = 1;$$
$$3a_3 + a_1 = 0, \quad \therefore a_3 = -\tfrac{1}{3};$$
$$5a_5 + 3a_3 = 0, \quad \therefore a_5 = \tfrac{1}{5}; \text{ etc.}$$
$$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \dots$$

Otherwise; since $y_1 = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \dots$ . . . . . (2)

and $\quad y_1 = a_1 + 3a_3 x^2 + 5a_5 x^4 \dots$

we get the same result by equating coefficients of $1, x^2, x^4 \dots$

NOTE.—If the student has begun the Integral Calculus, he will see that, since the integral of $\dfrac{1}{1+x^2}$ is $\tan^{-1} x$, he can obtain the expansion by merely integrating the series (2).

134. Ex. 3. $y = \sin(m \sin^{-1} x)$, *an odd function.*

$$\therefore y_1 = m \cos(m \sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}}.$$
$$\therefore \sqrt{1-x^2} \cdot y_1 = m \cos(m \sin^{-1} x) \quad \dots \quad (A)$$

Differentiating both sides,

$$\sqrt{1-x^2} \cdot y_2 - \frac{xy_1}{\sqrt{1-x^2}} = -m^2 \sin(m \sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}} = -\frac{m^2 y}{\sqrt{1-x^2}}$$

$$\therefore (1 - x^2)y_2 - xy_1 + m^2 y = 0, \text{ the differential equation} \quad . \quad (B)$$

There are two methods of proceeding from this point:—

First Method :—

Assume $y = a_1x + a_3x^3 + a_5x^5 \dots + a_{2n-1}x^{2n-1} + a_{2n+1}x^{2n+1} \dots$

$$\therefore\ y_1 = a_1 + 3a_3x^2 + 5a_5x^4 \dots + (2n-1)a_{2n-1}x^{2n-2}$$
$$+ (2n+1)a_{2n+1}x^{2n} \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$
$$y_2 = 3.2a_3x + 5.4.a_5x^3 \dots + (2n-1)(2n-2)a_{2n-1}x^{2n-3}$$
$$+ (2n+1)2na_{2n+1}x^{2n-1} \dots$$

$\therefore$ in (B),

$$(1-x^2)\{3.2a_3x + 5.4a_5x^3 \dots + (2n-1)(2n-2)a_{2n-1}x^{2n-3}$$
$$+ (2n+1)2na_{2n+1}x^{2n-1} \dots\}$$
$$- x\{a_1 + 3a_3x^2 + 5a_4x^4 \dots + (2n-1)a_{2n-1}x^{2n-2} + (2n+1)a_{2n+1}x^{2n} \dots\}$$
$$+ m^2\{a_1x + a_3x^3 + a_5x^5 \dots + a_{2n-1}x^{2n-1} + a_{2n+1}x^{2n+1} \dots\}$$
$$= 0, \text{ identically in } x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

From (C) and (A), putting $x = 0$, we have $a_1 = (y_1)_0 = m$.

Equating coefficients of x, x^3, ... x^{2n-1} in (D), we have

$$[\text{coeff. of } x]\ \ 3.2a_3 - a_1 + m^2a_1 = 0, \ \therefore\ a_3 = \frac{1-m^2}{3!}a_1 = \frac{(1-m^2)m}{3!},$$

$$[\ \ ,,\ \ x^3]\ \ (5.4a_5 - 3.2a_3) - 3a_3 + m^2a_3 = 0,$$
$$\therefore\ a_5 = \frac{9-m^2}{5.4}a_3 = \frac{(3^2-m^2)(1^2-m^2)m}{5!};$$

$$[\ \ ,,\ x^{2n-1}]\ \ \{(2n+1)2na_{2n+1} - (2n-1)(2n-2)a_{2n-1}\} - (2n-1)a_{2n-1}$$
$$+ m^2a_{2n-1} = 0,$$
$$\therefore\ a_{2n+1} = \frac{1}{(2n+1)2n}\left\{(2n-1)(2n-2) + 2n - 1 - m^2\right\}a_{2n-1}$$
$$= \frac{(2n-1)^2 - m^2}{(2n+1)2n}a_{2n-1}.$$

Since this is true for any value of n, it will be true if we change n into $n-1$ throughout the equation.

$$\therefore\ a_{2n-1} = \frac{(2n-3)^2 - m^2}{(2n-1)(2n-2)}a_{2n-3}.$$

So $$a_{2n-3} = \frac{(2n-5)^2 - m^2}{(2n-3)(2n-4)}a_{2n-5}, \text{ and so on.}$$

$$\therefore\ a_{2n+1} = \frac{\{(2n-1)^2 - m^2\}\{(2n-3)^2 - m^2\} \dots (3^2-m^2)(1^2-m^2)m}{(2n+1)!}$$

which is the coefficient in the general term.

Hence $$\sin(m\sin^{-1}x) = mx + \frac{m(1^2-m^2)}{3!}x^3 + \frac{m(1^2-m^2)(3^2-m^2)}{5!}x^5 + \dots$$

Second Method :—

Differentiating (B) n times (n *being odd*) by Leibnitz's Theorem, we have

$$(1 - x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n$$
$$- xy_{n+1} - ny_n$$
$$+ m^2y_n = 0.$$

Now put $x = 0$, and write A_n for $(y_n)_0$ [see Art. 129].

$$\therefore A_{n+2} + (m^2 - n^2)A_n = 0.$$

$$\therefore A_{n+2} = (n^2 - m^2)A_n = (n^2 - m^2)\{(n-2)^2 - m^2\}A_{n-2}$$
$$= \ldots = (n^2 - m^2)\{(n-2)^2 - m^2\} \ldots (3^2 - m^2)(1^2 - m^2)A_1 \quad . \text{(E)},$$

n being odd.

From (A), A_1 or $(y_1)_0 = m$.

But, by Maclaurin's Theorem, omitting even powers, since y is an *odd* function,

$$y = A_1x + \frac{A_3x^3}{3!} + \frac{A_5x^5}{5!} \ldots$$

Hence, putting $n = 1, 3, 5$, etc., in (E), and substituting in the above equation, we get the same series as before.

Cor.—Put $x = \sin\theta$, and the expansion becomes

$$\sin m\theta = m\sin\theta + \frac{m(1^2 - m^2)}{3!}\sin^3\theta + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!}\sin^5\theta + \ldots$$

or, $$= m\sin\theta - \frac{m(m^2 - 1^2)}{3!}\sin^3\theta + \frac{m(m^2 - 1^2)(m^2 - 3^2)}{5!}\sin^5\theta - \ldots,$$

This series can be shown to be convergent for any value of m.

Ex. 4. $y = \log(1 + e^x)$ . . . . (1)

Denoting the given function by $f(x)$, we have

$$f(-x) = \log(1 + e^{-x}) = \log\frac{1 + e^x}{e^x} = y - x.$$

Hence, if $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots$; we have, changing x to $-x$,

$$y - x = a_0 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 \ldots$$

$\therefore$ adding, halving, and transposing the x term,

$$y = a_0 + \frac{x}{2} + a_2x^2 + a_4x^4 \ldots$$
$$\therefore y_1 = \tfrac{1}{2} + 2a_2x + 4a_4x^3 \ldots$$
$$y_2 = 2a_2 + 12a_4x^2 \ldots$$

Now $y_1 = \dfrac{e^x}{1+e^x} = \dfrac{e^x}{e^y}$, from (1).

$$\therefore e^y y_1 = e^x.$$

Differentiating again, $\qquad e^y y_2 + e^y y_1^2 = e^x.$

$$\therefore y_2 + y_1^2 = \frac{e^x}{e^y} = y_1 \qquad . \quad . \quad (2)$$

$\therefore 2a_2 + 12a_4x^2 \ldots + (\tfrac{1}{2} + 2a_2x + 4a_4x^3 \ldots)^2 = \tfrac{1}{2} + 2a_2x + 4a_4x^3 \ldots,$

$\therefore$ equating coefficients,

[coefft. x^0] $\quad 2a_2 + \tfrac{1}{4} = \tfrac{1}{2}$; or $a_2 = \tfrac{1}{8}$;

[,, x^1] $\quad 2a_2 = 2a_2$, an identity, due to our having found a_1 otherwise;

[,, x^2] $\quad 12a_4 + 4a_2^2 = 0$, $\therefore a_4 = -\tfrac{1}{3}a_2^2 = -\tfrac{1}{192}$; etc.

Also $a_0 = (y)_0 = \log_e 2$.

$$\therefore \log(1+e^x) = \log_e 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2 - \tfrac{1}{192}x^4 \ldots$$

EXAMPLES XXII.

1. Expand by means of differential equations:—

(1) $\sin x$. $\qquad$ (2) $\log(1+x)$. $\qquad$ (3) $(a+x)^n$. $\qquad$ (4) $\sinh x$.

2. Given that $y = \sin^{-1} x$:—

(1) Prove that $(1-x^2)y_2 = xy_1$.

(2) Hence show that $\sin^{-1} x = x + \dfrac{x^3}{6} + \dfrac{3x^5}{40} \ldots$

(3) Show that $a_{n+2} = \dfrac{n^2}{(n+2)(n+1)} a_n$.

(4) Hence show that

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \ldots + \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n}\cdot\frac{x^{2n+1}}{2n+1} \ldots$$

(5) Show that $A_{n+2} = n^2 A_n$.

(6) Hence obtain the above expansion of $\sin^{-1} x$.

3. Given that $y = \sqrt{a^2 - x^2}\sin^{-1}\dfrac{x}{a}$:—

(1) Prove that $(a^2 - x^2)y_2 + (2 - y_1)x + y = 0$.

(2) Hence show that $y = x - \tfrac{1}{3}x^3/a^2 - \tfrac{2}{15}x^5/a^4 \ldots$

(3) Show that $a_{n+2} = \dfrac{n-1}{n+2}\,\dfrac{a_n}{a^2}$, except when $n = 1$.

(4) Hence show that

$$y = x - \frac{1}{3}\frac{x^3}{a^2} - \frac{2}{1.3.5}\cdot\frac{x^5}{a^4} - \frac{2.4}{1.3.5.7}\frac{x^7}{a^6}\dots - \frac{2.4.6\dots(2n-2)}{1.3.5\dots(2n+1)}\frac{x^{2n+1}}{a^{2n}}\dots$$

(5) Show that $A_{n+2} = \frac{n^2-1}{a^2}A_n$, except when $n = 1$; and that $A_0 = A_2 = A_4 = \dots = 0$.

(6) Hence obtain the above expansion of y.

4. Given that $y = e^{a\sin^{-1}x}$:—

(1) Prove that $(1 - x^2)y_2 = xy_1 + a^2y$.

(2) Hence show that $a_{n+2} = \frac{n^2+a^2}{(n+2)(n+1)}a_n$.

(3) Hence show that

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(1^2+a^2)x^3}{3!} + \frac{a^2(2^2+a^2)x^4}{4!} + \frac{a(1^2+a^2)(3^2+a^2)x^5}{5!}\dots$$

(4) Show that $A_{n+2} = (n^2 + a^2)A_n$.

(5) Hence expand y.

(6) Obtain a second expansion of $e^{a\sin^{-1}x}$ in ascending powers of $(a\sin^{-1}x)$; and show, by equating coefficients of a in the two series, that

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \dots, \text{ as in Ex. 2 (4).}$$

(7) Show, by equating coefficients of a^2, that

$$\tfrac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{1.3}\cdot\frac{x^4}{4} + \frac{2.4}{1.3.5}\cdot\frac{x^6}{6} + \frac{2.4.6}{1.3.5.7}\cdot\frac{x^8}{8} + \dots$$

(8) Putting $a = mi$, m being real, deduce that

$$\cos(m\sin^{-1}x) = 1 - \frac{m^2x^2}{2!} + \frac{m^2(m^2-2^2)x^4}{4!} - \frac{m^2(m^2-2^2)(m^2-4^2)x^6}{6!}\dots$$

$$\sin(m\sin^{-1}x) = mx - \frac{m(m^2-1^2)x^3}{3!} + \frac{m(m^2-1^2)(m^2-3^2)x^5}{5!} - \dots$$

(9) Show that

$$e^\theta = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \frac{5}{4!}\sin^4\theta + \frac{20}{5!}\sin^5\theta\dots$$

5. Expand $\cos(m\sin^{-1}x)$ by means of a differential equation. [See Ex. 4 (8).]

6. Show that $\sinh^{-1}x$, or $\log(x+\sqrt{1+x^2})$

$$= x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} - \dots$$

7. Show that $\sin(m\sinh^{-1}x)$

$$= mx - \frac{m(1^2+m^2)}{3!}x^3 + \frac{m(1^2+m^2)(3^2+m^2)}{5!}x^5 -$$

8. Show that $\sinh(m\sin^{-1}x)$

$$= mx + \frac{m(1^2+m^2)x^3}{3!} + \frac{m(1^2+m^2)(3^2+m^2)}{5!}x^5 + \dots$$

9. Show that $\sinh(m\sinh^{-1}x)$

$$= mx - \frac{m(1^2-m^2)}{3!}x^3 + \frac{m(1^2-m^2)(3^2-m^2)}{5!}x^5 - \dots$$

10. Show that $\sqrt{1+x^2}\sinh^{-1}x$

$$= x + \frac{1}{3}x^3 - \frac{2}{1.3}\cdot\frac{x^5}{5} + \frac{2.4}{1.3.5}\frac{x^7}{7} - \dots$$

11. If $y = \dfrac{x}{2}\dfrac{e^x+1}{e^x-1}$, show that $xy_1 + y(y-1) = \dfrac{x^2}{4}$.
Hence show that

$$y = 1 + B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} + B_3\frac{x^6}{6!} - B_4\frac{x^8}{8!} + \dots$$

where $B_1, B_2, B_3, B_4, \dots$ (called Bernouilli's numbers) are respectively $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \dots$

12. Putting xi for x in Ex. 11, deduce that

$$(1)\quad \frac{x}{2}\cot\frac{x}{2} = 1 - B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} - B_3\frac{x^6}{6!}\dots$$

$$(2)\quad \cot x = \frac{1}{x} - B_1\frac{2^2}{2!}x - B_2\frac{2^4}{4!}x^3 - B_3\frac{2^6}{6!}x^5\dots$$

13. Using the identity $\tan x = \cot x - 2\cot 2x$ (which prove), deduce that

$$\tan x = \frac{2^2(2^2-1)}{2!}B_1x + \frac{2^4(2^4-1)}{4!}B_2x^3 + \frac{2^6(2^6-1)}{6!}B_3x^5 + \dots$$

and show that it agrees with the expansion in Ex. 2 (5) of the preceding set.

14. Using the identity $\operatorname{cosec} x = \tan\frac{1}{2}x + \cot x$ (which prove), deduce that

$$\operatorname{cosec} x = \frac{1}{x} + \frac{2}{2!}B_1x + \frac{2(2^3-1)}{4!}B_2x^3 + \frac{2(2^5-1)}{6!}B_3x^5 + \dots$$

15. If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)y_2 - xy_1 = 2$.
Hence expand y into the series given above [Ex. 4 (7)].

16. Verify, by algebraical expansion, the series

$$\sin m\theta = m\sin\theta - \frac{m(m^2-1^2)}{3!}\sin^3\theta + \frac{m(m^2-1^2)(m^2-3^2)}{5!}\sin^5\theta - \dots$$

in the cases in which $m = 2$ and $\frac{1}{2}$ respectively, as far as three terms only.

***135. Strict Proof of Taylor's Theorem.**—We come now to the two conditions: (1) that $f(x+h)$ may be capable of expansion in positive integral ascending powers of h; (2) that the expansion may be convergent.

The method of treatment is as follows:—We shall first impose the condition that $f(x)$ and all its d.c.'s as far as the nth are finite and continuous *for values of* x *between two given values,* a *and* b, whatever may happen beyond those values. We shall then show that, if this is the case, $f(x+h)$ can be expanded into a series of n terms *plus* a single expression, called the *remainder after* n *terms;* and that if, on increasing n indefinitely, this remainder diminishes indefinitely, the infinite series will be convergent, and Taylor's Theorem will be true for that function—at least between the values, a and b, of x. The exceptions will be considered afterwards.

The investigation goes by the name of "Lagrange's Theorem on the Limits of Taylor's Series."

The proof involves the following—

***136. Lemma.**

If f(x) *be a function of* x, *such that both* f(x) *and* f′(x) *are finite and continuous between the values,* a *and* b, *of* x; *and if*

$$f(a) = f(b) = 0;$$

then f′(x) *will vanish for at least one value of* x *between* a *and* b.

Suppose $y = f(x)$ to be the curve in the figure. Then if $OM_1 = a$, $OM_2 = b$; we have $f(a) = f(b) = 0$. As x increases from a to b, y increases at first, but must of necessity diminish afterwards, since it is continuous and has to vanish again at M_2; hence at some point, A, it is a maximum, and $f'(x)$ vanishes at this point by Art. 113.

Again, suppose $OM_2 = a$, $OM_3 = b$; then, as before,

$$f(a) = f(b) = 0.$$

As x increases from OM_2 to OM_3, y diminishes at first and then increases; hence at some point, B, it is a minimum, and $f'(x) = 0$ at that point.

Again, between OM_3 and OM_4, $f'(x)$ vanishes three times; while

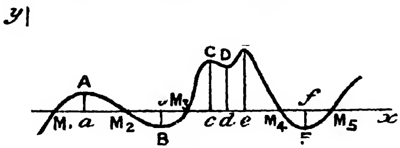

FIG. 22.

between OM_3 and OM_5, $f(x)$ itself vanishes, and $f'(x)$ vanishes four times.

All these cases are, of course, consistent with the truth of the lemma; and as they are typical of the various cases that may arise, we may consider the lemma established.

***137. Lagrange's Theorem on the Limits of Taylor's Series.**—We have seen that in numerous cases $f(x+h)$ can be expanded into the series $f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$.

Suppose we take n terms of the series, and examine the difference between their sum and $f(x+h)$. This difference is called, for obvious reasons, *the remainder after* n *terms*, and is, in fact,

$$f(x+h) - \left\{ f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) \right\}, \dagger \quad \text{(A)}$$

in which none of the terms are infinite for values of x between a and b, by the condition of Art. 135.

Let this remainder be denoted by $\frac{h^n}{n!} R$,† *i.e.* let

† *The remaining terms after the* nth *term are* $\frac{h^n}{n!} f^n(x) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(x) + \dots$ ad inf.

These are *not* equal to the *remainder after* n *terms*, i.e. (A), unless

$$f(x+h)-f(x)-hf'(x)-\frac{h^2}{2!}f''(x)-\ldots-\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x)=\frac{h^n}{n!}R\ . \quad \text{(B)}$$

Next, consider the expression, — obtained from (B) by transposing and putting z for h—

$$f(x+z)-f(x)-zf'(x)-\frac{z^2}{2!}f''(x)-\ldots-\frac{z^{n-1}}{(n-1)!}f^{(n-1)}(x)-\frac{z^n}{n!}R\ . \quad \text{(C)}$$

z being regarded as a variable; x and R being supposed constant.†

Now (C) vanishes identically if $z=0$; and also, from (B), if $z=h$.

Hence its d.c. with respect to z vanishes for some value of z between 0 and h (see Lemma), say $\theta_1 h$, where θ_1 is some proper fraction.

Differentiating (C) with respect to z, we get

$$f'(x+z)-f'(x)-zf''(x)-\ldots-\frac{z^{n-2}}{(n-2)!}f^{(n-1)}(x)-\frac{z^{n-1}}{(n-1)!}R\ . \quad \text{(D)}$$

But this vanishes when $z=0$, and when $z=\theta_1 h$.

Hence its d.c. with respect to z must vanish between $z=0$ and $z=\theta_1 h$, say $\theta_2 h$, where θ_2 is some smaller proper fraction.

Differentiating again in z, we get

$$f''(x+z)-f''(x)-\ldots-\frac{z^{n-3}}{(n-3)!}f^{(n-1)}(x)-\frac{z^{n-2}}{(n-2)!}R,$$

which vanishes when $z=0$, and when $z=\theta_2 h$.

This process can be repeated until we get

$$f^n(x+z)-R,$$

which vanishes between $z=0$ and $z=\theta_n h$, say θh, where θ is *some* proper fraction.

Taylor's Theorem is true. Hence, as the theorem is under proof, we cannot make the assumption.

† This is usually confusing. But we may notice (i) that R is the same function of x and h as before, *i.e.* we are not making any changes in it; (ii) that (C) is *not* an equation, but an expression which is not necessarily zero; *i.e.* (B) is not necessarily satisfied when z is put for h in a part only of the equation, the part R being unaltered; (iii) and that there is nothing to prevent us dealing with this or any other expression so long as we reason consistently with the hypothesis, which is that z is a variable quantity, while x and h are constants *for the time being*, and R depends on x and h, but not on z.

Hence $\qquad f^n(x+\theta h) - R = 0$, or $R = f^n(x+\theta h)$.

Substituting in (B), and transposing, we have finally

$$f(x+h) = f(x) + hf'(x) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}f^n(x+\theta h),$$

where θ is some proper fraction.

"The remainder after n terms" is $\dfrac{h^n}{n!}f^n(x+\theta h)$; and if on increasing n indefinitely this expression can, for given values of x and h, be made less than any assignable quantity, however small, then Taylor's Theorem will be true for that particular function, at least for the values of x and h given.

*138. Alternative Proof.

Again consider the expression (A), and let $x + h = a$, or $h = a - x$; it then becomes

$$f(a) - f(x) - (a-x)f'(x) - \frac{(a-x)^2}{2!}f''(x) - \frac{(a-x)^3}{3!}f'''(x)$$
$$- \frac{(a-x)^{n-1}}{(n-1)!}f^{(n-1)}(x). \qquad (1)$$

Putting this equal to $\dfrac{(a-x)^n}{n!}R$, consider the expression

$$f(a) - f(z) - (a-z)f'(z) - \frac{(a-z)^2}{2!}f''(z) - \frac{(a-z)^3}{3!}f'''(z) - \dots$$
$$- \frac{(a-z)^{n-1}}{(n-1)!}f^{(n-1)}(z) - \frac{(a-z)^n}{n!}R, \qquad (2)$$

in which the variable, z, is put for x, *except in* R, *which is supposed constant*, being a function of a, x, and n, only.

By reasoning similar to that above, since (2) vanishes when $z = x$, and also when $z = a$; its d.c. with respect to z vanishes for some value of z between x and a, say z_1.

Differentiating (2) with respect to z, we get

$$-f'(z) + \{f'(z) - (a-z)f''(z)\} + \left\{(a-z)f''(z) - \frac{(a-z)^2}{2!}f'''(z)\right\} + \dots$$
$$+ \left\{\frac{(a-z)^{n-2}}{(n-2)!}f^{(n-1)}(z) - \frac{(a-z)^{n-1}}{(n-1)!}f^n(z)\right\} + \frac{(a-z)^{n-1}}{(n-1)!}R,$$

which $\qquad = -\dfrac{(a-z)^{n-1}}{(n-1)!}f^n(z) + \dfrac{(a-z)^{n-1}}{(n-1)!}R;$

and this $= 0$, if $z = z_1$.

Dividing down by $\dfrac{(a-z)^{n-1}}{(n-1)!}$, which does not vanish when $z = z_1$, we get

$$R = f^n(z_1) = f^n(x + \theta h),$$

since $x + \theta h$ is *some* quantity between x and a; *i.e.* between x and $x + h$.

*139. Second Form of Remainder.

Put the expression (1) equal to $(a - x)R$, [i.e. hR]. Then, reasoning as above, we get

$$-\frac{(a-z)^{n-1}}{(n-1)!} f^n(z) + R = 0 \text{ when } z = x + \theta h;$$

$$\therefore R = \frac{\{a - (x+\theta h)\}^{n-1}}{(n-1)!} f^n(x + \theta h); \text{ or, since } a - x = h,$$

$$= \frac{(1-\theta)^{n-1} h^{n-1}}{(n-1)!} f^n(x + \theta h).$$

Since $1 - \theta$ is a proper fraction, $= \theta_1$ say, we may write

$$R = \frac{\theta_1^{n-1} h^{n-1}}{(n-1)!} f^n(x + \theta h), \text{ where } \theta + \theta_1 = 1.$$

$$\therefore f(x+h) = f(x) + hf'(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{\theta_1^{n-1} h^n}{(n-1)!} f^n(x + \theta h).$$

*140. Remainders in Maclaurin's Theorem.

Putting $x = 0$, and writing x for h, we get for the two remainders in Maclaurin's Theorem

$$(1)\ f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^n(\theta x).$$

$$(2)\ f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0)$$
$$+ \frac{\theta_1^{n-1} x^n}{(n-1)!} f^n(\theta x),$$

where $\theta + \theta_1 = 1$.

***141.** Provided that none of the quantities $f(x)$, $f'(x)$ ... in Taylor's Theorem, and $f(0)$, $f'(0)$ ... in Maclaurin's Theorem, become infinite, and provided that the remainder ultimately vanishes, we may continue the two series *ad infinitum*. For a fuller discussion of the convergency of the series, however, we must refer the student to other treatises.

***142. Geometrical Illustration.**—Consider the equation $f(x+h) = f(x) + hf'(x+\theta h)$.

Draw the curve $y = f(x)$; and let $OA = x$, $OB = x + h$.

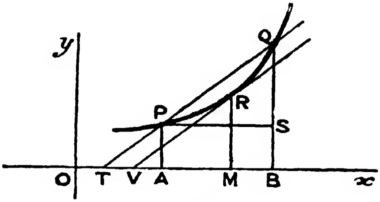

FIG. 23.

$$\text{Then } AP = f(x),\ BQ = f(x+h).$$

$$\therefore f(x+h) - f(x) = QB - PA = QS = h \tan QPS \quad . \quad . \quad (1)$$

Now at *some* point, R, between P and Q, we can draw a tangent parallel to the chord PQ; supposing this done, let $OM = x + \theta h$, θ being a proper fraction.

Then $\tan QPS = \tan RVM = f'(OM) = f'(x + \theta h)$.

$\therefore$ from (1) $\qquad f(x+h) - f(x) = hf'(x+\theta h)$.

Hence, conversely, this equation is equivalent to the statement, made above, concerning the point R.

*** 143. So-called Failure of Taylor's and Maclaurin's Theorems.**—The condition imposed in Art. 135 was, that $f(x)$ and all its d.c.'s up to the nth should not become infinite for any value of x between a and b.

Let c be a value of x between a and b.

Then

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + \ldots + \frac{h^n}{n!}f^n(c+\theta h), \quad . \quad (1)$$

and the series fails if $f(c)$, $f'(c)$, etc., are, any of them, infinite; since $f(c+h)$ is not necessarily infinite (though it might be for a particular value of h), whereas the right-hand side, if infinite, would be so for any value of h, since $f(c)$, $f'(c)$, . are independent of h.

***144. Ex. 1.** Let $f(x) \equiv \frac{1}{(x-c)^3}$, so that $f(c) = \infty$.

Then, by Taylor's Theorem,

$$\frac{1}{(x+h-c)^3} = \frac{1}{(x-c)^3} - \frac{3h}{(x-c)^4} + \frac{3.4.h^2}{1.2(x-c)^5} \cdots$$

If $x = c$, we get $1/h^3 = \infty$, so that the theorem fails, *but only for this value of* x. Since $1/h^3$ is a $-^{\text{ve}}$ power of h, and the right-hand side is a series of $+^{\text{ve}}$ powers, we might almost expect this absurdity.

If, however, $x = c + m$, we get

$$\frac{1}{(m+h)^3} = \frac{1}{m^3} - \frac{3h}{m^4} + \frac{3.4h^2}{1.2m^5} - \dots,$$

which may be easily verified by the Binomial Theorem.

Ex. 2. Let $f(x) \equiv x^{\frac{3}{2}}$.

Then $$(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}.h + \frac{3}{2}\cdot\frac{1}{2}x^{-\frac{1}{2}}\cdot\frac{h^2}{2!} + \dots$$

If $x = 0$, we get $h^{\frac{3}{2}} = 0 + 0 + \infty + \dots$, and the series fails at the third term which contains h^2, the next integral power to $h^{\frac{3}{2}}$.

The reason is similar to that given in Ex. 1.

If, however, we use Lagrange's form of remainder, and write

$$(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}h. + \frac{3.1}{2.2}\frac{h^2}{2!}(x+\theta h)^{-\frac{1}{2}},$$

we get, on putting $x = 0$,

$$h^{\frac{3}{2}} = \frac{3.1}{2.2}\frac{h^2}{2}(\theta h)^{-\frac{1}{2}} = \frac{3}{8\sqrt{\theta}}h^{\frac{3}{2}},$$

which is correct if $\frac{3}{8\sqrt{\theta}} = 1$; or $\theta = \frac{9}{64}$, *a proper fraction, as stipulated.*

***145.** The expansion $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$ fails *for any value of* x, if $f(0), f'(0) \dots$ are any of them infinite, unless we use Lagrange's form of remainder, which does not usually, however, give us a series at all, but merely an equation for finding θ.

As examples we have $1/x^3$ and $x^{\frac{3}{2}}$, discussed above.

Ex. 3. *Expand* cosec x *in ascending powers of* x *by Maclaurin's Theorem.*

Here $f(0) = \infty$, and the series fails at the first term.

In fact, by Ex. 3, Art. 121, $\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} \cdots$, which contains a *negative* power of x.

Since $x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \ldots$, containing $+^{ve}$ powers of x, the expansion of this by Maclaurin's Theorem will *not* fail.

Ex. 4. $y = \log x = \infty$, when $x = 0$; and the series fails.

But $x \equiv (1+x) \Big/ \left(1 + \frac{1}{x}\right)$.

$$\therefore \log x = \log(1+x) - \log\left(1 + \frac{1}{x}\right)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{3x^3} + \ldots$$

which is expanded in a double series of $+^{ve}$ and $-^{ve}$ powers of x.

The series, unfortunately, is divergent (and therefore useless) for all values of x except $x = 1$. We *can* obtain a series by using Lagrange's remainder for each of the expansions; but it is of little practical value, since one of the remainders will always be found to be the most important part of the series.

***146.** It may be observed that if $f(x)$ is infinite when $x = a$, but is finite for all other values of x in the neighbourhood of a, then

$$f'(a) = f''(a) = \ldots = \infty.$$

We shall show this by the aid of a figure as follows :—

If $OA = a$, and AB be a small distance; and if AC be infinite, while BD is finite; then $f(x)$ or PM must increase infinitely *somewhere* between BD and AC, and since the ordinate is only supposed to be absolutely infinite at A, it follows that the infinite increment must occur just at the point A. In fact, the tangent is evidently getting steeper and steeper as P approaches C, till it is ultimately perpendicular to Ox, when dy/dx or $\tan\theta$ becomes infinite.

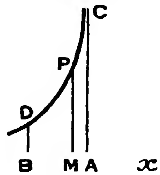

FIG. 24.

And if $f'(a) = \infty$, it follows from the same reasoning that $f''(a) = \infty$; and so on.

Ex. 1. $y = \log x$.

Here y_0, y_1, y_2, etc., are all infinite when $x = 0$.

Ex. 2. $y = \dfrac{1}{(x-a)^3}$.

Ex. 3. $y = \cot x$.

147. Application to Interpolation in Mathematical Tables.—In the case of functions, such as squares, logarithms, etc., which are frequently used in calculation, it is an advantage to draw up, for reference, tables giving the values of the functions for the different values of the variable, which latter proceed by regular and *small* intervals.

Interpolation is the process of finding values of the function for intermediate values of the variable.

Let $f(x)$ be one of these functions, and let h be the small interval between the different values of x in the table.

Then, *generally*, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots$

or $$f(x+h) - f(x) = hf'(x) + \frac{h^2}{2!}f''(x) + \ldots$$

Let a and $a+h$ be two consecutive values of x in the tables,

then $$f(a+h) - f(a) = hf'(a) + \frac{h^2}{2!}f''(a) + \ldots \quad . \quad . \quad (1)$$

Again, let $a + \theta h$ be an intermediate value of x; θ being therefore a proper fraction. Then

$$f(a+\theta h) - f(a) = \theta hf'(a) + \frac{\theta^2 h^2}{2!}f''(a) + \ldots \quad . \quad (2)$$

Now, if $OK = a$, $OL = a + h$, $OM = a + \theta h$, so that $PV = h$, $PN = \theta h$,

then $PK = f(a)$, $QL = f(a+h)$, $RM = f(a+\theta h)$

and $QV = f(a+h) - f(a)$; $RN = f(a+\theta h) - f(a)$.

Hence

$$\frac{RN}{QV} = \frac{f(a+\theta h) - f(a)}{f(a+h) - f(a)} = \frac{\theta hf'(a) + \frac{\theta^2 h^2}{2!}f''(a) + \ldots}{hf'(a) + \frac{h^2}{2!}f''(a) + \ldots} \text{ from (1) and (2)}$$

$= \theta$, *provided that the terms above and below, after the first, are negligible.*

$$\text{i.e. } \frac{RN}{QV} = \frac{PN}{PV} \text{ approximately.} \quad . \quad . \quad . \quad (3)$$

Hence [Euc. VI. 2] whatever the function, provided it is continuous, we may regard (in general) PQ as ultimately a portion of a straight line. This last statement is practically the *principle of proportional parts*.

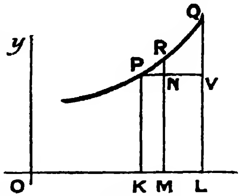

FIG. 25.

If we call PN and RN *partial* differences, as compared with PV and QV which we call *whole* differences, the principle may be thus stated [see (3)] :—

$$\frac{\text{Partial difference of function}}{\text{whole difference of function}} = \frac{\text{partial difference of variable}}{\text{whole difference of variable}}$$

Ex. *Given* $\left.\begin{aligned}\log 3\cdot6480 &= 0\cdot5620548\\ \log 3\cdot6481 &= 0\cdot5620667\end{aligned}\right\}$, *find* $\log 3\cdot64803$.

Here $OK = 3\cdot6480$, $\quad OL = 3\cdot6481$, $\quad OM = 3\cdot64803$,
$PK = 0\cdot5620548$, $QL = 0\cdot5620667$.

$$\therefore PN = 0\cdot00003, \quad PV = 0\cdot00010,$$
$$RN = ? \qquad QV = 0\cdot0000119.$$
$$\therefore \text{in (3) } RN = \frac{PN}{PV}QV = \frac{3}{10}QV = 0\cdot0000036.$$
$$\therefore RM = PK + RN = 0\cdot5620584 + 0\cdot0000036$$
$$= 0\cdot0000620.$$

148. Exceptions — Irregularity and Insensibility.— The proviso made above was, that the succeeding terms in Taylor's series should be negligible compared with the first.

If, as x approaches a certain value, this is not so, then the principle of proportional parts does not hold ; and the differences in the function for consecutive values of x are said to be *irregular* as x approaches this value.

For instance, if $f''(a)$ is large compared with $f'(a)$, so that the fraction $\dfrac{f''(a)}{f'(a)}$ is large, we have a condition for irregularity.

Ex. If $f(x) \equiv \sin x$, then $\dfrac{f''(x)}{f'(x)} = -\tan x$, which becomes large as x approaches 90°. Hence, in a table of natural sines there is irregularity when the angle is near 90°, and we cannot interpolate by the principle of

proportional parts. Of course, in this case, the value of the sine could be easily calculated directly.

149. Again, when $\dfrac{f(x+h)-f(x)}{h}$ is small, h being small, the differences in the function are small compared with the differences in the variable, and are said to be *insensible.* This, of course, occurs when $f'(x)$ is small.

Ex. $$\sin(x+h)-\sin x = h\cos x - \frac{h^2}{2}\sin x + \ldots;$$

and when x is near 90°, $\cos x$ is small: hence there is *insensibility*, as well as *irregularity*, in the differences when the angle is nearly a right angle.

EXAMPLES XXIII.

1. Find the two forms of remainder after n terms in the following:—

(1) $(x+a)^m$, $m>n$; (2) $\log(x+a)$; (3) e^x; (4) $\sin x$;
the expansions being in $+^{ve}$ integral powers of x.

2. Verify the lemma of Art. 136 in the following cases:—

(1) $y = x(x-1)$. (2) $y = x\sqrt{a^2-x^2}$.
(3) $y = (x-a)(x-2a)(x-3a)$. (4) $y = \sin x \cos x$.
(5) $y = x\cos x$.

3. Show that

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2(1-\theta x)^{3/2}},$$

where θ is a proper fraction.
If $x=\frac{1}{2}$, show that $\theta = 0{\cdot}57\ldots$

4. Show that $(x+h)^{5/3} = x^{5/3} + \frac{5}{3}hx^{2/3} + \frac{5.2}{3.6}\frac{h^2}{x^{1/3}} + \ldots$

When does the expansion fail? Using Lagrange's first form of remainder in the latter case, prove that $\theta = (\frac{8}{9})^3$.

5. Show that $\cot x$ and $\operatorname{cosec} x$ cannot be expanded in ascending *positive* integral powers of x. Given that $x\cot x$ and $x\operatorname{cosec} x$ can be so expanded [see Ex. XXII., 12, 14], what do you infer?

6. Show that Maclaurin's Theorem fails to expand $\sin x\sqrt{x}$.
Use Lagrange's form of remainder to show that

$$\sin x\sqrt{x} = \frac{3v}{8\theta^2}(\cos v - 3v\sin v) \text{ where } v \equiv (\theta x)^{3/2}.$$

7. If p and q are prime to each other, show that $f(x^{p/q})$ cannot in general be expanded in $+^{ve}$ integral powers of x.

ANSWERS.

1. (1) $\dfrac{m!}{n!\,(m-n)!}(\theta x + a)^{m-n}\,x^n$; $\dfrac{m!\,\theta_1{}^{n-1}}{(n-1)!\,(m-n)!}(\theta x + a)^{m-n}\,x^n$.

(2) $(-1)^{n-1}\dfrac{x^n}{n(a+\theta x)^n}$; $(-1)^{n-1}\dfrac{\theta_1{}^{n-1}x^n}{(a+\theta x)^n}$. (3) $\dfrac{x^n}{n!}e^{\theta x}$; $\dfrac{x^n\theta_1{}^{n-1}}{(n-1)!}e^{\theta x}$.

(4) $\dfrac{x^n}{n!}\sin\left(\theta x + \dfrac{n\pi}{2}\right)$; $\dfrac{\theta_1{}^{n-1}x^n}{(n-1)!}\sin\left(\theta x + \dfrac{n\pi}{2}\right)$.

5. That the d.c.'s of $x\cot x$ and $x\,\mathrm{cosec}\,x$ do not become infinite when $x = 0$.

CHAPTER XII.

INDETERMINATE FORMS.

150. When a function of x approaches any one of the forms

$$\frac{0}{0},\ 0\times\infty,\ \frac{\infty}{\infty},\ \infty-\infty,\ 1^{\infty},\ \infty^0,\ 0^0,$$

as x approaches a given value, the function is said to be *indeterminate in form*, or *undetermined*, or *illusory*, for that value of x.

But although indeterminate *in form*, the function has usually a determinate limit to which it approaches as x approaches the given value. (See Definitions, Art. 14).

The first three forms are practically one and the same, for

$$0\times\infty=0\times\frac{1}{0}=\frac{0}{0};\ \text{and}\ \frac{\infty}{\infty}=\frac{1}{0}\div\frac{1}{0}=\frac{0}{0}.$$

The fourth, $\infty-\infty$, can be reduced to the first by expressing it as a fraction with a common denominator. Thus, when $x=0$, $\operatorname{cosec} x-\cot x$ is of the form $\infty-\infty$; but it may be written $\dfrac{1-\cos x}{\sin x}$, which is of the form $\frac{0}{0}$.

The last three forms can be reduced to the first by taking logs; thus

$$\begin{aligned}\log 1^{\infty} &= \infty.\log 1=\infty\times 0=\tfrac{0}{0},\\ \log \infty^{0} &= 0.\log\infty=0\times\infty=\tfrac{0}{0},\\ \log 0^{0} &= 0.\log 0=0\times\infty=\tfrac{0}{0}.\end{aligned}$$

Hence all seven forms are practically reducible to the first form $\frac{0}{0}$.

151. Algebraical Method.—We have previously shown (Art. 16) that if $f(x)/\phi(x)$ be the quotient of two rational integral algebraical functions of x, then $\underset{x=a}{lt}\dfrac{f(x)}{\phi(x)}$ can be found by removing once, or more often, the common factor $x-a$.

If $f(x)$ and $\phi(x)$ be irrational we may rationalize by using the complementary surd, or a proper rationalizing factor; but the method of expansion is the more general one, and it can be applied to transcendental as well as algebraical functions. In this method we put $x = a + h$, expand in ascending powers of h, and divide down by h or a power of h, until, by putting $h = 0$, we obtain a determinate value, which is the required limit.

If $a = 0$, then we have to find $\underset{x=0}{lt}\dfrac{f(x)}{\phi(x)}$, and in this case we expand both $f(x)$ and $\phi(x)$ in ascending powers of x, and find the limit by dividing above and below by x, or as high a power of x as is necessary, and then putting $x = 0$.

Other algebraical methods will be given in the general examples below.

Ex. 1. *Find* $\underset{x=0}{lt}\dfrac{\log\cos x}{x\sin x} \equiv lt\, y$ *say.*

We have $y = \dfrac{\log\left(1 - \dfrac{x^2}{2}\cdots\right)}{x\left(x - \dfrac{x^3}{6}\cdots\right)} = \dfrac{-\dfrac{x^2}{2} + \text{higher powers}}{x^2 + \text{higher powers}} = \dfrac{-\frac{1}{2} + Kx}{1 + Lx}$, say.

Hence $\underset{x=0}{lt}\, y = -\frac{1}{2}$. [See Art. 92 (4); and Prop., Art. 91.]

Ex. 2. If $y = \dfrac{\sqrt{x^2 + a^2} - a\sqrt{2}}{\sqrt{x + 3a} - 2\sqrt{a}}$, *find* $\underset{x=a}{lt}\, y$.

The method of rationalization has been given above (Art. 16, Ex. 2). We give the other method.

Put $x = a + h$, so that $h = 0$ when $x = a$.

$$\therefore y = \frac{\sqrt{2a^2 + 2ah + h^2} - a\sqrt{2}}{\sqrt{4a + h} - 2\sqrt{a}} = \frac{a\sqrt{2}\left\{\left(1 + \dfrac{h}{a} + \dfrac{h^2}{2a^2}\right)^{\frac{1}{2}} - 1\right\}}{2\sqrt{a}\left\{\left(1 + \dfrac{h}{4a}\right)^{\frac{1}{2}} - 1\right\}}$$

$$= \sqrt{\frac{a}{2}}\,\frac{\left(1 + \dfrac{h}{2a}\cdots\right) - 1}{\left(1 + \dfrac{h}{8a}\cdots\right) - 1} = \sqrt{\frac{a}{2}}\cdot\frac{\frac{1}{2} + Kh}{\frac{1}{8} + Lh}.$$

$$\therefore lt\, y = \sqrt{\frac{a}{2}}\cdot\frac{\frac{1}{2}}{\frac{1}{8}} = 2\sqrt{2a}.$$

152. Compound Indeterminate Forms.—An expression which consists of the sum, difference, product, or any other function of one or more indeterminate forms is called a *compound indeterminate form*. To find the limit of such a form, we need only find the limit of each of the single forms which go to make up the compound form.

To prove this formally, let X and Y be two indeterminate forms, and let $lt X = a$, $lt Y = b$. Then we may put $X = a + \alpha$, $Y = b + \beta$, where α and β ultimately vanish.

We have

(1) $lt(X \pm Y) = lt(a + \alpha \pm b \pm \beta) = a \pm b = lt\, X \pm lt\, Y.$

(2) $lt\, XY = lt\{(a + \alpha)(b + \beta)\} = lt(ab + a\beta + b\alpha + \alpha\beta) = ab$
$= lt\, X . lt\, Y.$

(3) $$lt\frac{X}{Y} = lt\frac{a + \alpha}{b + \beta} = lt\frac{a}{b}\left(1 + \frac{\alpha}{a}\right)\left(1 + \frac{\beta}{b}\right)^{-1} = lt\frac{a}{b}\left(1 + \frac{\alpha}{a} - \frac{\beta}{b}\dots\right)$$
$$= \frac{a}{b} = \frac{lt\, X}{lt\, Y}.$$

(4) $$lt\, X^Y = lt(a + \alpha)^{b+\beta} = lt\, a^{b+\beta}\left(1 + \frac{\alpha}{a}\right)^{b+\beta}$$
$$= lt\, a^{b+\beta}\left(1 + \frac{b + \beta}{a}\alpha + \dots\right) = a^b \text{ by (2)} = (lt\, X)^{lt\, Y}.$$

And, similarly, we can show that for any algebraical function of X and Y, if $lt X$, $lt Y$, ... are finite, then $lt. f[X, Y \dots] = f[lt X, lt Y \dots]$.

The truth of the general statement follows readily from the proposition of Art. 91.

Ex. 3. *Find* $\displaystyle\underset{x=0}{lt}\frac{x^2 \log \tan x}{\sin^2 x \log x}$.

Here $y = \left(\dfrac{x}{\sin x}\right)^2 \cdot \dfrac{\log \tan x}{\log x}$.

The limit of the first factor is 1. To find that of the second, we have

$$lt\frac{\log \tan x}{\log x} = lt\frac{\log(\tan x/x) + \log x}{\log x}, \text{ noting that } \frac{\tan x}{x} = 1 \text{ ultimately},$$
$$= lt\frac{\log(\tan x/x)}{\log x} + 1 = 1.$$

$\therefore\ lt\, y = 1 \times 1 = 1.$

153. General Examples.

Ex. 4. $$\underset{x=1}{lt}\frac{1+\cos\pi x}{\tan^2\pi x}=lt\frac{2\cos^2\frac{\pi x}{2}\cos^2\pi x}{\sin^2\pi x}=lt\frac{2\cos^2\frac{\pi x}{2}\cos^2\pi x}{4\sin^2\frac{\pi x}{2}\cos^2\frac{\pi x}{2}}$$
$$=lt\frac{\cos^2\pi x}{2\sin^2\frac{\pi x}{2}}=\frac{1}{2}.$$

Ex. 5. *Let* $y=(1+\sin^2 x)^{\operatorname{cosec}^2 x}$; *to find* $(y)_0$. Form 1^∞.

We have $\log y=\operatorname{cosec}^2 x\log(1+\sin^2 x)=\dfrac{\sin^2 x+K\sin^4 x}{\sin^2 x}$
$$=1+K\sin^2 x=1 \text{ ultimately.}$$
$\therefore\ lt\, y=e^1=e.$

Or, putting $\operatorname{cosec}^2 x=n$,
$$(y)_0=\underset{n=\infty}{lt}\left(1+\frac{1}{n}\right)^n=e.$$

Ex. 6. *Find* $\underset{x=\infty}{lt}\dfrac{e^x}{x}$, $\underset{x=\infty}{lt}\dfrac{e^x}{x^n}$; *and* $\underset{x=\infty}{lt}\dfrac{a^x}{x}$, $\underset{x=\infty}{lt}\dfrac{a^x}{x^n}$, a *being greater than* 1 *by a finite quantity.*

(1) We have $\dfrac{e^{x+1}}{x+1}=\dfrac{e^x}{x}\cdot\dfrac{ex}{x+1}$; and when x increases indefinitely, $\dfrac{ex}{x+1}$ approaches indefinitely near to e, or 2·718 ....

Hence $\dfrac{e^{x+1}}{x+1}>\dfrac{e^x}{x}$ when x is large; that is to say, as we continually increase x by 1, the value of e^x/x is continually multiplied by a factor greater than 1 (and approaching 2·718 ...), and therefore $\underset{x=\infty}{lt}\dfrac{e^x}{x}=\infty$.

(2) Similarly, $\dfrac{e^{x+1}}{(x+1)^n}=\dfrac{e^x}{x^n}\cdot\dfrac{ex^n}{(x+1)^n}$; and, since $\underset{x=\infty}{lt}\dfrac{ex^n}{(x+1)^n}=e$, it follows, as before, that $\underset{x=\infty}{lt}\dfrac{e^x}{x^n}=\infty$.

(3) Again, $\dfrac{a^x}{x}=\dfrac{e^{x\log a}}{x}=\dfrac{e^{x\log a}}{x\log a}\log a=\dfrac{e^y}{y}\log a$, if $y=x\log a$.

If a is greater than 1 by a finite quantity, $\log a$ is finite.

Hence $$\underset{x=\infty}{lt}\frac{a^x}{x}=\underset{y=\infty}{lt}\frac{e^y}{y}\log a=\infty \text{ by (1).}$$

(4) Similarly, $\dfrac{a^x}{x^n}=\dfrac{e^{x\log a}}{x^n}=\dfrac{e^{x\log a}}{(x\log a)^n}\cdot(\log a)^n=\dfrac{e^y}{y^n}(\log a)^n$, if $y=x\log a$.
$$\therefore\ \underset{x=\infty}{lt}\frac{a^x}{x^n}=\underset{y=\infty}{lt}\frac{e^y}{y^n}(\log a)^n=\infty \text{ by (2).}$$

Ex. 7. *Find* $\underset{x=\infty}{lt}\dfrac{\log x}{x}$.

Let $x = e^y$; then if $x = \infty$, $y = \infty$.

$$\therefore \underset{x=\infty}{lt}\frac{\log x}{x} = \underset{y=\infty}{lt}\frac{y}{e^y} = 0, \text{ by Ex. 6, since } \underset{y=\infty}{lt}\frac{e^y}{y} = \infty.$$

Ex. 8. *Find* $\underset{x=0}{lt}\ x\log x$.

Let $x = \dfrac{1}{y}$; then, if $x = 0$, $y = \infty$.

$$\therefore \underset{x=0}{lt}\ x\log x = \underset{y=\infty}{lt}\frac{1}{y}\log\frac{1}{y} = -\underset{y=\infty}{lt}\frac{\log y}{y} = 0, \text{ by Ex. 7.}$$

Ex. 9. *Find* $\underset{x=0}{lt}\ x^x \equiv lt\ y$ *say.*

Since $y = x^x$, $\therefore \log y = x\log x = 0$ in the limit, by Ex. 8.

$$\therefore lt\, y = e^0 = 1.$$

NOTE.—These last four results are important, and the student should become familiar with them.

Ex. 10. *Find* $\underset{x=0}{lt}\ x^{\sin x}$.

We have $\log y = \sin x\log x = \left(x - \dfrac{x^3}{6}\cdots\right)\log x = x\log x\left(1 - \dfrac{x^2}{6}\cdots\right)$

$= 0$ in the limit, by Ex. 8.

$$\therefore lt\ y = e^0 = 1.$$

Ex. 11. *Find* $\underset{x=0}{lt}\dfrac{1 - x^{\sin x}}{x\log x}$.

Since $x^{\sin x}$ and $x\log x$ have 1 and 0 respectively for their limits, we get, by substitution, $\frac{1-1}{0} = \frac{0}{0}$ which is indeterminate in form. We cannot therefore adopt this method, but must expand.

$$\text{We have}\qquad y = \frac{1-\left\{1 + \sin x\log x + \dfrac{(\sin x\log x)^2}{2!}\cdots\right\}}{x\log x}$$

by the exponential theorem.

Now, $\sin x\log x = \left(x - \dfrac{x^3}{6}\cdots\right)\log x = x\log x.(1 + Kx)$, say.

$$\therefore y = \frac{-x\log x.(1 + Kx) + L(x\log x)^2}{x\log x} = -(1 + Kx) + L(x\log x).$$

$$\therefore lt\, y = -1, \text{ since } lt\, x\log x = 0, \text{ by Ex. 8.}$$

Ex. 12. $\underset{x=0}{lt}\left(1 + \dfrac{1}{x}\right)^x$. Form ∞^0.

We have $\log y = x \log\left(1 + \frac{1}{x}\right)$. We cannot expand $\log\left(1+\frac{1}{x}\right)$ in a convergent series unless $\frac{1}{x} < 1$; hence we write

$$\log y = x\{\log(1+x) - \log x\} = x\log(1+x) - x\log x = 0. \quad \text{[See Ex. 8.]}$$

$$\therefore \; lt\, y = e^0 = 1.$$

NOTE that $\underset{x=\infty}{lt}\; y = e$, while $\underset{x=0}{lt}\; y = 1$

Ex. 13. $\underset{\theta=0}{lt}\; \dfrac{\cot\theta \tan^{-1}(m\tan\theta) - m\cos^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta}$.

Here $y = \dfrac{\cot\theta\{m\tan\theta - \frac{1}{3}m^3\tan^3\theta + \ldots\} - m\cos^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta}$

$$= \frac{m - \frac{1}{3}m^3\tan^2\theta + \ldots - m\cos^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta} = \frac{m\sin^2\frac{1}{2}\theta - \frac{1}{3}m^3\tan^2\theta + K\tan^4\theta}{\sin^2\frac{1}{2}\theta}$$

$$= \frac{m\sin^2\frac{1}{2}\theta - \frac{4}{3}m^3\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta/\cos^2\theta + K_1\sin^4\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta},$$

for $\tan^4\theta$ contains $\sin^4\theta$, and therefore $\sin^4\frac{1}{2}\theta$, as a factor.

$$\therefore \; lt\, y = m - \tfrac{4}{3}m^3.$$

EXAMPLES XXIV.

Evaluate the following indeterminate forms:—

1. $\dfrac{\log(1-x^2)}{\log\cos x}$, $(x=0)$.

2. $\dfrac{x\log(1+x)}{1-\cos x}$, $(x=0)$.

3. $\dfrac{x-\sin x}{x(1-\cos x)}$, $(x=0)$.

4. $\dfrac{x^2\sin x}{2x-\sin 2x}$, $(x=0)$.

5. $\log\left(2-\dfrac{x}{a}\right)\cot(x-a)$, $(x=a)$.

6. $\cot^2 x \log\cos x$, $(x=0)$.

7. $\dfrac{\sin mx}{\sin nx}$, $(x=0)$.

8. $\dfrac{e^x - 1 - x}{\log(1+x) - x}$, $(x=0)$.

9. $\dfrac{2}{1-\cos x} - \dfrac{x\sin x}{(1-\cos x)^2}$, $(x=0)$.

10. $\operatorname{cosec} x - \dfrac{1}{x^2}\log(1+x)$, $(x=0)$.

11. $\dfrac{\sqrt{2x^2+7a^2} - 3a}{a - \sqrt{2ax - x^2}}$, $(x=a)$.

12. $\sqrt{\dfrac{a+x}{a-x}}\tan^{-1}\sqrt{a^2-x^2}$, $(x=a)$.

13. $\dfrac{\log \cos x^2}{\tan^4 x}$, $(x = 0)$. **14.** $\dfrac{\log (1 - \cos x)}{\log x}$, $(x = 0)$.

15. $\dfrac{x^a - a^x}{x - a}$, $(x = a)$. **16.** $\dfrac{\sin x - \sin a}{e^x - e^a}$, $(x = a)$.

17. $\dfrac{\log x - \log a}{\tan x - \tan a}$, $(x = a)$.

18. $\dfrac{\sqrt{a+x} + \sqrt{2x} - 2\sqrt{2a}}{\sqrt{a^2 + x^2} + \sqrt{2}\,x - 2\sqrt{2a}}$, $(x = a)$.

19. $\dfrac{1 - \tan x}{1 - \sqrt{2}.\sin x}$, $\left(x = \dfrac{\pi}{4}\right)$. **20.** $\dfrac{(x \sin^2\theta + a \cos^2\theta)^n - a^n}{x^n - a^n}$, $(x = a)$.

21. $(\cos x)^{\cot x}$, $(x = 0)$. **22.** $(1 - x^2)^{\frac{\sqrt{1-x^2}}{x}}$, $(x = 0)$.

23. $\left\{\tan\left(\dfrac{\pi}{4} + x\right)\right\}^{\cot x}$, $(x = 0)$. **24.** $(1 + \tan^2 x)^{\cot x}$, $(x = 0)$.

25. $(1 + \sin x)^{\frac{1}{x}}$, $(x = 0)$. **26.** $x(\log x)^n$, $(x = 0)$.

27. $\sin x \log \sin x$, $(x = 0)$. **28.** $x \log \tan x$, $(x = 0)$.

29. $x^{\tan x}$, $(x = 0)$. **30.** $x^{\frac{1}{x}}$, $(x = \infty)$.

31. $(\cot x)^{\tan x}$, $(x = 0)$. **32.** $(\sin x)^{\sin x}$, $(x = 0)$.

33. $(\cot x)^{\sin x}$, $(x = 0)$. **34.** $\dfrac{1 - x^x}{x \log x}$, $(x = 0)$.

ANSWERS.

1. 2. **2.** 2. **3.** $\frac{1}{3}$. **4.** $\frac{3}{4}$. **5.** $-\frac{1}{a}$. **6.** $-\frac{1}{2}$. **7.** $\frac{m}{n}$.

8. -1. **9.** $\frac{1}{3}$. **10.** $\frac{1}{2}$. **11.** ∞. **12.** $2a$. **13.** $-\frac{1}{2}$. **14.** 2.

15. $a^a(1 - \log a)$. **16.** $\dfrac{\cos a}{e^a}$. **17.** $\dfrac{\cos^2 a}{a}$. **18.** $\dfrac{1}{2\sqrt{a}}$. **19.** 2.

20. $\sin^2\theta$. **21.** 1. **22.** 1. **23.** e^2. **24.** e. **25.** e. **26.** 0. **27.** 0.

28. 0. **29.** 1. **30.** 1. **31.** 1. **32.** 1. **33.** 1. **34.** -1.

154. Method of Differential Calculus.—Since we have shown that all the indeterminate forms are reducible to the form $\frac{0}{0}$, we need only consider that form. We shall, however, give an alternative method for the form $\frac{\infty}{\infty}$.

155. Form $\frac{0}{0}$.

To find $\underset{x=a}{lt}\dfrac{f(x)}{\phi(x)}$, where $f(a) = \phi(a) = 0$.

Put $x = a + h$; then, if $y \equiv \dfrac{f(x)}{\phi(x)}$,

$$lt\, y = \underset{h=0}{lt}\frac{f(a+h)}{\phi(a+h)}, \text{ which may be written—}$$

$$= \underset{h=0}{lt}\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} \text{ since } f(a) = \phi(a) = 0$$

$$= \underset{h=0}{lt}\left\{\frac{f(a+h)-f(a)}{h} \div \frac{\phi(a+h)-\phi(a)}{h}\right\}$$

$$= \frac{f'(a)}{\phi'(a)},$$

where $f'(a)$ is the value of $f'(x)$ when $x = a$; and similarly for $\phi'(a)$.

Hence the rule :—*Differentiate* both *numerator and denominator* separately, *and put* x = a *in the result.*

*Otherwise—

We have $\dfrac{f(x+h)}{\phi(x+h)} = \dfrac{f(x) + hf'(x+\theta h)}{\phi(x) + h\phi'(x+\theta h)}$, by Taylor's Theorem

Put $x = a$; then, since $f(a) = \phi(a) = 0$,

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)}, \text{ having divided out by } h.$$

Now let h diminish indefinitely, and we have

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}; \text{ i.e. } \underset{x=a}{lt}\frac{f(x)}{\phi(x)} = \underset{x=a}{lt}\frac{f'(x)}{\phi'(x)}.$$

Ex. 1. $\underset{x=a}{lt}\dfrac{\sin(x^2-a^2)}{\sqrt{x}-\sqrt{a}} = \underset{x=a}{lt}\dfrac{2x\cos(x^2-a^2)}{1/2\sqrt{x}} = 2a \, . \, 2\sqrt{a} = 4a^{\frac{3}{2}}$.

Note.—This method is practically that of putting $x = a + h$, and dividing out by the vanishing factor h. See the first proof given above.

156. If $f'(a) = \phi'(a) = 0$, then the result is still of the form $\frac{0}{0}$, and we must repeat the operation until the numerator, or

denominator, or both, cease to vanish; the result being 0, ∞, or finite accordingly.

Ex. 2.

$$\underset{x=0}{lt}\frac{x^2 - x\sin x}{\frac{1}{2}x^2 - 1 + \cos x} = lt\frac{2x - \sin x - x\cos x}{x - \sin x} \quad \text{(still undetermined)}$$

$$= lt\frac{2 - \cos x - \cos x + x\sin x}{1 - \cos x.} \quad (\quad " \quad " \quad)$$

$$= lt\frac{2\sin x + \sin x + x\cos x}{\sin x} \quad (\quad " \quad " \quad)$$

$$= lt\frac{3\cos x + \cos x - x\sin x}{\cos x.} = 4.$$

NOTE.—We have differentiated above and below four times, and this corresponds to the fact that x^4 is a common factor, as may be seen by adopting the algebraical method.

157. Form $\frac{\infty}{\infty}$. Alternative Method.—We shall show that the rule for this form is the same as for the form $\frac{0}{0}$, as follows:—

Let $f(a) = \phi(a) = \infty$, and let $A = \underset{x=a}{lt}\frac{f(x)}{\phi(x)}$, or $\frac{f(a)}{\phi(a)}$.

Then $lt\frac{f(x)}{\phi(x)} = lt\frac{1}{\phi(x)}\Big/\frac{1}{f(x)}$,

which is of the form $\frac{0}{0}$; and therefore, by Art. 155, the expression

$$= lt\frac{\phi'(x)}{[\phi(x)]^2}\Big/\frac{f'(x)}{[f(x)]^2} = lt\left\{\frac{f(x)}{\phi(x)}\right\}^2\frac{\phi'(x)}{f'(x)} = lt\left\{\frac{f(x)}{\phi(x)}\right\}^2.lt\frac{\phi'(x)}{f'(x)},$$

by Art. 152 (2).

$$\text{i.e. } A = A^2\, lt\frac{\phi'(x)}{f'(x)} \qquad (1)$$

Hence, *if* A *be neither zero nor infinite*, we may divide out by it, and

$$\therefore 1 = A.lt\frac{\phi'(x)}{f'(x)}, \text{ or } A = lt\frac{f'(x)}{\phi'(x)}, \text{ by Art. 152 (3) } [X = 1]$$

i.e. $\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}$ as in the previous case, *so that the rule is the same for the form $\frac{\infty}{\infty}$ as for the form $\frac{0}{0}$.*

158. Exceptional Cases.—If we consider the alternative solution of equation (1) above, we can only state that A is *either* 0 *or* ∞, but we cannot say which.

But (1), suppose A or $\underset{x=a}{lt}\dfrac{f(x)}{\phi(x)}=0.$

Add 1 to both sides, then $lt\dfrac{f(x)+\phi(x)}{\phi(x)}=1.$ [Art. 152 (1).]

Now, since the limit of this latter expression is *not* zero, we may employ the rule above; hence also

$$1=lt\frac{f'(x)+\phi'(x)}{\phi'(x)}=lt\frac{f'(x)}{\phi'(x)}+1. \quad \text{[Art. 152 (1).]}$$

$$\therefore\ lt\frac{f'(x)}{\phi'(x)}=0.$$

Similarly (2), if $A=\infty$, then $\dfrac{1}{A}$ or $lt\dfrac{\phi(x)}{f(x)}=0$, and by case (1) we know that therefore $lt\dfrac{\phi'(x)}{f'(x)}=0$; i.e. $lt\dfrac{f'(x)}{\phi'(x)}=\infty$.

Hence, *in any case*, $\dfrac{f'(x)}{\phi'(x)}$ *has the same limit as* $\dfrac{f(x)}{\phi(x)}$, *when* $x=a$.

Ex. 3. $\underset{x=0}{lt}\ x\log x=lt\dfrac{\log x}{1/x}\left\{\text{form }\dfrac{\infty}{\infty}\right\}=lt\dfrac{1}{x}\Big/\left(-\dfrac{1}{x^2}\right)=lt\,(-x)=0.$

Ex. 4. $\underset{x=0}{lt}\ x^{\sin x}=lt\,y$, say.

We have $\log y=\sin x\log x=\dfrac{\log x}{\operatorname{cosec} x}\left\{\text{form }\dfrac{\infty}{\infty}\text{ when } x=0\right\}$;

$$\therefore\ lt\log y=lt\frac{1}{x}\Big/(-\operatorname{cosec} x\cot x),$$

which may be conveniently transformed to

$$-lt\frac{\sin^2 x}{x\cos x}=-lt\frac{\sin x}{x}.lt\frac{\sin x}{\cos x}=(-1).0=0;$$

$$\therefore\ lt\,y=e^0=1.$$

159. General Examples.

Ex. 5. $\underset{x=0}{lt}\dfrac{1-x^{\sin x}}{x\log x}\left\{\text{form }\dfrac{0}{0};\text{ see Art. 153, Exs. 8 and 10}\right\}$

$$= lt\, \frac{-(\sin x . x^{\sin x - 1} + x^{\sin x} \log x \cos x)}{\log x + 1}$$

[Art. 47, Ex. 7]

$$= -\, lt\, x^{\sin x} . lt\, \frac{\sin x + x \cos x \log x}{x(\log x + 1)}$$

$$= -\, lt\, \frac{\sin x + x \cos x \log x}{x(\log x + 1)}, \text{ since } lt\, x^{\sin x} = 1 \text{ [Ex. 4]},$$

$$= -\, lt\, \frac{\sin x}{x(\log x + 1)} - lt\, \frac{\cos x \log x}{\log x + 1}$$

$$= -\, lt\, \frac{\sin x}{x} . \frac{1}{\log x + 1} - lt \cos x + lt\, \frac{\cos x}{\log x + 1}$$

$$0 \qquad - 1 \qquad + \qquad = -1.$$

Ex. 6. $\underset{x=0}{lt}\, (\log x)^{\log(1-x)} = lt\, y$, say.

We have $lt \log y = lt \log(1 - x) . \log \log x = lt\, \dfrac{\log \log x}{1/\{\log(1-x)\}}$

$$= lt\, \frac{1}{x \log x} \Big/ \frac{1}{\{\log(1-x)\}^2} . \frac{1}{1-x} = lt\, \frac{\{\log(1-x)\}^2 (1-x)}{x \log x}$$

$$= lt\, \frac{\{\log(1-x)\}^2}{x \log x} - lt\, \frac{\{\log(1-x)\}^2}{\log x} = A - B, \text{ say.}$$

Now $B = 0$ at once; and $A = lt\, \dfrac{-2\log(1-x)}{1-x} \Big/ (\log x + 1) = 0$;

$$\therefore\ lt \log y = 0; \quad \therefore\ lt\, y = e^0 = 1.$$

EXAMPLES XXV.

Evaluate the following by the method of the Calculus:—

1. $\dfrac{\log(1+x)}{\sin x}$, $(x = 0)$.

2. $\dfrac{\cos^{-1} x}{\sqrt{1 - x^2}}$, $(x = 1)$.

3. $\dfrac{\sin x - \sin a}{e^x - e^a}$, $(x = a)$.

4. $\dfrac{x^a - a^x}{x - a}$, $(x = a)$.

5. $\dfrac{\log \sin x}{\log \tan x}$, $(x = 0)$.

6. $\dfrac{e^x - 1}{\log(1 + x)}$, $(x = 0)$.

7. $\dfrac{e^x - 1 - x}{\log(1 + x) - x}$, $(x = 0)$.

8. $\dfrac{e^x - 1 - x - \frac{1}{2}x^2}{\log(1 + x) - x + \frac{1}{2}x^2}$, $(x = 0)$.

9. $\dfrac{x \cos x - \sin x}{\sin 2x - 2 \sin x}$, $(x = 0)$.

10. $\dfrac{1 - \cos x}{x \log(1 + x)}$, $(x = 0)$.

11. $\dfrac{\sinh x}{x}$, $(x = 0)$.

12. $x^{\frac{1}{x-1}}$, $(x = 1$, and $x = \infty)$.

13. $(\sin x)^{\tan x}$, $(x = 0)$.

14. $\{1 + \log(1 + x)\}^{\frac{1}{e^x - e^{-x}}}$, $(x = 0)$.

15. $(\cos x)^{\frac{x}{x - \sin x}}$, $(x = 0)$.

16. $\dfrac{\sqrt{2a + x} - \sqrt{2x + a}}{\sqrt{4a - x} - \sqrt{4x - a}}$, $(x = a)$.

17. $\dfrac{\log \sin x}{1 - e^{1 - \sin x}}$ $\left(x = \dfrac{\pi}{2}\right)$.

18. $\left(\dfrac{\log x}{x}\right)^{\frac{1}{x}}$, $(x = \infty)$.

19. $\cos x \log \tan x$, $\left(x = \dfrac{\pi}{2}\right)$.

20. $\dfrac{a^3 \sin^3 x - x^3 \sin^3 a}{x^3 \sin(a - x)}$, $\left(\begin{matrix} x = a \text{ and} \\ x = 0 \end{matrix}\right)$.

21. $\dfrac{x^m \sin nx - x^n \sin mx}{x^m \sin mx - x^n \sin nx}$, $(m = n)$.

22. $\left\{\dfrac{1}{2}\left(\sqrt{\dfrac{a}{x}} + \sqrt{\dfrac{x}{a}}\right)\right\}^{\frac{1}{(x-a)}}$, $(x = a)$.

Evaluate the following by any method :—

23. $\dfrac{x^3 - 2x^2 + 1}{2x^3 - x - 1}$, $(x = 1)$.

24. $\dfrac{x^{11} - 6x^5 + x + 4}{x^{13} + 5x^{10} - 9x^5 + 3}$, $(x = 1)$.

25. $\dfrac{x^6 - 6x^4 + 21x^2 - 24x + 8}{x^6 - 15x^2 + 24x - 10}$, $(x = 1)$.

26. $\dfrac{x^a + x^b - 2}{a^c + b^c - a - b}$, $(x = 1)$.

27. $\dfrac{1}{x}\left(e^{a \tan^{-1} x} - e^{b \tan^{-1} x}\right)$, $(x = 0)$.

28. $\dfrac{\tan^2 x \log(1 + x)}{x(e^x - 1)^2}$, $(x = 0)$.

29. $\dfrac{\log \cos x^2}{\tan^4 x}$, $(x = 0)$.

30. $\sin^{-1} \sqrt{\dfrac{a - x}{a + x}} \cdot \operatorname{cosec} \sqrt{a^2 - x^2}$, $(x = a)$.

31. $\dfrac{\log_{\sin x}(\cos x)}{\log_{\sin x/2}\left(\cos \dfrac{x}{2}\right)}$, $(x = 0)$.

32. $a \cot ax - b \cot bx$, $(x = 0)$.

33. $(1 + \log x)^{\frac{1}{x-1}}$, $(x = 1)$.

34. e^x/x^n, $(x = \infty)$, (a) when n is a $+^{\text{ve}}$ integer, and (b) when n is any $+^{\text{ve}}$ quantity.

35. $\left\{\cos\left(\dfrac{a}{x} - 1\right)\right\}^{1/(x-a)^2}$, $(x = a)$.

36. $\dfrac{\sin^4 x - \tan^4 x}{(1 + \cos x)(1 - \cos x)^3}$, $(x = 0)$.

37. $\dfrac{\sqrt{1-x}\cos^{-1}x}{\log x}$, $(x=1)$. 38. $\dfrac{e^{\operatorname{cosec}^2 x}}{(1+x^2)^{1/x^4}}$, $(x=0)$.

39. $$\frac{\left(\frac{\pi}{2}-x\right)\log\sin x}{e^{\cos x}-1+\log\left(1+x-\frac{\pi}{2}\right)}.\left(x=\frac{\pi}{2}\right).$$

40. $\{1+\phi(x)\}^{\psi(x)}$, $(x=a)$, (1) if $\phi(a)=0$, $\psi(a)=\infty$, and $\phi(a).\psi(a)=m$; (2) if $\phi(a)=\infty$, $\psi(a)=0$, and $\phi(a).\psi(a)=m$.

41. $\dfrac{\log(e^{ax}+e^{-ax})-\log(e^x+e^{-x})}{a+\cos ax-a\cos x-1}$, when $a=1$, and when $x=0$.

42. If $y=x\cot x$, prove that $(y)_0=1$, $(y_1)_0=0$, $(y_2)_0=-\frac{2}{3}$.

43. If $y=x\operatorname{cosec}x$, find $(y_2)_0$.

44. $\dfrac{1}{x}(1+x)^{\frac{1}{x}-1}-\dfrac{1}{x^2}(1+x)^{\frac{1}{x}}\log(1+x)$, $(x=0)$.

45. $\dfrac{\left(1+\frac{1}{x}\right)^x-1}{x\log x}$, $(x=0)$.

46. $\left\{\dfrac{\sin x}{x}\log(e+x^2)-\dfrac{x}{\tan x}\cos^n x\right\}\Big/\sin x\log(1+x)$, $(x=0)$.

ANSWERS.

1. 1. 2. 1. $\dfrac{\cos a}{e^a}$. 4. $a^a(1-\log a)$. 5. 1. 6. 1.

7. -1. 8. $\frac{1}{2}$. 9. $\frac{1}{3}$. 10. $\frac{1}{2}$. 11. 1. 12. e; 1. 13. 1.

14. $e^{\frac{1}{2}}$. 15. e^{-3} 16. $\frac{1}{6}$. 17. 1. 18. 1. 19. 0.

20. $\dfrac{3\sin^2 a(\sin a-a\cos a)}{a}$; $\dfrac{a^3-\sin^3 a}{\sin a}$. 21. $\dfrac{\log x\sin nx-x\cos nx}{\log x\sin nx+x\cos nx}$.

22. $e^{1/8a^2}$. 23. $-\frac{1}{6}$. 24. -1. 25. $-\frac{1}{5}$. 26. $\dfrac{a+b}{a\log a+b\log b}$.

27. $a-b$. 28. 1. 29. $-\frac{1}{2}$. 30. $1/2a$. 31. 4. 32. 0.

33. e. 34. ∞. [See proof in Ex. 6, Art. 153.] 35. $e^{-1/2a^2}$. 36. -8.

37. $-\sqrt{2}$. 38. $e^{\frac{1}{6}}$. 39. $\frac{3}{2}$. 40. e^m; 1. 41. $\dfrac{x\tanh x}{1-x\sin x-\cos x}$, $\dfrac{a+1}{a}$.

43. $\dfrac{1}{3}$. 44. $-\dfrac{e}{2}$. 45. -1. 46. $\dfrac{1}{e}+\dfrac{3n+1}{6}$.

CHAPTER XIII.

PARTIAL DIFFERENTIATION.

Partial Differential Coefficient—Total Differential.

160. Let u be a function of several variables, x, y, z, ...; *i.e.* let $u = f(x, y, z, \ldots)$.

Suppose that x, y, z, etc., each receive an infinitesimal increment, and it is required to find the resulting increment of u.

For simplicity we shall consider the cases of first two and then three variables, whence the general case can be easily inferred.

161. Two Independent Variables. — Let $u = f(x, y)$. Suppose x to become $x + \Delta x$, and y to become $y + \Delta y$; in consequence of which u becomes $u + \Delta u$. Then, since $u + \Delta u$ is the same function of $x + \Delta x$ and $y + \Delta y$ as u is of x and y, we may write

$$u + \Delta u = f(x + \Delta x, y + \Delta y).$$

Putting h for Δx, and k for Δy, we have

$$\Delta u = f(x + h, y + k) - f(x, y) \quad . \quad . \quad . \quad . \quad (1)$$

It is convenient to suppose $f(x, y)$ to increase to $f(x + h, y + k)$ in *two* stages, first increasing y by itself, and then x by itself; so that we pass from $f(x, y)$ to $f(x, y + k)$ and thence to $f(x + h, y + k)$.

Hence

$$\Delta u = \{f(x + h, y + k) - f(x, y + k)\} + \{f(x, y + k) - f(x, y)\}$$

which we shall put

$$= \frac{f(x + h, y + k) - f(x, y + k)}{h} . h + \frac{f(x, y + k) - f(x, y)}{k} . k.$$

Now the limit of the second fraction is the d.c. of $f(x, y)$ with respect to y, *supposing* y *alone to vary.* This is called the *partial diff. co. of* f(x, y) *or* u *with respect to* y, *and is written* $\partial u/\partial y$.

The limit of the first fraction is the d.c. of $f(x, y+k)$, or—since k is ultimately infinitesimal—$f(x, y)$ with respect to x, *supposing* x *alone to vary*, and is called *the partial diff. co. of* f(x, y) *or* u *with respect to* x, *and is written* $\partial u/\partial x$.

Also Δu, h, and k, are written du, dx, and dy, so that we have ultimately

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Here du is called the *total differential of* u *when* x *and* y *both vary.*

In the general case *both* x and y are independent variables, for not only x, but y, may be chosen arbitrarily, as also may their increments dx and dy. They are thus independent of each other. If, however, an equation exist between x and y, so that x and y, dx and dy, are *not* independent of each other, the equation (A) will still be true.

162. We now give an example worked by first principles, and also by means of (A).

Ex. *Let* $u = x^2y - xy^3$.

Then
$$\begin{aligned}\Delta u &= (x+h)^2(y+k) - (x+h)(y+k)^3 - x^2y + xy^3\\ &= (x^2 + 2xh \ldots)(y+k) - (x+h)(y^3 + 3y^2k \ldots) - x^2y + xy^3\\ &= x^2y + 2xyh + x^2k \ldots - xy^3 - 3xy^2k - y^3h \ldots - x^2y + xy^3\\ &= (2xy - y^3)h + (x^2 - 3xy^2)k,\end{aligned}$$
neglecting higher powers of h and k.

Hence, in the limit, $du = (2xy - y^3)dx + (x^2 - 3xy^2)dy$.

Otherwise :—

Treating y as if it were constant

$$\partial u/\partial x = 2xy - y^3;$$

and treating x as if it were constant

$$\partial u/\partial y = x^2 - 3xy^2;$$

whence from (A), $\quad du = (2xy - y^3)dx + (x^2 - 3xy^2)dy.$

163. Three Independent Variables.

Again, let $u = f(x, y, z)$, and suppose x, y, z respectively

increased to $x+h$, $y+k$, and $z+l$; in consequence of which u is increased to $u+\Delta u$.

We shall suppose $f(x, y, z)$ to be increased in three stages, so that we pass in succession from $f(x, y, z)$ to $f(x, y, z+l)$, $f(x, y+k, z+l)$, and $f(x+h, y+k, z+l)$.

Then
$$\Delta u = f(x+h, y+k, z+l) - f(x, y, z)$$
$$= \frac{f(x+h, y+k, z+l) - f(x, y+k, z+l)}{h}.h$$
$$+ \frac{f(x, y+k, z+l) - f(x, y, z+l)}{k}.k$$
$$+ \frac{f(x, y, z+l) - f(x, y, z)}{l}.l, \text{ as in Art. 161.}$$

Hence, ultimately,
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \quad . \quad . \quad . \quad \text{(B)}$$
where $\partial u/\partial x$ means the d.c. of u with respect to x, supposing x alone to vary; and so for the others.

Note.—As $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ are in general finite, the statement of Art. 91 is justified.

164. If $u = f(v)$ where $v = \phi(x, y)$, we have
$$du = f'(v)dv = f'(v)\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right).$$

Ex. $u = \log \sin \sqrt{x^2 - y^2}$.

Put $x^2 - y^2 = v$.
$$\therefore dv = 2xdx - 2ydy.$$
$$\text{Also } u = \log \sin \sqrt{v}.$$
$$\therefore du = \frac{\cos \sqrt{v}}{\sin \sqrt{v}} \frac{1}{2\sqrt{v}} dv = \frac{\cot \sqrt{v}}{2\sqrt{v}}.dv$$
$$= \frac{xdx - ydy}{\sqrt{x^2 - y^2}} \cot \sqrt{x^2 - y^2}.$$

165. Differentiation of an Implicit Function.

We have seen that if $u = f(x, y)$, then
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

and that this result is true whether x and y vary quite independently of each other, or whether each is a function of the other.

Now, suppose we are given $u \equiv f(x, y) = 0$; then, since u is always 0, x and y are the only variables. Hence they must each be a function of the other; also $du = 0$.

Hence, from (A), $\quad \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = 0$;

$$\therefore \frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}.$$

166. This last result appears incorrect at first sight, since dy/dx is apparently equal to what looks like $-\dfrac{dy}{dx}$. But the "δu" *of the numerator is not the same as the* "δu" *of the denominator;* the former is the *partial* increment of u, caused by the variation of x separately, and the latter that by y separately. Moreover, since the total increment of u is zero, we should expect one "δu" to be equal, *but opposite in sign,* to the other "δu." The partial d.c.'s are, however, always kept as differential coefficients, and not treated as the ratio of differentials.

167. The advantage of this method is that it enables us to find dy/dx without having previously to find y in terms of x from the equation $f(x, y) = 0$. But then we get dy/dx in terms of x and y, and not in terms of x alone.

Ex. If $u \equiv 3x^4 - x^2y + 2y^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad$ (1)

$\partial u/\partial x = 12x^3 - 2xy$; $\quad \partial u/\partial y = -x^2 + 6y^2$.

$$\therefore \frac{dy}{dx} = \frac{12x^3 - 2xy}{x^2 - 6y^2}.$$

To find this in terms of x alone, we should have to use the original equation (1), which would give as much trouble as adopting the "explicit" method.

But suppose we wish to find when $dy/dx = 0$.

This gives $12x^3 - 2xy = 0$.

$\therefore$ (i) $x = 0$, and $\therefore$ from (1), $y = 0$,

(ii) $y = 6x^2$, and $\therefore$ from (1),

$(3 - 6)x^4 + 432x^6 = 0$,

$\therefore x = 0$, or $\pm\frac{1}{12}$; whence $y = 0$, or $\frac{1}{24}$.

The points of the curve (1), at which $dy/dx = 0$, are the origin, and the points $(\frac{1}{12}, \frac{1}{24})$, $(-\frac{1}{12}, \frac{1}{24})$.

Similarly, we can find the points at which $\frac{dy}{dx}$ is infinite. [Compare Art. 61, Ex. 5.]

168. The result in Art. 163 is true for any infinitesimal increments whatever of x, y, and z. It will therefore still be true if x, y, z are functions of some other quantity v.

If v be taken as the independent variable, an arbitrary increment δv in v will produce corresponding increments δx, δy, δz in x, y, z; and these in turn will produce the total increment δu in u.

Now, there is nothing partial in these increments of x, y, and z, i.e. no supposition as to certain of the quantities being constant. Thus δx is the *absolute* change in x produced by the increment δv in v, and similarly for the others.

Hence, dividing throughout by dv, we have from (B), Art. 163,

$$\frac{du}{dv} = \frac{\partial u}{\partial x}\frac{dx}{dv} + \frac{\partial u}{\partial y}\frac{dy}{dv} + \frac{\partial u}{\partial z}\frac{dz}{dv}$$

where dx/dv, dy/dv, dz/dv are *total* d.c.'s, as the notation indicates.

Cor.—If $u = f(x, y)$, and y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

169. Simple Cases.—The following simple cases of partial differentiation are very important, and may be easily verified.

(1) If $u = x \pm y$, $du = dx \pm dy$.

(2) If $u = xy$, $\quad du = ydx + xdy$.

(3) If $u = \frac{x}{y}$, $\quad du = \frac{1}{y}dx - \frac{x}{y^2}dy = \frac{ydx - xdy}{y^2}$.

(4) If $u = x^y$, $\quad du = yx^{y-1}dx + x^y \log x\, dy$.

It will be seen that these cases are an extension of the fundamental rules of differentiation; for here x and y *may be utterly unconnected*. But if x and y be regarded as functions of the

same variable (t), then the above results still hold, and are in fact the fundamental rules.

Thus if $x = f(t)$, $y = \phi(t)$, we have

$$(1)\ u = f(t) \pm \phi(t), \quad du = f'(t)dt \pm \phi'(t)dt,$$
$$\therefore du/dt = f'(t) \pm \phi'(t).$$
$$(2)\ u = f(t).\phi(t), \quad du = f'(t)dt.\phi(t) + f(t).\phi'(t)dt.$$
$$\therefore du/dt = f'(t)\phi(t) + f(t)\phi'(t),$$

and so on. [Compare (4) with Ex. 7, Art. 47.]

170. Examples on the Preceding Results.

Ex. 1. $u = x^2y - xy^3$.

$$du = 2xdx.y + x^2dy - dx.y^3 - x.3y^2dy$$
$$= (2xy - y^3)dx + (x^2 - 3xy^2)dy, \text{ as in Art. 160.}$$

Ex. 2. $u = \sqrt{x + \sqrt{x^2 + y^2}}$.

$$\therefore u^2 = x + \sqrt{x^2 + y^2} = x + \sqrt{v} \text{ say, where } v = x^2 + y^2,$$
$$dv = 2(xdx + ydy).$$

$$2udu = dx + \frac{dv}{2\sqrt{v}}$$
$$= dx + \frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \frac{(x + \sqrt{x^2 + y^2})dx + ydy}{\sqrt{x^2 + y^2}}.$$
$$\therefore du = \frac{(x + \sqrt{x^2 + y^2})dx + ydy}{2\sqrt{x + \sqrt{x^2 + y^2}}.\sqrt{x^2 + y^2}}$$

Ex. 3. *If* $\left.\begin{array}{ll} x = r\cos\theta & \ldots\ldots (1) \\ y = r\sin\theta & \ldots\ldots (2) \end{array}\right\}$ *and* $y = f(x)$ . . . . . (3)

to find dy/dx *and* d^2y/dx^2 *in terms of* r, θ, $dr/d\theta$, *and* $d^2r/d\theta^2$.

We shall denote $dr/d\theta$ by r', and $d^2r/d\theta^2$ by r''.

Since we have three equations from which we could eliminate x and y, giving an equation in r and θ only, it follows that r may be regarded as a function of θ alone; as also appears from the fact that r and θ are the polar co-ordinates of a point on the curve $y = f(x)$.

Again, since r is a function of θ, then from (1) and (2) we can regard x and y as functions of θ alone.

Now, by Art. 168, Cor.,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = r'\cos\theta - r\sin\theta.$$
$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = r'\sin\theta + r\cos\theta.$$

Hence, $$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}.$$

Again, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)\Big/\dfrac{dx}{d\theta}$; which, on using the rule for a quotient, and substituting for $\dfrac{dx}{d\theta}$, becomes $\dfrac{X}{(r'\cos\theta - r\sin\theta)^3}$ say,

where $X = (r''\sin\theta + r'\cos\theta + r'\cos\theta - r\sin\theta)(r'\cos\theta - r\sin\theta)$
$- (r''\cos\theta - r'\sin\theta - r'\sin\theta - r\cos\theta)(r'\sin\theta + r\cos\theta)$,

which will reduce to $\quad r^2 - rr'' + 2r'^2$.

Hence $$\frac{d^2y}{dx^2} = \frac{r^2 - rr'' + 2r'^2}{(r'\cos\theta - r\sin\theta)^3}.$$

171. Higher Partial Derivatives.

Let $u = f(x, y)$. Then since $\partial u/\partial x$ and $\partial u/\partial y$ are functions of x and y, we may, as in total differentiation, repeat the operation of partial differentiation with respect to x or y.

Thus (1) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$ is the partial d.c. of $\dfrac{\partial u}{\partial x}$ with respect to x, i.e. on the supposition that y is constant; and is written $\dfrac{\partial^2 u}{\partial x^2}$.

(2) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$ is the partial d.c. of $\dfrac{\partial u}{\partial x}$ with respect to y, i.e. on the supposition that x is constant; and is written $\dfrac{\partial^2 u}{\partial y\partial x}$.

(3) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ which is similarly defined, is written $\dfrac{\partial^2 u}{\partial x\partial y}$, and

(4) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)$ is written $\dfrac{\partial^2 u}{\partial y^2}$.

Similarly for higher derivatives; thus
$$\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\right\} \text{ is written } \frac{\partial^3 u}{\partial x\partial y^2}.$$

172. To prove that $\dfrac{\partial^2 u}{\partial y\partial x} = \dfrac{\partial^2 u}{\partial x\partial y}$.

Since $\dfrac{\partial u}{\partial x}$ is the limit of $\dfrac{f(x+h, y) - f(x, y)}{h}$, when $h \doteq 0$;

$\therefore \dfrac{\partial^2 u}{\partial y \partial x}$, or $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$, is the limit of

$$\left\{\frac{f(x+h, y+k)-f(x, y+k)}{h}-\frac{f(x+h, y)-f(x, y)}{h}\right\} \div k,$$

$$\text{or } \frac{f(x+h, y+k)-f(x, y+k)-f(x+h, y)+f(x, y)}{hk}.$$

when both $h = 0$ and $k = 0$.

Similarly, $\dfrac{\partial^2 u}{\partial x \partial y}$ can be shown to be the limit of the same fraction; which proves the proposition.

EXAMPLES XXVI.

1. Find du in terms of dx and dy in the following examples:—

(1) $u = x^3 - 3xy^2 + 2y^3$. (2) $u = (2x^2 - 3y^2)^3$.

(3) $u = \sin(ax + by)$. (4) $u = e^{xy}$.

(5) $u = \log \tan \dfrac{y}{x}$. (6) $u = \log(x + \sqrt{x^2 + y^2})$.

(7) $u = \sqrt{\dfrac{x+y}{x-y}}$.

2. Find dy/dx in the following examples:—

(1) $x^3 + y^3 = 3axy$. (2) $b^2x^2 + a^2y^2 = a^2b^2$.

(3) $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = 1$.

3. If $x = \sqrt{1 + t}$, $y = \sqrt{1 - t}$, and $u = x^3 + 3x^2y + 2y^3 - 6y$, prove that
$$\frac{du}{dt} = \frac{3}{2}(x + y).$$

4. If $x = r\cos\theta$, prove that $d\theta = \dfrac{xdr - rdx}{r\sqrt{r^2 - x^2}}$.

5. If $x = r\cos\theta$, $y = r\sin\theta$, y being a function of x, prove that:—

(1) $r(yd\theta + dx) = xdr$. (2) $ydx - xdy = -r^2d\theta$.

(3) $dx^2 + dy^2 = dr^2 + r^2d\theta^2$.

(4) $\left(\dfrac{d^2x}{d\theta^2}\right)^2 + \left(\dfrac{d^2y}{d\theta^2}\right)^2 = (r'' - r)^2 + 4r'^2$.

(5) $\dfrac{d^2y}{d\theta^2}\dfrac{dx}{d\theta} - \dfrac{d^2x}{d\theta^2}\dfrac{dy}{d\theta} = r^2 - rr'' + 2r'^2$.

$$(6)\ \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\bigg/\frac{d^2y}{dx^2}=\frac{(r'^2+r^2)^{\frac{3}{2}}}{r^2-rr''+2r'^2}.$$

6. A straight moves so that the triangle which it cuts off from two fixed lines (not at right angles) is of constant area. If x and y be the intercepts cut off from the lines, prove that $\frac{dy}{dx}=-\frac{y}{x}$.

7. The vertex, A, of a triangle, ABC, moves on a circle through B and C, which are fixed. Prove that at any time

$$\cos C\,db+\cos B\,dc=0.$$

Give also a geometrical proof.

8. A straight line, BC, of fixed length, slides between two fixed lines, Ax, Ay, cutting off a variable triangle, ABC.

Show that at any time $\cos C\,db+\cos B\,dc=0$.

Give also a geometrical proof, and show that this question is merely another statement of the preceding question.

9. If the sides b, c, and the included angle A of any triangle, ABC, be each increased infinitesimally, prove that the resulting increment of a is given by

$$da=\cos C\,db+\cos B\,dc+b\sin C\,dA,$$

Answers.

1. (1) $3(x-y)\{(x+y)dx-2ydy\}$. (2) $6(2x^2-3y^2)^2(2xdx-3ydy)$.

(3) $\cos(ax+by).(adx+bdy)$. (4) $e^{xy}(ydx+xdy)$.

(5) $2\operatorname{cosec}\frac{2y}{x}\cdot\frac{xdy-ydx}{x^2}$. (6) $\frac{dx}{\sqrt{x^2+y^2}}+\frac{ydy}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})}$.

(7) $\frac{xdy-ydx}{(x+y)^{\frac{1}{2}}(x-y)^{\frac{3}{2}}}$.

2. (1) $\frac{x^2-ay}{ax-y^2}$. (2) $-\frac{b^2x}{a^2y}$. (3) $-\sqrt{\frac{1-y^2}{1-x^2}}$.

173. Homogeneous Functions—Dimensions.—When a quantity, x, is raised to some power, say the nth, where n is any quantity, then x^n is said to be of *n dimensions* in x, or of the *nth degree* in x.

If a be a numerical coefficient, then ax^n is still of n dimensions in x.

Again, ax^py^q is said to be of p dimensions in x, and q dimensions in y, or (if we do not wish to distinguish between x and y) of $p+q$ dimensions in x and y.

Similarly ax^p/y^q is of $p-q$ dimensions in x and y.

An important case is ax^n/y^n, which is of *no dimension*, or of *zero degree*, in x and y.

NOTE.—x^py^q, which is of $p+q$ dimensions, may be written $x^{p+q}.(y/x)^q$, in which the degree of x is the same as that of the whole term; as it should be, since $(y/x)^q$ is of no degree in x and y.

174. A group of terms, all of the same degree, is called a *homogeneous expression*, e.g. $x^3-2x^2y+\dfrac{x^4}{y}+y^3$. This expression is of the *third* degree; hence, taking x^3 outside a bracket, the expression inside must be of zero degree, and must therefore involve y/x, or its powers ($+^{ve}$ or $-^{ve}$), only.

Thus $$x^3-2x^2y+\frac{x^4}{y}+y^3=x^3\left\{1-2\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^{-1}+\left(\frac{y}{x}\right)^3\right\}.$$

Generally, a rational integral algebraical homogeneous expression of the nth degree in x and y can be written in the form $x^nf(y/x)$; thus
$$p_0x^n+p_1x^{n-1}y+\ldots+p_rx^{n-r}y^r+\ldots+p_ny^n$$
$$=x^n\left\{p_0+p_1\left(\frac{y}{x}\right)+\ldots+p_r\left(\frac{y}{x}\right)^r+\ldots+p_n\left(\frac{y}{x}\right)^n\right\},$$
and x^n represents the degree of the expression.

175. But a wider definition may be given. If $f(y/x)$ be any function of x, then if $f(y/x)$ is capable of expansion in powers of y/x, each term of the expansion will be of no degree in x and y, and therefore also will $f(y/x)$ be of no degree.

Should $f(y/x)$ be incapable of expansion the preceding remarks cannot be applied; but we still regard it as of no degree in x and y, and we shall find that we are justified in doing so when we come to apply Euler's Theorem below, if indeed any justification is necessary.

It follows from the above that $x^n f(y/x)$, where f is *any* function, is a *homogeneous function of the* n*th degree in* x *and* y.

176. Similar remarks apply to terms and expressions of three or more variables.

For example, $2x^3y^2z - 3x^2y^4 + \dfrac{x^2y^6}{z^2} - \dfrac{z^9}{2x^2y}$ is homogeneous in x, y, and z, and is of the 6th degree. It may be written

$$x^6\left\{2\left(\frac{y}{x}\right)^2\left(\frac{z}{x}\right) - 3\left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^6\left(\frac{z}{x}\right)^{-2} - \frac{1}{2}\left(\frac{y}{x}\right)^{-1}\left(\frac{z}{x}\right)^9\right\},$$

and, as we should expect, the expression in brackets is of no dimensions in x, y, and z, but is of the form $f(y/x, z/x)$. We might also have taken y^6 or z^6 outside.

Hence we conclude that $x^n f(y/x, z/x)$ represents a *homogeneous function of the* n*th degree in* x, y, *and* z.

177. Examples.

Ex. 1. If u and v be two homogeneous functions of the *same* degree in x and y, we may write

$$u = x^n f(y/x), \quad v = x^n \phi(y/x).$$

$\therefore \dfrac{u}{v} = f\left(\dfrac{y}{x}\right)\Big/\phi\left(\dfrac{y}{x}\right) = \dfrac{f(m)}{\phi(m)}$, if m is the ratio of y to x; and $\dfrac{u}{v}$ is of no degree in x and y.

If $x = r\cos\theta$, $y = r\sin\theta$, then $m = \tan\theta$.

$$\therefore \frac{u}{v} = \frac{f(\tan\theta)}{\phi(\tan\theta)}.$$

Ex. 2. $\dfrac{x^4 - 3x^3y - xy^3}{2x^3y - x^2y^2 - 3y^4} = \dfrac{1 - 3m - m^3}{2m - m^2 - 3m^4}$, which could be obtained by putting $x = 1$ and $y = m$.

Ex. 3. *Express* $\tan 3\theta$ *in terms of* $\tan\theta$.

Let $s = \sin\theta$, $c = \cos\theta$, $t = \tan\theta$.

Then $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta} = \dfrac{3s - 4s^3}{4c^3 - 3c}$.

Make the numerator and denominator homogeneous and of the same degree, by writing $s^2 + c^2$ for 1, and we get

$$\frac{3s(s^2 + c^2) - 4s^3}{4c^3 - 3c(s^2 + c^2)} = \frac{3sc^2 - s^3}{c^3 - 3cs^2} = \frac{3t - t^3}{1 - 3t^2}.$$

Ex. 4. *Express* $\cos\theta + 3\sin^2\theta$ *in terms of* $\tan\theta$.

We have $c + 3s^2 = \dfrac{c(s^2 + c^2)^{\frac{1}{2}} + 3s^2}{s^2 + c^2} = \dfrac{(t^2 + 1)^{\frac{1}{2}} + 3t^2}{t^2 + 1}$.

178. Euler's Theorem of Homogeneous Functions

(A) Prop.—*If* u *be a homogeneous function of the* n*th degree in* x *and* y, *then shall*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

(1) Let u be a rational integral algebraical function;
thus $u = p_0x^n + p_1x^{n-1}y + \ldots + p_rx^{n-r}y^r + \ldots + p_{n-1}xy^{n-1} + p_ny^n$

Then

$$x\frac{\partial u}{\partial x} = np_0x^n + (n-1)p_1x^{n-1}y + \ldots + (n-r)p_rx^{n-r}y^r + \ldots + p_{n-1}xy^{n-1}$$

$$y\frac{\partial u}{\partial y} = p_1x^{n-1}y + \ldots + r.p_rx^{n-r}y^r + \ldots + (n-1)p_{n-1}xy^{n-1} + np_ny^n$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = np_0x^n + np_1x^{n-1}y + \ldots + np_rx^{n-r}y^r + \ldots + np_{n-1}xy^{n-1} + np_ny^n = nu.$$

(2) Let $u = x^nf(y/x)$, where n is *any* quantity and $f(y/x)$ is *any* function of y/x.

Then

$$\frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - x^{n-2}yf'\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = x^nf'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right) = x^{n-1}f'\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^nf\left(\frac{y}{x}\right) = nu.$$

Cor.—If $n = 0$, *i.e.* if $u = f\left(\frac{y}{x}\right)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

* **(B) Prop.**—*If* u *be a homogeneous function in* x, y, z ... *of the* n*th degree, then shall*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots = nu.$$

This is not a difficult extension of the preceding, and we leave the proof to the student.

179. Any rational integral algebraical function of x and y may be arranged in homogeneous groups. If u_r denote a homogeneous group of the rth degree, the expression may be written

$$u = u_n + u_{n-1} + \dots + u_1 + u_0.$$

Hence, for *any* rational integral algebraical function of x and y,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu_n + (n-1)\,u_{n-1} + \dots + u_1.$$

EXAMPLES XXVII.

1. If $y = mx$, write down the values of:—

(1) $\dfrac{x-2y}{3x+y}$. (2) $\dfrac{x^3-3xy^2}{3x^2y-y^3}$. (3) $\dfrac{\sqrt{x}(\sqrt{x}+\sqrt{y})}{\sqrt{y}(\sqrt{x}-\sqrt{y})}$.

(4) $\dfrac{x\sqrt{x}+\sqrt{x^3+y^3}}{\sqrt{xy}\,(\sqrt{x}+\sqrt{y})}$. (5) $2\log(\sqrt{x}+\sqrt{y}) - \log(x+y)$.

2. Express in terms of $\tan\theta$:-

(1) $\dfrac{3\sin\theta-5\cos\theta}{2\sin\theta-\cos\theta}$. (2) $\dfrac{\sin\theta-\cos^3\theta}{\cos\theta-\sin^3\theta}$. (3) $\dfrac{2+\cos 2\theta}{1-\sin 2\theta}$.

(4) $\dfrac{\sin\theta+\cos^2\theta}{\cos\theta+\sin^2\theta}$. (5) $\dfrac{1}{a+b\cos\theta}$. (6) $\dfrac{\sin^2\theta+\cos^3\theta}{\sin\theta\,(1+\cos\theta)}$.

3. If $x^2+y^2=1$, express *purely* as a function of y/x ($=m$, say):—

(1) x. (2) y. (3) $\dfrac{1}{1+x}$. (4) $\dfrac{a+bx}{a-bx}$. (5) $1+\sqrt{x}$.

4. State whether the following functions are homogeneous or not; and, homogeneous, state the degree:—

(1) $\dfrac{x^3-y^3}{\sqrt{x}-\sqrt{y}}$. (2) $\left(1+\dfrac{x}{\sqrt{x^2+y^2}}\right)\Big/\sqrt{xy}(\sqrt{x}+\sqrt{y})$.

(3) $y^3\log\dfrac{2x+y}{x-2y}$. (4) $\sin\dfrac{2x^2-y^2}{\sqrt{x}+\sqrt{y}}$.

(5) $3\log(x^{\frac{1}{3}}+y^{\frac{1}{3}})^x - 2\log(x^{\frac{1}{2}}+y^{\frac{1}{2}})^x$.

(6) $\left\{\log\dfrac{x^n+y^n}{x^m+y^m}+(m-n)\log x\right\}\div x^m$.

5. Verify Euler's Theorem in the following examples:—

(1) $u = x + y$. (2) $u = x^m y^n$.

(3) $u = \sqrt{x} + \sqrt{y}$. (4) $u = 3x^3 + xy^2 - y^3$.

(5) $u = \dfrac{x-y}{x+y}$ (6) $u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

(7) $u = \sin^{-1}\left(1 + \dfrac{y}{x}\right)$. (8) $u = x^3 \log \dfrac{y}{x}$.

(9) $u = y^{-2} e^{\frac{x}{y}}$. (10) $u = x^2 y - 2xyz - xz^2$.

$$(11)\ u = x^2 \log\left(1 + \sqrt{\frac{y}{x}} + \sqrt{\frac{z}{x}}\right).$$

$$(12)\ u = x^m y^n \sin \frac{ay^2 + bz^2}{x^2}.$$

6. If $u = \log \dfrac{x^3 + y^3}{x^2 + y^2}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

7. If $u = \sin(\sqrt{x} + \sqrt{y})$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tfrac{1}{2}(\sqrt{x} + \sqrt{y}) \cos(\sqrt{x} + \sqrt{y}).$$

8. If $u = \sin^{-1} \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial u} = 0$.

9. If $u = f(v)$, v being homogeneous and the nth degree in x and y, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf'(v).$$

Hence if $u = \log v$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n$.

ANSWERS.

1. (1) $\dfrac{1-2m}{3+m}$. (2) $\dfrac{1-3m^2}{3m-m^2}$ (3) $\dfrac{1+\sqrt{m}}{\sqrt{m}(1-\sqrt{m})}$.

(4) $\dfrac{1+\sqrt{1+m^3}}{\sqrt{m}(1+\sqrt{m})}$. (5) $2\log(1+\sqrt{m}) - \log(1+m)$.

2. (1) $\dfrac{3t-5}{2t-1}$. (2) $\dfrac{t^3+t-1}{1+t^2-t^3}$. (3) $\dfrac{3+t^2}{(1-t)^2}$. (4) $\dfrac{t\sqrt{1+t^2}+1}{\sqrt{1+t^2}+t^2}$.

(5) $\dfrac{\sqrt{1+t^2}}{a\sqrt{1+t^2}+b}$. (6) $\dfrac{t^2\sqrt{1+t^2}+1}{t\sqrt{1+t^2}(\sqrt{1+t^2}+1)}$.

3. (1) $-\dfrac{1}{\sqrt{1+m^2}}$. (2) $\dfrac{}{\sqrt{1+m^2}}$. (3) $\dfrac{\sqrt{1+m^2}}{\sqrt{1+m^2}+1}$.

(4) $\dfrac{a\sqrt{1+m^2}+b}{a\sqrt{1+m^2}-b}$. (5) $1+\dfrac{}{(1+m^2)^{\frac{1}{2}}}$

4. (1) Yes; $\frac{5}{2}$. (2) Yes; $-\frac{3}{2}$. (3) Yes; 3. (4) No.

(5) Yes; 1. (6) Yes; $-m$.

CHAPTER XIV.

MAXIMA AND MINIMA.

180. We have already considered a few simple examples of maxima and minima in Chapter X. (*q.v.*). We shall now enter more fully into the subject, treating it (1) algebraically, (2) geometrically, and (3) by the method of the Calculus.

181. Function of a Single Variable—Definition.—If y be a function of x which first increases up to a certain value and then decreases, the variable being supposed to increase uniformly throughout; then this value of y is said to be *a maximum*. If the function first decreases and afterwards increases, it is said to be *a minimum*.

A function of x may have several *maxima* and *minima*; thus, if the curve in Fig. 22 (Art. 136) represent $f(x)$, the max^a. and min^a. of $f(x)$ [or its singular, or turning, values] are represented by the max. and min. ordinates Aa, Bb, etc.

It will be noticed that max^a. and min^a. occur alternately; Aa, Cc, Ee being max^a., and Bb, Dd, Ff min^a. This will be proved below.

182. Alternative Definition.—When $f(a)$ is $\begin{bmatrix}\text{greater}\\ \text{less}\end{bmatrix}$ than both $f(a+h)$ and $f(a-h)$ when h is finite, however small, then $f(a)$ is called a $\begin{bmatrix}\text{max.}\\ \text{min.}\end{bmatrix}$ value of $f(x)$.

183. Algebraical Treatment.—This method is directly based on the fact that certain fundamental functions are restricted in value, *i.e.* have max. or min. values.

Thus, (1) x^2 can never be —^{ve}; hence 0 is a min. value of x^2.

Or, if $y = \sqrt{x}$, the min. value of x is 0.

(2) if $y = \sin x$, the max. value of y is $+1$, and the min., -1.

(3) if $y = a^x$, the min. of y is 0.

(4) if $y = \log x$, the min. of x is 0.

Imaginary values are obviously excluded.

We may remark that in the case of a square root *the double sign will generally be supposed to be attached.* Hence in (1) above we do not say that 0 is the min. value of y, for it can have *any* value, $+^{ve}$ or —^{ve}.

Again, in (3), $y = 0$ when $x = -\infty$. This is an exceptional case, for here the *ascent* in the value of y is not accompanied by a previous *descent*. The same thing, too, can happen in the case of a discontinuity.

The above remarks apply to (4) if x and y be interchanged.

184. The following points will be useful :—

If a is a *max.* of $f(x)$, then (1) $\dfrac{1}{a}$ is a *min.* of $\dfrac{1}{f(x)}$;

(2) $-a$ is a *min.* of $-f(x)$;

(3) $a \pm b$ is a *max.* of $f(x) \pm b$;

(4) $\phi(a)$ is a *max.* of $\phi f(x)$, *provided that $\phi(y)$ is a function which increases with* y.

Similarly for a min. of $f(x)$.

For example, in the expression $2 - \dfrac{a}{2a - f(x)}$, if a be a max. of $f(x)$, then the *min.* of $2a - f(x)$ is a;

$$\therefore \text{ the } max. \text{ of } \frac{a}{2a - f(x)} \text{ is } \frac{a}{a}, \text{ or } 1;$$

$$\therefore \text{ the } min. \text{ of } 2 - \frac{a}{2a - f(x)}, \text{ is } 2 - 1, \text{ or } 1.$$

185. Examples.

Ex. 1. If $x^2 + y^2 = a^2$, so that $y = \sqrt{a^2 - x^2}$; then, for real values of y, x cannot be $> a$ nor $< -a$. Hence, $+a$ is a max. value of x, and $-a$ a min. value.

Similarly, $x = \sqrt{a^2 - y^2}$, and $+a$ and $-a$ are max. and min. values of y.

Hence, in the circle $x^2 + y^2 = a^2$, the max. numerical value of x or y is a, the radius.

Ex. 2. $\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2 2\theta$. Hence $\frac{1}{4}$ is a max., and 0 is a min.

Ex. 3. If $y = a^2\cos^2\phi + b^2\sin^2\phi$ $(a > b)$; then

$$y = (a^2 - b^2)\cos^2\phi + b^2.$$

The max. and min. values of $\cos^2\phi$ are 1 and 0; hence

(1) when $\cos^2\phi = 1$, y is a max., and $= a^2$.

(2) when $\cos^2\phi = 0$, y is a min., and $= b^2$.

Here $\sqrt{y}$ is the length of a semi-diameter of an ellipse referred to the centre as origin, ϕ being the eccentric angle. Hence a and b, the major and minor semi-axes, are the maximum and minimum respectively.

Ex. 4. $y = a\cos\phi + b\sin\phi$.

Put $a = k\cos\alpha$, $b = k\sin\alpha$, so that $k^2 = a^2 + b^2$.

Then $y = k\cos(\phi - \alpha)$, the max. and min. values of which are $\pm k$, or $\pm\sqrt{a^2 + b^2}$.

Hence, if AA', BB' be the axes of an ellipse, and OM, PM the coordinates of a point P on the curve, the max. value of $OM + PM$ is AB.

Or again, if a and b represent two forces at right angles to each other, y will be the sum of the resolved parts along a line making an angle ϕ with the direction of a. And the max. value of y occurs when the line is taken in the direction of their resultant, for its value is then $\sqrt{a^2 + b^2}$; also $\phi = \alpha = \tan^{-1}\frac{b}{a}$.

Again, if $x = a\cos\phi$, $y = b\sin\phi$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; hence the question might be stated thus:—

"Find the max. of $x + y$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;" the answer being $\sqrt{a^2 + b^2}$.

Ex. 5. $y = \log\sin(x + \alpha) + \log\sin(x - \alpha)$.

Here $y = \log\{\sin(x + \alpha)\sin(x - \alpha)\}$

$$= \log\frac{\cos 2\alpha - \cos 2x}{2}.$$

Now, $\cos 2x$ cannot be $> \cos 2\alpha$; and when $\cos 2x = \cos 2\alpha$, $y = -\infty$; hence there is no finite minimum. But when $\cos 2x = -1$, y is a max., and $= \log\frac{1 + \cos 2\alpha}{2} = \log\cos^2\alpha = 2\log\cos\alpha$; and this occurs when

; or $(2n + 1)\frac{\pi}{2}$. [See Art. 184 (4).]

EXAMPLES XXVIII.

1. Find the max. and min. values (stating which) of the following functions:—

(1) $2 + 3x^2$. (2) $a - bx^2$. (3) $1 + (x - 1)^2$.

(4) $1 + \cos x$. (5) $a - b \sin^2 x$. (6) $a + b \operatorname{cosec}^2 x$.

(7) $a - b \operatorname{cosec} x$. (8) $a^x + b$. (9) $1 - 2^x$.

(10) $a^{x-a} - a$. (11) $\sqrt{4 - x^2}$. (12) $\sqrt{a^2 + x^2}$.

(13) $1 - \dfrac{1}{2 - \sin x}$

2. Find the max. and min. values (stating which) of y in the following:—

(1) $y = \log (a \pm \sqrt{a^2 - x^2})$. (2) $a^2y^2 + b^2x^2 = a^2b^2$.

(3) $y = 2 \sin^2 x + 8 \cos^2 x$. (4) $y = \sin x + \cos x$.

(5) $y = 3 \sin x + 4 \cos x$. (6) $y = \sin (x + 2\alpha) + \sin (x + 2\beta)$

(7) $y = a + \dfrac{\quad}{\log x}$. (8) $y = \tan^{-1} \sqrt{1 - x^2}$.

(9) $y = \dfrac{\sin x - \sin \alpha}{\sin x}$. (10) $y = \sin^3 x \cos^3 x$.

(11) $y = \sqrt{\dfrac{\sin (x - \alpha) \cos (x + \alpha)}{\sin x \cos x}}$.

ANSWERS.

1. (1) Min., 2. (2) Max., a. (3) Min., 1. (4) Max., 2; min., 0.
(5) Min., $a-b$; max., a. (6) Min., $a+b$. (7) Max., $a-b$; min., $a+b$.
(8) Min., b. (9) Max., 1. (10) Min., $-a$. (11) Max., $+2$; min., -2.
(12) Min., $+a$; max., $-a$. (13) Min., 0; max., $\frac{2}{3}$.

2. (1) Max., $\log 2a$. (2) Max., b; min., $-b$. (3) Max., 8; min., 2.
(4) Max., $\sqrt{2}$; min., $-\sqrt{2}$. (5) Max., 5; etc. (6) Max., $2 \cos (\alpha - \beta)$; etc.
(7) Max., a, when $x = 0$; min., a, when $x = \infty$. (8) Max., $\frac{\pi}{4}$; min., $-\frac{\pi}{4}$.
(9) Max., $1 - \sin \alpha$; min., $1 + \sin \alpha$. (10) Max., $\frac{1}{8}$; min., $-\frac{1}{8}$.
(11) Max., $\cos \alpha \sim \sin \alpha$; min., $-(\cos \alpha \sim \sin \alpha)$; min., $\cos \alpha + \sin \alpha$; max., $-(\cos \alpha + \sin \alpha)$.

186. The following two methods are applicable to algebraical expressions of the form $\dfrac{ax^2 + 2hx + b}{a'x^2 + 2h'x + b'}$, which includes very important particular cases, in which one or more of the coefficients a, b, etc., is zero. For example, if $a' = h' = 0$, and $b' = 1$; we get $ax^2 + 2hx + b$.

First Method.

187. From the formula $(x + y)^2 = (x - y)^2 + 4xy$ . . (1)
we have four cases, x and y being supposed variable quantities.

(A) If $x + y = 2a$, a constant, we have

$$4a^2 = (x - y)^2 + 4xy,$$

$\therefore$ $4xy < 4a^2$, since $(x - y)^2$ is always $+^{ve}$, unless $x = y$, in which case xy has its *maximum* value, viz. a^2; and $x = y = a$.

(B) If $x - y = 2a$, $\therefore$ $(x + y)^2 = 4a^2 + 4xy$;

$$\therefore\ 4xy > -4a^2 \text{ unless } x = -y = a,$$

in which case xy is a *minimum*, viz. $-a^2$.

NOTE.—In both of these cases a may be $+^{ve}$ or $-^{ve}$.

(C) If $xy = a^2$, $\therefore$ $(x + y)^2 = (x - y)^2 + 4a^2$,

$$\therefore\ (x + y)^2 > 4a^2, \text{ unless } x = y;$$

$\therefore$ $x + y$ is *numerically* $> 2a$, *i.e.* it cannot lie between $+ 2a$ and $- 2a$. Hence $+ 2a$ will be a *minimum*, and $- 2a$ a *maximum* value of $x + y$; which occur when $x = y = \pm a$ respectively.

(D) If $xy = a^2$, $\therefore$ $(x - y)^2 = (x + y)^2 - 4a^2$;

$$\therefore\ (x - y)^2 > -4a^2, \text{ unless } x + y = 0.$$

But if $x + y = 0$, then $(x - y)^2$ is $-^{ve}$, giving imaginary results. Hence the minimum value of $(x - y)^2$ is 0; and therefore $x - y$ may have any value, $+^{ve}$ or $-^{ve}$, i.e. *it has no finite max. or min. value.*

188. The cases (A) and (C) are the two to be remembered, for (B) can be deduced from (A), thus :—

Put $y = -z$; then from (A)

$$x - z = 2a, \quad \therefore 4x(-z) < 4a^2, \text{ or } -xz < a^2.$$

Hence $xz > -a^2$, except when $x = y$, i.e. $x = -z = a$, in which case the minimum of xz is $-a^2$.

189. We now give a formal statement of cases (A) and (C) :—

(I) *If* x + y = 2a, *a constant, the maximum value of* xy *is* a^2 (*the square of half the sum*), x *being equal to* y.

(II) *If* xy = a^2, *a constant, the minimum value of* x + y *is* + 2a (*twice the square root of the product*) ; *and the maximum is* − 2a ; x *being equal to* y.

Both of these results follow from the fact that the A. mean of x and y is greater than the G. mean, except when $x = y$, *in which case they become equal.* And if $\frac{x+y}{2} = \sqrt{xy}$, we have (I) $xy = \left(\frac{x+y}{2}\right)^2$; (II) $x+y=2\sqrt{xy}$. This may be a help to the memory.

190. Examples worked on Formula (I).

Ex. 1. The max. rectangle of given perimeter is a square.

For, if x and y be the sides, and $4a$ the given perimeter, then

$$2x + 2y = 4a, \text{ or } x + y = 2a,$$

$\therefore$ the area xy is a max. when $x = y = a$.

Ex. 2. $y = 6 + x - x^2$.

Since $y = (x + 2)(3 - x)$, and $(x + 2) + (3 - x) = 5$, a constant,

$\therefore$ by the rule, the max. $= (\frac{1}{2}.5)^2 = \frac{25}{4}$.

Ex. 3. $y = x^2 - x - 6 = (x + 2)(x - 3)$.

Here we might use (B) since $(x + 2) - (x - 3) = 1$, a constant.

But we can use (A) or (I) thus:—

$-y = (x + 2)(3 - x)$, the max. of which is (by Ex. 2) $\frac{25}{4}$.

Hence the *min.* of y is $-\frac{25}{4}$. [Art. 184 (2).]

Ex. 4. $y = \dfrac{1}{(a + x)(b - x)}$.

Let $z = (a + x)(b - x)$; then, since $(a + x) + (b - x) = a + b$ (const.),

$$\therefore \text{ max. of } z = \left(\frac{a + b}{2}\right)^2;$$

$$\therefore \textit{ min. } \text{of } y = \left(\frac{2}{a + b}\right)^2.$$

Ex. 5. $y = (ax + b)(c - dx)$.

To eliminate x from $ax + b$ and $c - dx$, we have

$$d(ax + b) + a(c - dx) = bd + ac \text{ (const.)},$$

$$\therefore \text{ max. of } ad(ax + b)(c - dx) \text{ is } \left(\frac{bd + ac}{2}\right)^2;$$

$$\therefore \text{ (1) if } ad \text{ is } +^{\text{ve}}, \text{ max. of } y \text{ is } \frac{(bd + ac)^2}{4ad};$$

$$\text{(2) if } ad \text{ is } -^{\text{ve}}, \textit{ min. } \text{of } y \text{ is } \frac{(bd + ac)^2}{4ad},$$

since we are dividing down by a negative quantity.†

Ex. 6. $y = x^2 + px + q$.

Since $y = \left(x + \frac{p}{2}\right)^2 + \frac{4q - p^2}{4}$, the min. of y is evidently $\frac{4q - p^2}{4}$

Otherwise, if $y = (x - \alpha)(x - \beta)$, α and β being the roots of $x^2 + px + q = 0$, then $-y = (x - \alpha)(\beta - x)$ the max. of which is, by (1), $\left(\frac{\beta - \alpha}{2}\right)^2$

$, p^2 - 4q$

$$\therefore \text{ the min. of } y \text{ is } \frac{4q - p^2}{4}.$$

Ex. 7. $y = x(x + 1)(x + 2)(x + 3) = (x^2 + 3x)(x^2 + 3x + 2)$
$= z(z + 2)$, if $z = x^2 + 3x$.

$\therefore -y = -z(z + 2)$; and since $-z + (z + 2)$ is constant, the max. of $-y$ is 1; hence the min. of y is -1.

This method does not, however, give the other max. or min. values, which will hereafter be shown to exist.

Ex. 8. $y = \log x + \log(2 - x) = \log x(2 - x)$. The max. of $x(2 - x)$ is 1; hence the max. of y is 0. [Art. 184 (4).]

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† Thus, if the max. of ax is b, a being $-^{\text{ve}}$, then $ax < b$, $\therefore x > \frac{b}{a}$ (since a is $-^{\text{ve}}$); hence the *min.* of x is b/a. [See Art. 184 (2).]

EXAMPLES XXIX.

1. Find, *by formula* (I), the max. and min. values of:—

(1) $\sin^2\theta\cos^2\theta$.

(2) $x(a-x)$.

(3) $(2-x)(x+4)$.

(4) $\sqrt{(a-x)(a+x)}$.

(5) $(x-1)(x-2)$.

(6) x^2-5x+6.

(7) $(2x-3)(2-3x)$.

(8) $\dfrac{1}{(a-bx)(c-dx)}$.

(9) $(x-a)x(x+a)(x+2a)$.

(10) $\dfrac{1}{1-x}+\dfrac{2}{2x+3}$.

(11) $\dfrac{a}{ax+b}-\dfrac{c}{cx+d}$, $[a/b>c/d$, and a, b, c, d all $+^{ve}]$.

(12) $x\sqrt{a^2-x^2}$.

(13) $\left(x+\dfrac{1}{x}\right)\sqrt{a^2-\left(x-\dfrac{1}{x}\right)^2}$.

(14) $1\div\sqrt{\log_{10}(2-\sin x)(3\sin x+1)}$, given $\log 7=0{\cdot}845$; $\log 3=0{\cdot}477$; $\log 2=0{\cdot}301$.

(15) $\sqrt{(\cos\theta-\cos\alpha)(\cos\beta-\cos\theta)}$, $(\alpha>\beta)$.

(16) $\sqrt{(a+2b\sin^2\theta)(2b\cos^2\theta-a)}$, $(a<b)$.

2. Show that $(4+\cos x)(1-\cos x)$ has 0 for a minimum, but that the maximum as determined by (I) is imaginary. [See Ex. XXXI, 2 (8), Answer.]

3. A given straight line is divided into two parts, find when the rectangle contained by the two parts is a maximum.

4. Show, by means of formula (I), that the maximum rectangle inscribable in a given circle is a square.

5. If (x, y) be a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, find the maximum value of xy. Hence, find the area of the maximum rectangle inscribable in an ellipse.

6. If r, r' be the focal distances of a point on an ellipse, find the maximum value of rr'. Interpret this.

7. P is a point on the base BC of a given isosceles triangle, vertex A. PM, PN are drawn parallel to AB, AC respectively. Find the area of the parallelogram $AMPN$ when it is a maximum.

8. Find the area of the maximum rectangle two of whose sides are on the axes of co-ordinates, and whose other two sides meet on the line $3x + 5y = 60$.

9. Show that in the cardioide $r = a(1 - \cos\theta)$, the maximum value of x is $a/4$ [see Ans.].

ANSWERS.

1. (1) Max., $\frac{1}{4}$; min., 0. (2) Max., $a^2/4$. (3) Max., 9.

(4) Max., a; min., $-a$. (5) Min., $-\frac{1}{4}$. (6) Min., $-\frac{1}{4}$. (7) Max., $\frac{25}{24}$.

(8) Max., $-4bd/(ad - bc)^2$, if bd is +ve. (9) Min., $-a^4$. (10) Min., $\frac{8}{5}$.

(11) Max., $\dfrac{-4ac}{(ad - bc)^2}$. (12) Max., $\dfrac{a^2}{2}$; min., $-\dfrac{a^2}{2}$.

(13) Max., $\dfrac{a^2 + 4}{2}$; etc. (14) Min., 1·3; max., − 1·3.

(15) Max., $\sin\dfrac{\alpha + \beta}{2}\sin\dfrac{\alpha - \beta}{2}$; etc. (16) Max., b; etc.

3. When square. **5.** $ab/2$; $2ab$. **6.** a^2; $rr' = CD^2$, CD being conjugate to CP.

7. $\frac{1}{4}b^2 \sin A$, where $b = AB$; or, half the triangle. **8.** 60.

9. $r + a\cos\theta =$ const.; use (I).

191. Examples worked on Formula (II).

Ex. 1. $y = x + \dfrac{1}{x}$.

Here $x \cdot \dfrac{1}{x} = 1$ (constant); ∴ the max. and min. values occur when $x = \dfrac{1}{x}$, i.e. $x = \pm 1$. Hence we have $+2$ a min., and -2 a max.

Ex. 2. $y = \dfrac{(x - 1)(x - 5)}{x + 2}$.

Put $x + 2 = z$, whence $x = z - 2$.

$$\therefore y = \frac{(z - 3)(z - 7)}{z} = \frac{z^2 - 10z + 21}{z} = z + \frac{21}{z} - 10.$$

Now, the max. and min. values of $z + \dfrac{21}{z}$ are given by $\pm$(2 sq. root of product), *i.e.* $\pm 2\sqrt{21}$.

∴ we have $+ 2\sqrt{21} - 10$ a min., and $- 2\sqrt{21} - 10$ a max.

Ex. 3. $y = \dfrac{(x-a)(x-b)}{x-c}$.

Put $x - c = z$; $\quad \therefore y = \dfrac{\{z+(c-a)\}\{z+(c-b)\}}{z}$

$$= \text{const.} + z + \frac{(c-a)(c-b)}{z}.$$

Now, if $(c-a)(c-b)$ is $+^{ve}$, the max. and min. values occur when

$$z = \frac{(c-a)(c-b)}{z}; \text{ or } z = \pm\sqrt{(c-a)(c-b)}.$$

Taking the upper sign, y becomes

$$\frac{\{\sqrt{(c-a)(c-b)}+(c-a)\}\{\sqrt{(c-a)(c-b)}+(c-b)\}}{\sqrt{(c-a)(c-b)}};$$

and dividing the factors of the numerator by $\sqrt{c-a}$ and $\sqrt{c-b}$ respectively we get $(\sqrt{c-a}+\sqrt{c-b})^2$ for the *min.*

Similarly, taking the lower sign, we obtain $(\sqrt{c-a}-\sqrt{c-b})^2$ for the *max.*

Ex. 4. $y = \dfrac{2x^2+5x+8}{x^2+2x+3} = 2 + \dfrac{x+2}{x^2+2x+3} = 2 + \dfrac{1}{v}$ say;

$$\therefore v = \frac{x^2+2x+3}{x+2} = x + \frac{3}{x+2} = x+2+\frac{3}{x+2} - 2.$$

For v we have $+2\sqrt{3}-2$ a min.; and $-2\sqrt{3}-2$ a max.

$\therefore$ for y we have $2 + \dfrac{1}{2\sqrt{3}-2}$ a *max.*; and $2 - \dfrac{1}{2\sqrt{3}+2}$ a *min.*

These reduce respectively to $\frac{1}{4}(9 \pm \sqrt{3})$.

Note.—For the general case the method given below is preferable.

EXAMPLES XXX.

1. *Use formula* (II) to find the max. and min. values of:—

(1) $x + \dfrac{a^2}{x}$.

(2) $x + 1 + \dfrac{4}{x+1}$.

(3) $\sin\theta + \dfrac{1}{4\sin\theta}$.

(4) $\sec\theta \operatorname{cosec}\theta\ [= \tan\theta + \cot\theta]$.

(5) $e^x + e^{2a-x}$.

(6) $\cosh x$.

(7) $x + \dfrac{1}{x-2}$.

(8) $9x + 1 + \dfrac{2}{2x-3}$.

(9) $\dfrac{(x+a)(x+b)}{x}$. (10) $ax + b + \dfrac{bd}{cx - d}$.

(11) $3 - x - \dfrac{1}{x}$. (12) $\dfrac{(1-3x)(x-3)}{x}$.

(13) $\dfrac{(2x-1)^2}{x+3}$. (14) $\sqrt{\left(\dfrac{1}{x} - a\right)\left(x - a\right)}$.

(15) $\sqrt{(e^x - e^a)(e^{-x} - e^a)}$. (16) $\sqrt{(\tan x - \tan \alpha)(\cot x - \tan \alpha)}$.

(17) $\dfrac{x}{1+x^2}$. (18) $\dfrac{x-2}{x^2}$.

(19) $\sqrt{-\dfrac{(3x+8)(5x+4)}{2x+7}}$. (20) $e^{\cosh x}$.

2. Show that $x - \dfrac{1}{x}$ has no finite max. or min. values.

3. A rectangle has a given area, find when its perimeter is a minimum.

4. Use any of the previous methods to find the max. or min. values of:—

(1) $\sqrt{\tan \theta} + \sqrt{\cot \theta}$. (2) $\log(2x-3) + \log(2-3x)$.

(3) $\log(2x-3) + \log(3x-2)$. (4) $\dfrac{x-1}{x} \quad \dfrac{x}{x-3}$.

(5) $\dfrac{x-1}{x-2} - \dfrac{x-3}{x-4}$. (6) $\dfrac{x}{ax^2+bx+c}$.

(7) $\dfrac{(x-1)(x-4)}{(x-2)(x-3)}$.

5. Find the max. or min. values of $\sqrt{\dfrac{(x-p)(x-q)}{x-r}}$, stating the condition.

6. Show that the maximum value of $\dfrac{x-a}{x-b} - \dfrac{x-c}{x-d}$ is $\dfrac{(\sqrt{a-b} - \sqrt{c-d})^2}{b-d}$, provided that $b > d$, and $(a-b)(c-d)$ is $+^{\text{ve}}$.

7. Show that $\dfrac{(x-a)(x-c)}{(x-b)(x-d)}$ only admits of max. and min. values, when $(a-b)(b-c)(c-d)(d-a)$ is $+^{\text{ve}}$.

ANSWERS.

NOTE.—The *min.* will be written first, and the *max.* second.

1. (1) $\pm 2a$. (2) ± 4. (3) ± 1. (4) ± 2. (5) $\pm 2e^a$. (6) ± 1.

(7) 4, 0. (8) $20\frac{1}{2}$, $8\frac{1}{2}$. (9) $(\sqrt{a} \pm \sqrt{b})^2$. (10) $\dfrac{(\sqrt{bc} \pm \sqrt{ad})^2}{c}$.

(11) 5, 1. (12) 16, 4. (13) 0, -56. (14) $1 \pm a$. (15) $1 \pm e^a$.

(16) $1 \pm \tan a$. (17) $\mp \frac{1}{2}$. (18) Max., $\frac{1}{3}$. (19) 7, 2. (20) e, $1/e$.

3. When square.

4. (1) ± 2. (2) Max., log 25/24. (3) No real max. or min.

(4) None. (5) Min., 2. (6) $1/(b \mp 2\sqrt{ac})$. (7) Min., 9.

5. $\sqrt{r-p} \pm \sqrt{r-q}$, if $(r-p)(r-q)$ is $+^{ve}$.

Second Method.

192. Let $u = \dfrac{ax^2 + 2hx + b}{a'x^2 + 2h'x + b'}$ . . . . (1)

(which we have stated includes simpler forms) be the expression whose max. and min. values we require to find, x being the only variable.

Before giving the method, we may mention that the expression

$$\frac{ax^2 + 2hxy + by^2}{a'x^2 + 2h'xy + b'y^2}, \quad \ldots \ldots \quad (2)$$

involving two variables, can be put in the previous form, since the numerator and denominator are *homogeneous and of the same degree.*

Thus, put $x = yz$ (z being the *ratio* of x to y) and it becomes $\dfrac{az^2 + 2hz + b}{a'z^2 + 2h'z + b'}$, the same form as (1). Hence we can find the max. and min. values of (2) as well as of (1).

193. Simpler Case.

Let $u = ax^2 + 2hx + b$.

Solve for x in terms of u. Then

$$ax^2 + 2hx + (b - u) = 0;$$

whence $x = \dfrac{-h \pm \sqrt{h^2 - a(b-u)}}{} = \dfrac{-h \pm \sqrt{h^2 - ab + au}}{}$

Now, x is real only if $h^2 - ab + au >$ or $= 0$.

(i) Let a be $+^{ve}$, then $u >$ or $= \dfrac{ab - h^2}{a}$; *i.e.* the *min.* value of u for real values of x is $\dfrac{ab - h^2}{a}$.

(ii) Let a be $-^{ve}$, then $au >$ or $= ab - h^2$.

But dividing down by the $-^{ve}$ quantity a, the sign of the inequality is changed,

$$\therefore\ u < \text{or} = \frac{ab - h^2}{a}$$

$$\therefore \text{ the } max. \text{ value of } u \text{ is } \frac{ab - h^2}{a}.$$

(iii) Let $a = 0$, then there are no finite maxima or minima. The expression, in fact, becomes $2hx + b$.

Otherwise :— $ax^2 + 2hx + b = a\left\{\left(x + \dfrac{h}{a}\right)^2 + \dfrac{ab - h^2}{a^2}\right\}$

$= a\left(x + \dfrac{h}{a}\right)^2 + \dfrac{ab - h^2}{a}, >$ or $< \dfrac{ab - h^2}{a}$, according as a is $+^{ve}$ or $-^{ve}$.

In the former case we get a minimum when $x = -\dfrac{h}{a}$, and in the latter case a maximum.

194. General Case.

Let $u = \dfrac{ax^2 + 2hx + b}{a'x^2 + 2h'x + b'}$; and solve for x in terms of u. Then we have $(a - a'u)x^2 + 2(h - h'u)x + b - b'u = 0$;

giving $x = \dfrac{-(h - h'u) \pm \sqrt{\{(h - h'u)^2 - (a - a'u)(b - b'u)\}}}{a - a'u}$. (1)

As above, for real values of x we must have

$$(h - h'u)^2 - (a - a'u)(b - b'u) > \text{or} = 0,$$

i.e. $(h'^2 - a'b')u^2 + (ab' + a'b - 2hh')u + (h^2 - ab) >$ or $= 0$.

If α, β be the roots which make the above quadratic expression vanish ($\alpha > \beta$, suppose), we have

$$(h'^2 - a'b')(u - \alpha)(u - \beta) > \text{or} = 0.$$

(i) If $h'^2 - a'b'$ be $+^{ve}$ [*i.e.* if $a'x^2 + 2h'x + b'$ can be resolved into two *real* linear factors],

$$u - \alpha \text{ and } u - \beta \text{ must be both } +^{ve} \text{ or both } -^{ve},$$

i.e. $u > \alpha$ or $< \beta$, for real values of x; *i.e.* it cannot lie between α and β.

Hence, $u = \alpha$ is a *min.*, and $u = \beta$ is a *max.*

Solving the quadratic in u, α and β will be found to be respectively

$$\frac{-(ab'+a'b-2hh') \pm \sqrt{\{(ab'-a'b)^2-4(ah'-a'h)(hb'-h'b)\}}}{2(h'^2 - a'b')} \quad . \quad (2)$$

(ii) If $h'^2-a'b'$ be $-^{ve}$ [*i.e.* if $a'x^2 + 2h'x + b'$ cannot be resolved into real linear factors], then $(u - \alpha)(u - \beta)$ must be $-^{ve}$; *i.e.* $u < \alpha$ and $> \beta$; *i.e.* it must lie between α and β.

Hence $u = \alpha$ is a *max.*, and $u = \beta$ is a *min.*

NOTE.—Here α is to be taken as the *greater* root; but, as a matter of fact, when $h'^2 - a'b'$ is $-^{ve}$, the greater root is that in which the *lower* sign is taken in (2) That is to say, the α and β in case (ii) are respectively the β and α in case (i).

(iii) If $h'^2 - a'b' = 0$ [*i.e.* if $a'x^2 + 2h'x + b'$ is a perfect square, or a simple multiple of one], then α and β are infinite, and there are no finite maxima or minima.

195. Value of x.—Since, for the max. and min. values of u, the quantity under the radical sign in (1) vanishes, we have

$$x = -\frac{h - h'u}{a - a'u}, \text{ or } (a - a'u)\,x + (h - h'u) = 0 \ . \qquad (3)$$

Eliminate u between this and the original equation

$$(a - a'u)x^2 + 2(h - h'u)x + (b - b'u) = 0\text{; thus}$$

From (3), $(a - a'u)x^2 + (h - h'u)x \qquad = 0$;

$\therefore$ by subtraction $(h - h'u)x + (b - b'u) = 0 \quad . \quad . \quad . \quad (4)$

From (3) and (4) we have respectively

$$ax + h = (a'x + h')u,$$
$$\text{and } (h'x + b')u = hx + b\text{;}$$
$$\therefore (ax + h)(h'x + b') = (a'x + h')(hx + b),$$

$$\text{or } (ah' - a'h)x^2 + (ab' - a'b)x + (hb' - h'b) = 0,$$

$$\text{giving } x = \frac{-(ab' - a'b) \pm \sqrt{\{(ab' - a'b)^2 - 4(ah' - a'h)(hb' - h'b)\}}}{2(ah' - a'h)}.$$

[Compare with (2).]

*196. Exceptions.

It only remains to consider the cases in which α, β are real and equal, or imaginary. In the above, of course, they are supposed to be real and unequal.

(*a*) If $\alpha = \beta$, the condition for real values of x becomes

$$(h'^2 - a'b')(u - \alpha)^2 > \text{or} = 0.$$

But in case (i) this is always satisfied, and in case (ii) it is only satisfied when $u = \alpha$. Hence, in case (i) there is neither a max. nor min.; and in case (ii) we get $u = \alpha$ (a constant) as the only possible value of u for which x is real, which points to the original expression being independent of x, i.e. $a/a' = h/h' = b/b'$. This condition causes the expression under the radical sign in (2) to vanish.

(*b*) If α and β are imaginary, then, remembering that these are the values of u which are supposed to make the expression under the radical sign in (1) vanish, it follows that the latter cannot vanish, and, therefore, for all values of u will be either $+^{ve}$ or $-^{ve}$. In the first case, as in (*a*), there is no restriction to the value of u, *i.e.* there are no max. or min. values; and the second case is inadmissible, as we should then have the square root of a negative quantity, so that no value of u could be found to correspond to a real value of x, which is obviously not the case if the coefficients a, h, b, a', h', b' are real.

197. Examples worked.

Ex. 1. $u = \dfrac{(x+7)(x+2)}{(x-1)(x-5)} = \dfrac{x^2 + 9x + 14}{x^2 - 6x + 5}.$

Then $(u - 1)x^2 - (6u + 9)x + 5u - 14 = 0$ . . . . (1)
and for real values of x we have

$$(6u + 9)^2 - 4(u - 1)(5u - 14) \text{ must be } +^{ve};$$

or

$$16u^2 + 184u + 25 > 0.$$

Solving as a quadratic equation, the max. and min. values are given by

$$u = -0{\cdot}14 \text{ or } -11{\cdot}36.$$

Now, since for real values of x,

$$16u^2 + 184u + 25, \text{ i.e. } 16(u + 0{\cdot}14)(u + 11{\cdot}36) > 0,$$

it follows that u must be algebraically $> -0{\cdot}14$ or $< -11{\cdot}36$.

$$\therefore\ -0{\cdot}14 \text{ is a } \textit{min.}, \text{ and } -11{\cdot}36 \text{ is a } \textit{max.}$$

Also from (1), $x = \dfrac{6u+9}{2(u-1)}$ [since the surd has vanished] $= -3{\cdot}7$ and $2{\cdot}4$ respectively.

Ex. 2.

$$u = a^2\cos^4\theta + b^2\sin^4\theta\ [a > b] = \frac{a^2}{4}(1+\cos 2\theta)^2 + \frac{b^2}{4}(1-\cos 2\theta)^2$$

$$= \tfrac{1}{4}\left[(a^2+b^2)\cos^2 2\theta + 2(a^2-b^2)\cos 2\theta + a^2 + b^2\right]$$

$$= \tfrac{1}{4}\left[lx^2 + 2mx + l\right] \text{ say, where } l \equiv a^2+b^2,\ m \equiv a^2-b^2, \text{ and } x \equiv \cos 2\theta,$$

$$= \frac{1}{4l}\left\{(lx+m)^2 + l^2 - m^2\right\};$$

and since $4l$ *is* $+^{ve}$, u is a min. when $lx + m = 0$, and its value

$$= \frac{l^2-m^2}{4l} = \frac{(a^2+b^2)^2-(a^2-b^2)^2}{4(a^2+b^2)} = \frac{a^2b^2}{a^2+b^2}.$$

To find the value of $\cos\theta$ corresponding to this value of u, we have

$$lx + m = 0, \quad \therefore x = -\frac{m}{l} = -\frac{a^2-b^2}{a^2+b^2},$$

$$\text{i.e. } \cos 2\theta = 2\cos^2\theta - 1 = -\frac{\quad}{a^2+b^2}.$$

$$\therefore \cos^2\theta = \frac{b^2}{a^2+b^2} \text{ and } \cos\theta = \frac{b}{\sqrt{a^2+b^2}},$$

which is less than 1, so that θ is real. θ evidently $= \tan^{-1}\dfrac{a}{b}$.

Again, it is evident that u must have some max. value. To find it, we have

$$u = \frac{1}{4l}\left\{(lx+m)^2 + l^2 - m^2\right\} = \frac{(lx+m)^2}{4l} + \frac{l^2-m^2}{4l}.$$

Since $4l$ *is* $+^{ve}$, u will be a max. when $lx + m$ is a max.; *i.e.* when x or $\cos 2\theta$ is a max.; *i.e.* when $\cos 2\theta = 1$.

Hence, the required max. is $\dfrac{(l+m)^2 + l^2 - m^2}{4l} = \dfrac{l+m}{2} = a^2$.

EXAMPLES XXXI.

1. Find the max. and min. values, by the second method, of:—

(1) $x + \frac{1}{x}$. (2) $x(a + x)$.

(3) $(x - 1)(x - 3)$. (4) $1 + \frac{2}{x} + \frac{3}{x^2}$.

(5) $\frac{1 - x^2}{1 + x^2}$. (6) $\frac{(1 + x)^2}{(1 - x)(1 + 2x)}$.

(7) $\frac{1 - 2x}{(1 - x)(1 + 2x)}$. (8) $\frac{1}{x - 2} - \frac{1}{4x + 3}$.

(9) $\frac{1}{ax + b} - \frac{1}{cx + d}$. (10) $\frac{a^2}{\sin^2 \phi} + \frac{b^2}{\cos^2 \phi}$.

2. Use *any* algebraical method to find the max. and min. values of:—

(1) $\log (x^2 + \sqrt{x^2 + 1})$. (2) $e^{a \operatorname{cosec} x}$.

(3) $x^2 \sqrt{a^4 - x^4}$. (4) $\sqrt{a + x} + \sqrt{a - x}$.

(5) $x + \sqrt{a^2 - x^2}$. (6) $3 - x - x^2$.

(7) $\frac{x - 1}{(x - 2)(x - 5)}$. (8) $(\sin x - 2)(8 - \sin x)$.

(9) $\frac{(x + a)(x + b)}{(x - a)(x - b)}$.

3. If $x^2 + y^2 = c^2$, find the max. and min. values of $x + y$ and $x - y$; (1) by using the identity $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$; (2) by turning to polars. Give a geometrical interpretation.

4. If $b^2x^2 + a^2y^2 = a^2b^2$, find the max. and min. values of $bx + ay$.

5. If $x + y = c$, find the minimum value of $x^2 + y^2$; interpret the result geometrically.

6. If $bx + ay = ab$, find the minimum value of $x^2 + y^2$; interpret the result geometrically.

7. If $x^2 + y^2 = c^2$, find the max. and min. values of $bx + ay$.

8. If $ax + by + cz = d$, use the identity

$$(ax + by + cz)^2 + (cy - bz)^2 + (az - cx)^2 + (bx - ay)^2 = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$$

to find the minimum value of $x^2 + y^2 + z^2$.

$\dfrac{\quad}{ax + b} \cdot \dfrac{\quad}{cx + d}$ has no max. or min. values unless a and c have different signs.

10. P is a point on the hypotenuse AB of a right-angled triangle, ABC; find when the sum of the squares of the distances from P on AC and BC is a minimum.

Answers.

Note.—The *min.* will be written first, and the *max.* second.

1. (1) ± 2. (2) Max., $a^2/4$. (3) Min., -1. (4) Min., $\frac{2}{3}$.
(5) ∓ 1. (6) $\frac{8}{9}$, 0. (7) None. (8) $-\frac{1}{11}, -\frac{9}{11}$.
(9) $(\sqrt{a} \mp \sqrt{c})^2/(bc - ad)$. (10) $(a \pm b)^2$

2. (1) Min., 0. (2) $e^{\pm a}(a, +^{ve})$. (3) $\mp \frac{1}{2}a^4$. (4) $\mp 2\sqrt{a}$.
(5) Putting $x = a \sin \theta$, we get $\mp a\sqrt{2}$. Or, $u^2 = a^2 + 2x\sqrt{a^2 - x^2}$.
(6) Max., $\frac{13}{4}$. (7) $-\frac{1}{9}, -1$.
(8) $\sin x = 5$, inadmissible. But $u = 9 - (5 - \sin x)^2$; $\therefore$ min. $= 9 - 6^2 = -27$; max. $= 9 - 4^2 = -7$. Similarly in Ex. XXIX., 2, min. $= 0$; max. $= 6$.
(9) $-(\sqrt{a} \mp \sqrt{b})^2/(\sqrt{a} \pm \sqrt{b})^2$.

3. $\mp c\sqrt{2}$ for either. The sum of the chords drawn from the ends of the diameter of a circle is a max. when they are equal.

4. $\mp ab\sqrt{2}$.

5. (*a*) $\dfrac{c^2}{2}$. (*b*) The min. rad. vector of the line $x + y = c$ is the $\perp^r$ from the origin.

6. (*a*) $\dfrac{a^2b^2}{a^2 + b^2}$, from which we could deduce Ex. 2 in the text, by putting $x = a \cos^2 \theta$, $y = b \sin^2 \theta$; (*b*) [see Ex. 5 (*b*)]. [*Otherwise*, see Ex. 7.]

7. $\pm c\sqrt{a^2 + b^2}$. Use the identity

$$(bx + ay)^2 + (ax - by)^2 = (a^2 + b^2)(x^2 + y^2).$$

8. $d^2/(a^2 + b^2 + c^2)$, when $x/a = y/b = z/c$.

10. When PC is $\perp^r$ to AB.

198. Geometrical Treatment.

There are two distinct methods which should be noticed :—

The *first method* consists in proving that a certain value of the magnitude under consideration is greater than all neighbouring values.

The *second method* is practically that of the Differential Calculus, and consists in asserting the fact that a max. or min. is a *stationary* value (*i.e.* at such a point two consecutive values of the magnitude are equal) † and drawing conclusions therefrom. It is more powerful than the first method, but it must previously be shown (though it is often obvious) that there *does* exist either a max. or min., for the stationary value might be due to a point of inflexion. [See Arts. 115 and 207.]

199. Examples on First Method.

Ex. 1. AB *is the section of a plane mirror*; CED *is the path of a ray of light from a candle placed at* C *to an eye at* D, *after reflection at* E; *the law of reflection being that the angles* CEA, DEB, *are equal. Prove that the path is a minimum.*

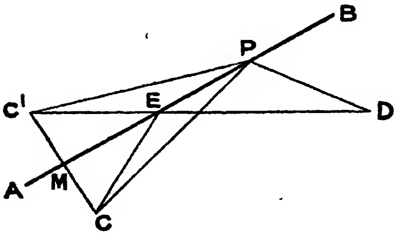

FIG. 26.

Let C' be the image of C, so that AB bisects CC' at right angles. Then, since the distance of any point P, on AB, from C or C' is the same, we have $CP + PD = C'P + PD$, which is obviously a min. when P is at E, where $C'ED$ is a straight line. And in this case we have the law of reflection obeyed, viz. that the angles AEC, BED are equal.

Cor.—If CD be $\parallel$ to AB, then CED, CPD will be two triangles of equal area, of which CED is isosceles, since CE is now equal to ED. Hence, of all triangles of given area and base, the isosceles triangle has the least perimeter.

NOTE.—An ellipse, foci C and D, can be drawn to touch AB in E, since $\angle CEA = \angle BED$.

† This is equivalent to the condition $dy/dx = 0$, given in Art. 113.

Ex. 2. If the mirror in Ex. 1 be a cylinder of which AB is a right section as in the figure; then by drawing an ellipse to touch AB in E, the law of reflection is obeyed, as may be seen by drawing the tangent FEG.

If P be a point on AB; join CP, cutting the ellipse in Q, and join QD, PD.

Then

$PQ + PD > QD$; add CQ.

$\therefore CP + PD > CQ + QD$,

i.e. $> CE + ED$,

which is therefore a minimum.

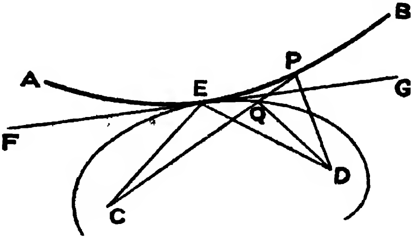

FIG. 27.

NOTE 1.—If, however, the curve AB be concave to C and D, such that it falls *within* the ellipse, then $CE + ED$ will be a maximum.

NOTE 2.—If the curve is the ellipse itself, then any point whatever on the curve satisfies the condition.

Ex. 3. AB *and* AC *are two given straight lines, and* O *is a given point. Through* O *draw a straight line* DOE, *cutting off an isosceles triangle. Prove that the rectangle* DO.OE *is a minimum,* i.e. *less than the same thing for any other line through* O.

Draw a circle touching AB, AC in D and E. Draw $PROSQ$, some other line through O.

Then, since $PO.OQ > RO.OS > DO.OE$ [Euc. III. 35], the proposition follows.

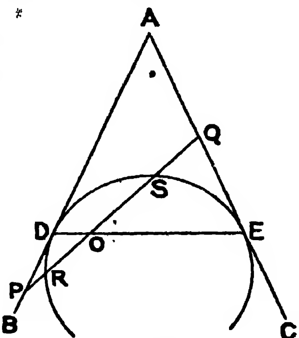

FIG. 28.

200. Examples on the Second Method.—The three preceding examples can be worked by this method; we shall take the last two, in a different form, however.

Ex. 1. *Given a curve* AB, *convex to* C *and* D, *two points in its plane. Find a point* P *on it such that* CP + PD *is a minimum.*

Let Q and Q' be two points indefinitely close together, one on each side of the point required. Then, since $CP + PD$ is slightly less than either $CQ + QD$ or $CQ' + Q'D$, it will be possible to choose the points

Q, Q' so that the last two sums are equal. Hence, supposing this done, we have

$$CQ + QD = CQ' + Q'D;$$

$\therefore$ Q and Q' are two near points on an ellipse, foci C and D.

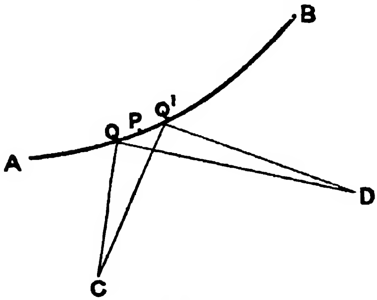

Fig. 29.

In the limit, the chord QQ' becomes a tangent, both to the given curve and the ellipse; hence P can be found by drawing an ellipse, foci C and D, to touch the curve in P, as in Ex. 2 above.

Ex. 2. O *is a given point between* AB *and* AC: *to draw a straight line through* O *such that the rectangle under the segments cut off by* AB, AC *shall be a minimum.*

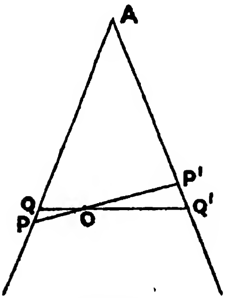

Fig. 30.

Let POP', QOQ' be two very near positions, one on each side of that required.

Then, by reasoning similar to that in the preceding example, $PO.OP' = QO.OQ'$;

$\therefore$ P, Q, P', Q', are concyclic; so that, when in the limit P and Q coincide, AB, AC become tangents to the circle. Hence, if Q, Q' be the final positions, $AQ = AQ'$; which gives the construction, as in Ex. 3 above.

201. Further Examples by either Method.

Ex. 3. *Of all figures bounded by a perimeter of given length, the circle has the maximum area.*

First, we shall show that if two sides, BC, CA, of a triangle be given,

the area is a maximum when C is a right angle. For, make BC the base; then the altitude is evidently a maximum when CA is at right angles to BC; hence, since the base does not vary, the area will be a maximum at the same time.

Next, let $ACBD$ be the perimeter of the given figure, and let ACB be one half of it. Take any point, C, on the perimeter, and suppose the segments on AC, CB to be *rigidly* attached to each line, and movable with it, C being a loose joint,—then, if the angle ACB be not a right angle, we can, by what has gone before, increase the area of the triangle (and hence, that of the curvilinear figure ACB) by opening out or closing in—as the case may be—the angle at C until it is a right angle. We then take another point on the circumference and do the same thing. By repeating the operation *ad infinitum*, we shall arrive at the semicircle, since the angle in a semicircle is a right angle.

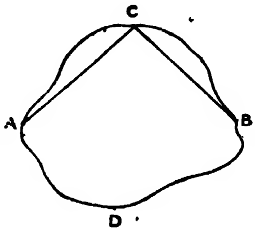

FIG 31.

Also, since the lower part of the perimeter, ADB, is equal to the upper part, it follows that, by keeping the semicircle ACB fixed, and making ADB into another semicircle, we get a maximum area in this case also.

Hence the whole figure, which is now a circle, has the maximum area for the given perimeter.

The proof given above, as also that of the theorem which follows, appears to have been given first by Robert Simson (1747).

Ex. 4. *Of all quadrilaterals formed by four given straight lines, that which is cyclic has the max. area.*

For, starting with the cyclic quadrilateral and its circumscribed circle, suppose the four segments *rigidly* attached to each side and movable with it; also suppose A, B, C, D to be loose joints. Then, if the quadrilateral be distorted by a pressure at D and B say, the circle (but *not* the segments) will be distorted into another figure of equal perimeter. Hence its area will be lessened, by Ex. 3 above.

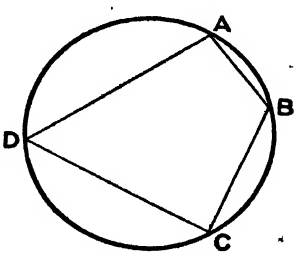

FIG. 32.

Take away the four segments which are unaltered in shape; hence the quadrilateral has been diminished in area; that is, the cyclic quadrilateral has the max. area.

This can also be proved by Trigonometry.

Ex. 5. OA, OB *are fixed lines,* C *a fixed point.*
Draw through C *a minimum straight line bounded by* OA, OB.

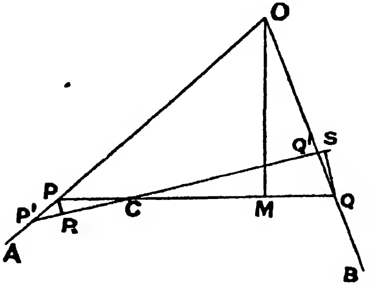

FIG. 33.

If PQ, $P'Q'$ be two consecutive positions such that $PQ = P'Q'$; we have, drawing PR, QS $\perp^r$ to $P'Q'$, $PC = RC$, and $QC = SC$, ultimately. And $P'R + RQ' = PQ = RC + CS$,

$$\therefore P'R = Q'S;$$

i.e. $PR \cot OP'Q' = QS \cot OQ'P'$. Or, since $OP'Q'$, $OQ'P'$ are ultimately equal to OPQ, OQP respectively,

$$\frac{\cot OPQ}{\cot OQP} = \frac{QS}{PR} = \frac{CQ}{CP}.$$

But if OM be drawn $\perp$ to PQ, $\dfrac{\cot OPQ}{\cot OQP} = \dfrac{PM}{QM}$;

$\therefore \dfrac{CQ}{CP} = \dfrac{PM}{QM}$; whence by comp^n. we get $MQ = CP$. [See Ex. Art. 94.]

Hence a hyperbola, asymptotes OA, OB, can be drawn passing through C and M.

The construction for finding PQ is thus:—On OC as diameter, describe a circle, and through C draw hyperbola, asymptotes OA, OB, cutting the circle in M. Then OMC is a right angle, and also by the properties of a hyperbola, $PC = MQ$. Hence PQ is the required minimum line through C.

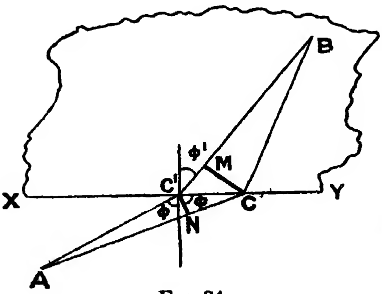

FIG. 34.

Ex. 6. A *and* B *are any two points respectively in air and a given transparent medium bounded by a plane,* XY.

Find the relation between the directions in either medium along which a ray of light must travel from A *in order to reach* B *in the shortest time;* v *and* v′ *being its velocity in the two media respectively.*

Let the plane of the paper be that containing A and B, and which is $\perp$

to the plane XY. Let ACB, $AC'B$ be two very near paths, such that the times along these paths are equal.

Drawing CM, $C'N \perp$ to $C'B$ and CA respectively, we have ultimately $AN = AC'$, $BM = BC$, to the 2nd order of infinitesimals. Hence, time along NC = time along $C'M$; or

$$NC \quad C'M. \quad , \frac{v}{v'} \quad \frac{NC}{C'M} \quad \frac{\sin\phi}{\sin\phi'} = \mu,$$

the refractive index of the second medium with respect to air.

Since v/v' is constant for all values of ϕ, it follows that $\sin\phi/\sin\phi'$ or μ is constant for all directions from A; and that, moreover, the law of refraction is based on "the principle of least time."

EXAMPLES XXXII.

1. Given the four sides of a parallelogram, find when its area is a maximum.

2. Given two sides of a triangle, when is its area a maximum?

3. Given one side of a triangle, and the opposite angle, find when the area is a maximum.

4. Show that the maximum triangle inscribed in (or the minimum triangle described about) a circle is equilateral.

5. Show (1) that the maximum rectangle, (2) the maximum quadrilateral, inscribed in a circle is a square.

6. Show that the maximum polygon inscribed in a circle is regular.

7. Given one side of a triangle and its area, prove that the opposite angle is a maximum when the triangle is isosceles.

8. A and B are two points outside a circle, and P is a point on the circumference. Find when the triangle is (1) a max., (2) a min. If B is inside the circle, how will the question be modified?

9. A and B are two points on opposite sides of a straight line CD, on which a point P is taken. Show that the max. of $AP \sim BP$ is equal to the distance between A and the image of B as reflected by CD. Show also that a hyperbola, foci A and B, can be drawn to touch CD at P in its final position.

10. AB is a fixed chord of a circle, and P is any point on the circumference. Find when $AP^2 + PB^2$ is (1) a max., (2) a min.; and (3) find when the rectangle $AP.PB$ is a max.

11. A and B are *any* two points, and P is a point on a circle. Find when $AP^2 + PB^2$ is (1) a max., (2) a min. What exceptions are there?

12. A quadrilateral is inscribed in a circle. Find when the sum of the rectangles contained by the pairs of opposite sides is a maximum.

13. From a given point O is drawn a straight line cutting two given straight lines in P and Q, both on the same side of O. Find when the rectangle $OP.OQ$ is a minimum.

14. A and B are two given points, and CD a given straight line; find a point P on CD such that the angle APB is a maximum.

15. Draw a tangent to a given circle such that the length intercepted between two given tangents is a minimum.

16. Given a point between two straight lines, draw through it a straight line cutting off a minimum triangle.

17. A straight line of given length slides between two straight lines; find when the triangle cut off is a maximum.

18. A chord of given length slides on an ellipse; find the positions at which max. and min. segments will be cut off.

19. Show that the minimum triangle inscribed about an ellipse has each side bisected at the point of contact with the curve.

20. Show that the maximum triangle inscribed in an ellipse has each side parallel to the tangent at the opposite vertex.

21. A straight line moves parallel to a given direction and cuts two non-intersecting curves in P and Q respectively. Show that when the intercepted portion PQ is a max. or min. the tangents at P and Q are parallel.

22. Find a condition that the distance between two points, one on each of two non-intersecting curves, is a max. or min.

23. A chord PQ of a circle, centre C, moves parallel to itself; show that the triangle CPQ is a max. when PCQ is a right angle. Hence, show that the max. triangle contained by a chord of a circle and the bounding radii is half the square on the radius.

24. From a fixed point A on a circle, a perpendicular AY is drawn to the tangent at P. Show that the triangle APY is a max. when the angle APY is that of an equilateral triangle.

ANSWERS.

1. When right-angled. 2. When right-angled. 3. When isosceles.

8. When the tangent at P is parallel to AB. If B is within, we have two maxima.

10. Bisect AB in M; then $AP^2 + PB^2 = 2AM^2 + 2MP^2 =$ a max. or min. with MP. Exception: when M is the centre of the circle.

11. See preceding answer.

12. When right-angled, by Euc. VI. D.

13. When OPQ cuts off an isosceles triangle.

14. Describe a circle through AB to touch CD in P.

16. The point bisects the line. **17.** When isosceles.

18. See preceding question; the chord is parallel to either axis.

22. The tangents at the points are both perpendicular to the line joining them.

202. Method of the Differential Calculus.

We have already given a method of dealing with ordinary functions in Chapter X. The following method is more thorough, as it enables us to deal with certain exceptions.

Let $f(x)$ be a finite and continuous function of x; that is, within the vicinity of the value $x = a$.

Then, by definition, Art. 182, $f(a)$ is a max. or min. according as $f(a+h) - f(a)$ and $f(a-h) - f(a)$ are *both* $-^{ve}$ or *both* $+^{ve}$, h being indefinitely small. This may be seen by drawing the graph $y = f(x)$; in the figure AM is $f(a)$, and $A'M'$ is $f(a \pm h)$.

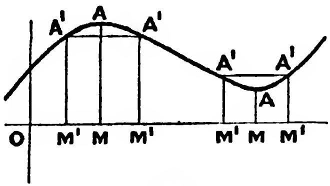

Fig. 35.

Now, by Taylor's Theorem,

$$f(a+h) - f(a) = +hf'(a) + \frac{h^2}{2!}f''(a+\theta h)\dagger \quad . \qquad (1)$$

† If the student has not read Lagrange's Theorem (Art. 137), he may use the ordinary formula

$$f(a+h) - f(a) = hf'(a) + \frac{h^2}{2!}f''(a) + \dots,$$

on the understanding that the series on the right is convergent. Lagrange's expression avoids this condition.

$$f(a-h)-f(a) = -hf'(a) + \frac{h^2}{2!}f''(a-\theta h) \quad . \quad . \quad (2)$$

But if h be small enough, the second terms in (1) and (2) on the right-hand side are insignificant compared with the first.

Hence $\left.\begin{aligned} f(a+h)-f(a) &= +hf'(a) \\ f(a-h)-f(a) &= -hf'(a) \end{aligned}\right\}$ approximately.

But this is contrary to the condition for a max. or min., which states that $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are to be of the *same* sign. The only possible way of reconciling these conditions is to make

$$f'(a) = 0.$$

In this case, expanding by Taylor's Theorem to an extra term,

$$f(a+h)-f(a) = \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a+\theta h)$$

$$f(a-h)-f(a) = \frac{h^2}{2!}f''(a) - \frac{h^3}{3!}f'''(a-\theta h).$$

And if h be small enough, the second terms on the right may be neglected, and the condition above mentioned is satisfied.

Hence *for a max.*, f''(a) *is* $-^{ve}$; *for a min.*, f''(a) *is* $+^{ve}$.

203. Exceptions.

If $f''(a)$ happens to vanish along with $f'(a)$, we have

$$f(a+h)-f(a) = +\frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f''''(a+\theta h)$$

$$f(a-h)-f(a) = -\frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f''''(a-\theta h).$$

Hence the condition for a max. or min. is that

$$f'''(a) = 0\,;$$

and it can easily be shown that for a $\left\{\begin{smallmatrix}\text{max.}\\ \text{min.}\end{smallmatrix}\right\}$ $f''''(a)$ must be $\mp^{ve}$, respectively.

If $f'(a), f''(a) \ldots f^{n-1}(a)$ all simultaneously vanish, but $f^n(a)$ does *not* vanish; then, if n be odd, there is neither a max. nor a min.; if n be even, there will be a $\left\{\begin{smallmatrix}\text{max.}\\ \text{min.}\end{smallmatrix}\right\}$ if $f^n(a)$ is $\mp^{ve}$.

Ex. 1. $y = (x-3)^2(x+1)+3$.

dy/dx or $f'(x) = 2(x-3)(x+1)+(x-3)^2 = (x-3)(3x-1) = 0$ for a max. or min.

$$\therefore x = 3 \text{ or } \tfrac{1}{3}.$$

$$d^2y/dx^2 \text{ or } f''(x) = 3x - 1 + 3(x-3) = 6x - 10.$$

$$\therefore f''(3) = +^{ve}, \text{ and } x = 3 \text{ gives a min.}$$

$$f''(\tfrac{1}{3}) = -^{ve}, \quad ,, \quad x = \tfrac{1}{3} \quad ,, \quad \text{max.}$$

Ex. 2. $y = x^5 - 3x^4 + 2$.

$y' = x^3(5x-12) = 0$, if $x = 0$ or $\frac{12}{5}$.

$y'' = x^2(20x-36) = 0$, if $x = 0$; but is $+^{ve}$ if $x = \frac{12}{5}$, giving a *min.*

$y''' = 60x^2 - 72x = 0$, if $x = 0$.

$y'''' = 120x - 72 = -^{ve}$, if $x = 0$.

$$\therefore x = 0 \text{ gives a } max.$$

NOTE.—Since $f'(x+h) = f'(x) + hf''(x) + \dfrac{h^2}{2!}f'''(x+\theta h)$,

$$f'(x-h) = f'(x) - hf''(x) + \dots$$

we have, putting $x = a$ so that $f'(a) = 0$,

$$\left.\begin{aligned} f'(a+h) &= +hf''(a) \\ f'(a-h) &= -hf''(a) \end{aligned}\right\} \text{approximately,}$$

i.e. $f'(x)$ *changes sign from* $+^{ve}$ *to* $-^{ve}$ *as* x *increases and passes through* a, *for a max.; and from* $-^{ve}$ *to* $+^{ve}$ *for a min.*

204. Other Exceptions—$f'(a) = \infty$, and $f'(a)$ discontinuous.

We now come to the cases in which $f(x)$ or $f'(x)$ are not continuous.

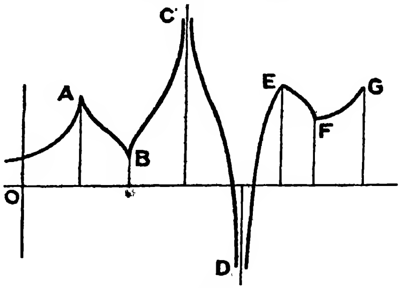

FIG. 36.

The annexed figure shows the variations which might possibly occur.

Thus at A and B, dy/dx is infinite, but y is finite; at C and D, dy/dx is infinite, as also y; while at E and F dy/dx is discontinuous; and at G the curve suddenly stops, so that y is discontinuous. In each of these cases, however, y is either a max. or min.

With the exception of the last point, in these cases we may adopt the method of showing that dy/dx changes sign on passing through the turning value.

Ex. 1. $y^3 = (x - a)^2$

$$\therefore\ y = (x-a)^{\frac{2}{3}},\quad y' = \frac{2}{3}\cdot\frac{1}{(x-a)^{\frac{1}{3}}} = \infty\text{, if } x = a.$$

$$\left.\begin{array}{ll}\text{Put } x = a + h, & \therefore\ y' \text{ is } +^{\text{ve}} \\ \phantom{\text{Put }} x = a - h, & \therefore\ y' \text{ is } -^{\text{ve}}\end{array}\right\}\ \text{hence } y \text{ is a min. when } x = a.$$

Ex. 2. Let $dy/dx = 2e^{-e^{1/(x-a)}} - 1$.

We must note that $e^{-\infty} = 0$, while $e^{+\infty} = \infty$.

Hence, if x be $< a$ and approach a, dy/dx approaches $2 - 1$ or 1, and is $+^{\text{ve}}$; and if x be $> a$ and approach a, dy/dx approaches $2e^{-\infty} - 1$ or -1, and is $-^{\text{ve}}$. Hence $f'(a)$ is discontinuous, but $f(a)$ is, nevertheless, a *maximum*.

205. Rational Functions—Theory of Maxima and Minima.

Prop.—*In a rational integral algebraical function of the* n*th degree the greatest number of turning values is* n − 1, *and these are alternately maxima and minima.*

The term "integral" implies that x can never appear in a denominator.

Let $y \equiv p_n x^n + p_{n-1}x^{n-1} + \ldots + p_1 x + p_0$ be such a function.

Then $dy/dx = np_n x^{n-1} + \ldots + p_1 = 0$; an equation of the $(n-1)$th degree, which has $n - 1$ roots real or imaginary.

The greatest number of turning values occurs when all the roots are real.

Let $dy/dx = np_n(x - a_1)(x - a_2) \ldots (x - a_{n-1}) = 0$; . (1)
which is equivalent to the previous equation, $a_1, a_2, \ldots$ being in ascending order of magnitude.

Suppose that when $x < a_1$, dy/dx is $+^{\text{ve}}$. Then, as x passes through a_1, dy/dx changes from $+^{\text{ve}}$ to $-^{\text{ve}}$, giving a *max.*; again, as x passes through a_2, dy/dx changes from $-^{\text{ve}}$ to $+^{\text{ve}}$, giving a *min.*; and so on. This shows that maxima and minima occur

alternately. Moreover, if the n roots of $y = 0$ are all real, it follows from Art. 136 that the $n - 1$ turning values occur severally *between* the n points for which $y = 0$.

206. Prop.—*The greatest number of points of inflexion is* n − 2.

This easily follows from the fact that the equation $d^2y/dx^2 = 0$, which is one condition for a point of inflexion (see Art. 115), is of the $(n - 2)$th degree. Moreover, since d^2y/dx^2 bears the same relation to dy/dx as dy/dx does to y, it follows, by reasoning similar to the above, that points of inflexion will occur between maxima and minima, and that dy/dx will at these points be alternately a maximum and minimum. Thus, in the figure, A, B, C ... are the turning points of y, and A', B', C' ... are the points of inflexion.

207. Case of Two or More Equal Roots.

(A) Let $a_1 = a_2$ in equation (1) above.

Then $dy/dx = np_n(x - a_1)^2 (x - a_3) \dots (x - a_{n-1})$.

Since $(x - a_1)^2$ does *not* change sign as x passes through a_1, dy/dx will not change sign, and therefore there will be no turning value at $x = a_1$.

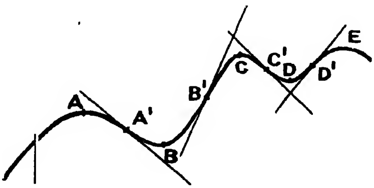

FIG 37.

Or, since dy/dx contains $(x - a_1)^2$ as a factor, we can easily see by differentiation that d^2y/dx^2 will contain $x - a_1$ as a factor, and will therefore vanish with dy/dx. But as d^3y/dx^3 will *not* contain this factor, it will *not* vanish with dy/dx. Hence, by Art. 203, $x = a_1$ gives no turning value; but there will be a point of inflexion.

This is illustrated in Figs. 37 and 38. A and B, which correspond to $x = a_1$ and $x = a_2$, have coincided, and A' which comes between is included. Hence the point of inflexion has become

horizontal; while, the slope from A to B having vanished, the turning values at A and B have coincided; and the curve, though for a moment horizontal, again rises, so that there is practically no turning value.

We may note that (*a*) in Fig. 37 the roots a_1, a_2 giving the turning values are real and different; (*b*) in Fig. 38 they are real and equal; (*c*) in Fig. 39 they are imaginary, for the tangent at

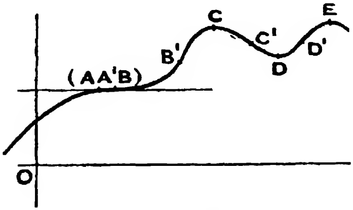

FIG. 38.

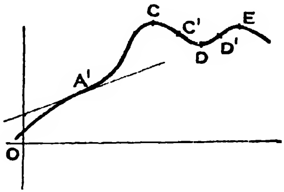

FIG. 39.

A' is not horizontal, so that dy/dx is not zero. Hence (*b*) is the border line between real and imaginary turning values.

(B) If $a_1 = a_2 = a_3$, then $dy/dx = np_n(x - a_1)^3(x - a_4) \ldots$, and as dy/dx now *does* change sign as x passes through a_1, there will be a turning value.

Again, we can show that d^3y/dx^3 will vanish, but not d^4y/dx^4. Hence the two points of inflexion, A' and B', will coincide and *neutralize* each other as the max^{a}. and min^{a}., A and B, did in the preceding case. That is to say, there will not be a point of inflexion at B', but there will be a turning value, which agrees with Art. 203. Such a point is called a *point of undulation* (Fig. 40).

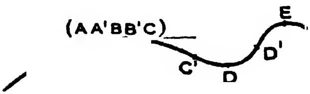

FIG. 40.

208. Miscellaneous Examples.

Ex. 1. *Find the turning values of* $(3 - x)\{\sqrt{1 + x^2} + x\}$.

Let $y = (3 - x)\{\sqrt{1 + x^2} + x\}$.

$\therefore \log y = \log (3 - x) + \log \{\sqrt{1 + x^2} + x\}$.

$$\frac{1}{y}y_1 = \frac{1}{3-x} + \frac{1}{\sqrt{1+x^2}} = \frac{-\sqrt{1+x^2}+3-x}{(3-x)\sqrt{1+x^2}};$$

$$\therefore y_1 = \frac{(-\sqrt{1+x^2}+3-x)(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}}.$$

$\therefore$ for a max. or min. we have (1) $-\sqrt{1+x^2}+3-x=0$,

$$\text{or } 1+x^2 = 9-6x+x^2, \text{ or } x = \tfrac{4}{3}.$$

(2) $\sqrt{1+x^2}+x=0$, or $1+x^2=x^2$ (inadmissible).

To discriminate between max^{a}. and min^{a}., we may consider the sign of y_1 when x is $< \frac{4}{3}$, and when $> \frac{4}{3}$.

We may note that the factors $\sqrt{1+x^2}+x$ and $\frac{1}{\sqrt{1+x^2}}$ do not change sign, even when x is $-^{\text{ve}}$; while $-\sqrt{1+x^2}+3-x$ has only one root, viz. $x=\frac{4}{3}$; hence y_1 only changes sign at this point. We need only, therefore, choose two values of x, one on each side of $\frac{4}{3}$, and see whether y_1 changes from $+$ to $-$ or from $-$ to $+$. Putting $x=0$, then $y_1=4$; and putting $x=2$, $y_1=-\sqrt{5}+1$.

Hence y_1 changes from $+$ to $-$ on passing through 0; therefore y is a max. Its value is $(3-\frac{4}{3})(\frac{5}{3}+\frac{4}{3})$ or 5.

Ex. 2. *Find the turning values of* y *if*

$$9y^2+6xy+4x^2-24y-8x+4=0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Solving for x in terms of y, we have

$$4x^2+2x(3y-4)+9y^2-24y+4=0.$$

Now x is real if $(3y-4)^2-4(9y^2-24y+4)>0$.

The expression $= -27y^2+72y = 9y(8-3y)$.

Hence y must be >0 and $<\frac{8}{3}$; *i.e.* 0 is a min. and $\frac{8}{3}$ a max.

Otherwise :—Differentiating with respect to x, we have

$$(18y+6x-24)y'+6y+8x-8=0 \quad . \quad . \quad . \quad . \quad (2)$$

For max. or min. put $y'=0$;

$$\therefore 6y+8x-8=0, \text{ or } 3y+4x-4=0 \quad . \quad . \quad . \quad (3)$$

Combining with (1), since $3y=-4(x-1)$,

$$16(x-1)^2-8x(x-1)+32(x-1)+4(x-1)^2=0.$$

$$\therefore 20(x-1)^2 = 8(x-4)(x-1).$$

$\therefore$ (1) $x=1$, *whence from* (3) $y=0$,

(2) $5(x-1)=2(x-4)$, $\therefore x=-1$, whence $y=\frac{8}{3}$.

To discriminate, we have by differentiating (2), and remembering that $y' = 0$,

$$(18y + 6x - 24)y'' + 8 = 0.$$

(1) If $x = 1$, $y = 0$, then y'' is $+^{ve}$. $\therefore y$ is a min.

(2) If $x = -1$, $y = \frac{8}{3}$, then y'' is $-^{ve}$. $\therefore y$ is a max.

Ex. 3. *A variable sphere is described with its centre on the surface of a fixed sphere; find for what value of the radius the area of its surface intercepted by the fixed sphere is greatest.*

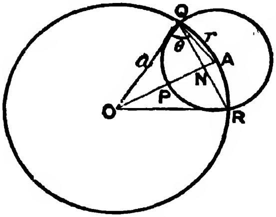

FIG. 41.

Let $\angle AQO = \theta$; $AQ = r$; $OQ = a$.

Then $r = 2a\cos\theta$.

Area of surface QPR = area of corresponding zone of circumscribed cylinder

$= 2\pi r . PN = 2\pi r . r(1 - \cos\theta) = \frac{\pi}{a}r^2(2a - r)$

$= y$ say.

$$y_1 = \frac{\pi}{a}(4ar - 3r^2) = 0 \text{ for max. or min.}$$

$$\therefore r = 0 \text{ or } \frac{4a}{3}.$$

Again, $y_2 = \frac{\pi}{a}(4a - 6r) = -^{ve}$ if $r = \frac{4a}{3}$. Hence this value of r gives the maximum surface.

Ex. 4. *If chords of an ellipse be drawn through a fixed point in the minor axis, prove that the minor axis and the chord parallel to the major axis are either maximum or minimum chords. If they be equal to each other, prove that they are both minimum chords, and that the maximum chord is an arithmetic mean between the major axis and the latus rectum of the ellipse.* (*Sci. and Art.*, 1886.)

Let ϕ and ϕ' be the eccentric angles of the extremities of the chord PQ; $OC = c$, C being the fixed point on the minor axis, and O the centre.

The equation to PQ is $\dfrac{x - a\cos\phi}{a(\cos\phi' - \cos\phi)} = \dfrac{y - b\sin\phi}{b(\sin\phi' - \sin\phi)}$.

This passes through $C(0, c)$ if

$$-b\cos\phi(\sin\phi' - \sin\phi) = (c - b\sin\phi)(\cos\phi' - \cos\phi)$$

$$\text{or } c(\cos\phi' - \cos\phi) = b\sin(\phi - \phi')$$

$$\text{or } c\sin\frac{\phi + \phi'}{2} = b\cos\frac{\phi - \phi'}{2}. \qquad (1)$$

Also $u^2 \equiv PQ^2$

$$= a^2(\cos\phi - \cos\phi')^2 + b^2(\sin\phi - \sin\phi')^2$$

$$= 4\sin^2\frac{\phi \sim \phi'}{2}\left\{a^2\sin^2\frac{\phi+\phi'}{2} + b^2\cos^2\frac{\phi+\phi'}{2}\right\}$$

$$= [\text{from } (1)]\ \frac{4}{b^2}\left(b^2 - c^2\sin^2\frac{\phi+\phi'}{2}\right)\left\{a^2\sin^2\frac{\phi+\phi'}{2} + b^2\cos^2\frac{\phi+\phi'}{2}\right.$$

$$= \frac{4}{b^2}(b^2 - c^2v)\{(a^2 - b^2)v + b^2\}, \text{ where } v \equiv \sin^2\frac{\phi+\phi'}{2},$$

$$= \frac{4}{b^2}(b^2 - c^2v)(a^2e^2v + b^2)$$

$$= \frac{4}{b^2}[b^4 + b^2(a^2e^2 - c^2)v - a^2c^2e^2v^2] \qquad (2)$$

Now, it must be remembered that v is $+^{ve}$ and cannot be > 1; hence its extreme values are 1 and 0.

Putting u^2 in the form $\frac{4}{b^2}[A - (B - Cv)^2]$ we can see that $v = 0$ gives *either* a max. or min., according to the sign of C, and depending on whether C is $>$ or $< B$; and similarly for $v = 1$. Hence the turning values are

(i) $u^2 = 4b^2$ when $v = 0$,

$u = 2b =$ the minor axis.

(ii) $u^2 = \frac{4}{b^2}[b^4 + b^2(a^2e^2 - c^2) - a^2c^2e^2]$, when $v = 1$,

$$= \frac{4}{b^2}(b^2 + a^2e^2)(b^2 - c^2) = \frac{4a^2}{b^2}(b^2 - c^2).$$

But if we put $y = c$ in the equation to the ellipse, we get

$$b^2x^2 + a^2c^2 = a^2b^2, \text{ or } x^2 = \frac{a^2}{b^2}(b^2 - c^2).$$

Hence $u =$ the chord through C parallel to the major axis.

Again, if these are equal, then

$$4b^2 = \frac{4a^2}{b^2}(b^2 - c^2),$$

or $\qquad a^2c^2 = b^2(a^2 - b^2) = a^2b^2e^2,$

or $\qquad c^2 = b^2e^2.$

$\therefore$ in (2) $u^2 = \frac{4}{b^2}[b^4 + a^2b^2e^4v - a^2b^2e^4v^2]$

$= 4[b^2 + a^2e^4v(1 - v)]$, which is a min. for each of the extreme values, 1 and 0, of v.

To find the max., we have

$$u^2 = 4b^2 + a^2e^4 - a^2e^4(2v-1)^2$$
$$= 4b^2 + a^2e^4 \text{ when } 2v = 1 \text{ or } v = \tfrac{1}{2}. (<1)$$
$$= 4b^2 + a^2\left(\frac{a^2-b^2}{a^2}\right)^2 = \frac{(a^2+b^2)^2}{a^2}.$$

$\therefore u = a + \dfrac{b^2}{a} = \dfrac{1}{2}\left\{2a + \dfrac{2b^2}{a}\right\} = A$. mean of the major axis and latus rectum.

EXAMPLES XXXIII.—MISCELLANEOUS.

1. Employ the method of the Calculus to find the max. and min. values of:—

(1) $x^2 - 2x$. (2) $x + \dfrac{1}{x}$. (3) $x^2(x-3)$.

(4) $x^3(x-4)$. (5) $x^{2n}(x-2n-1)$. (6) $x^{2n-1}(x-2n)$.

(7) $x^8 - 2x^4$. (8) $\tan^2 x(\tan^2 x + 3)$.

(9) $\tan^4 x + \sec^4 x$. (10) $5x^6 + 6x^5 - 15x^4 + 12$.

(11) $(x-3)^2\sqrt{1+x^2}$. (12) $(\sqrt{1+x^2} - x)(x+2)$.

(13) $\sqrt[3]{7\sin x + \sin 2x}$.

2. Show that $\cos x - \cot x$ has no max. or min. values.

3. P is a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$, PN being the ordinate. Find when the triangle OPN has a max. area.

4. The perimeter of a circular sector is b; find the radius when the area is a max.

5. SP is the radius vector of a parabola, focus S; PR, drawn at right angles to SP, meets the axis in R. Show that the min. of SR is $4l$, l being the semi-latus rectum.

6. An open tank is to be constructed with a square base and vertical sides, so as to contain a given quantity of water. Show that the expense of lining it with lead is least when the depth is half of the width.

7. Find a point P on the parabola $y^2 = 4ax$ such that the perpendicular on the tangent at P from a point on the axis distant h from the vertex, may be a minimum. What is the geometrical meaning of the result?

8. Show that the shortest line which can be drawn to bisect the triangle ABC is of length $\sqrt{2bc}\sin\dfrac{A}{2}$, A being the least angle.

9. Find when the perpendicular from the origin to the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is a minimum.

10. It is required to construct a conical bell-tent of given internal cubical capacity so that the superficial area of the canvas may be a minimum. Find the ratio between the height and the diameter of the base.

11. Find the max. and min. values of $x + y$, where $mx^2 + ny^2 = a^2$, m and n being $+^{ve}$.

12. O is a fixed point without a circle, A one of the extremities of the diameter through O, OQQ' a chord through O; find its position when the area of the triangle QAQ' is a maximum.

13. From a fixed point, A, on the circumference of a circle of radius c, the perpendicular Ay is let fall on the tangent at P; prove that the max. area of the triangle APy is $\frac{3}{8}c^2\sqrt{3}$.

14. A length l of wire is cut into two parts which are respectively bent into the form of a circle and square. Show that the sum of the areas is a minimum when a side of the square is twice the radius of the circle.

15. The least perimeter of a parallelogram described about an ellipse, with sides parallel to conjugate diameters, is $4\sqrt{2(a^2+b^2)}$.

Find the max. and min. values of :--

16. $\dfrac{3\cos\theta - \sin\theta}{1 - \sin\theta}$.

17. $(x - 3)^3(x^2 - 3x - 3)$.

18. $\dfrac{(x+a)(x+b)}{(x-a)(x-b)}$.

19. $(\log x)/x$.

20. $(1/x)^x$.

21. $2(1 - x)\sin\alpha - \cos(2x - \alpha)$.

22. $a^{2n}\cos^n x + b^{2n}\sec^n x$.

23. Show that the expression $\dfrac{1 - \sqrt{1 - x}}{x}$ has no max. or min. values.

24. Find the max. and min. values of $\cos(\theta + \alpha)\cos(\theta - \alpha)\cos\theta$.

25. Given the whole surface of a cone, including the base, find the height and vertical angle when the volume is a max.

26. Find the position of that normal to a given ellipse which is at the greatest possible distance from the centre.

27. Given the vertical angle A of a triangle, and the radius R of the circum-circle, show that the max. area of the triangle is $R^2\sin A(1 + \cos A)$.

28. A conical vessel of given *small* thickness, open at the top, is to be

cast from a given weight of metal; find the semi-vertical angle when the volume is a maximum.

29. A straight road runs along the edge of a common, and a person situated at P on the common at a perpendicular distance $PM(=a$ yards) from the road, wishes to go to a point B on the road b yards from M. Find, in terms of a, b, u and v, the least number of minutes that he can take to go from P to B, if he walks u yards and v yards per minute respectively on the common and road.

30. Prove that $x=\pi/3$ will make $\cos^{-1}(a\sin x)+2\cos^{-1}(a\cos\frac{1}{2}x)$ a min. if $0<a<1$, but a max. if $a>1$.

31. A window 5 feet high is such that its sill is 20 feet above the level of the ground. How far from the wall containing the window must a person stand, so that the height of the window may subtend the greatest angle at his eye, supposing the latter to be 5 feet from the ground.

32. At one of the foci of a level elliptical enclosure, whose semi-axes are 30 feet and 50 feet respectively, stands a flagstaff 60 feet high, whose upper third is painted white. Find two points on the major axis, one on each side of the flagstaff, at which the white portion subtends the greatest angle, the eye of the observer being at a height of $5\frac{1}{2}$ feet above the level of the enclosure.

Find also the points on the minor axis where the maximum angle is subtended.

33. A right circular cone is circumscribed to a hemisphere, the base of the cone being in the same plane with the base of the hemisphere. Find the form of the cone when (1) its convex surface, (2) its total surface, is a minimum; and in the latter case, show that the surface of the cone is double of the convex surface of the hemisphere.

34. Show that the maximum area of the rectangle, formed by two parallel tangents to an ellipse, and two lines drawn perpendicular to them through the foci, is equal to twice the square on the semi-major axis.

35. The radius of the base of a right cone is 5 inches, and the altitude of the cone is $5\sqrt{2}$ inches. Show that the greatest parabolic section of the cone has an area of $37\frac{1}{2}$ square inches.

36. If SP and SQ be two focal distances in an ellipse inclined to each other at an angle α, find the greatest and least values of the area of the triangle PSQ.

37. A tetrahedron of given altitude stands upon a given base, the distance of the foot N of the perpendicular from the vertex on the base, from one of the sides of that base, being also given. Prove that when the

surface of the tetrahedron is the least possible, the distances of N from the other two sides of the base must be equal.

Hence show that when a tetrahedron of given volume has least surface, the foot of the perpendicular from each vertex upon the opposite triangular face must be the centre of the circle inscribed in that face.

38. Divide twenty-one into three parts a, b, c, in continued proportion, such that $3a + 6b + 4c$ may be a maximum.

39. $ABCD$, $A'B'C'D'$ are two equal post-cards, of which AB, $A'B'$ are the shorter edges; and $AB = A'B' = a$. A', B' are made to move along DA and AB respectively. If P be the intersection of BC and $B'C'$, show that the maximum value of BP is $\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}a$, in which case the cards are inclined to each other at an angle $\cos^{-1}(2 \sin \pi/10)$, and AB is divided by B' in medial section, as in Euc. II. 11.

ANSWERS.

1. (1) Min., -1. (2) Min., 2; max., -2. (3) Max., 0; min., -4.
(4) Min., -27. (5) Max., 0; min., $-(2n)^{2n}$. (6) Min., $-(2n-1)^{2n-1}$.
(7) Min., -1; max., 0; min., -1. (8) Min., 0. (9) Min., 1.
(10) Min., -100; max., 12; min., 8. (11) Max., $-5\sqrt{5}/4$; min., $-2\sqrt{2}$.
(12) Max., $\frac{5}{2}$. (13) Max., $\sqrt{15}/2$; min., $-\sqrt{15}/2$.

3. When $x = a/\sqrt{2}$. **4.** $b/4$.

7. $2\sqrt{a(h-a)}$; the point $(0, h)$ is the foot of the normal at P.

9. When the tangent is at the end of the minor axis.

10. 1; $\sqrt{2}$. **11.** $\pm a\sqrt{(m+n)/mn}$.

12. QQ' subtends a right angle at the centre; $QOA = \sin^{-1}(c/\sqrt{2}l)$, where c = radius, l = distance from O to centre.

16. Max., 5. **17.** Max., 81; min., $-\frac{567}{3125}$.

18. Max., $-(\sqrt{a}+\sqrt{b})^2/(\sqrt{a}-\sqrt{b})^2$; min., $-(\sqrt{a}-\sqrt{b})^2/(\sqrt{a}+\sqrt{b})^2$.

19. Max., $1/e$. **20.** Max., $e^{1/e}$.

21. Min^a., $2\{1-(n\pi+\alpha)\}\sin\alpha - \cos\alpha$; max^a., $2\{1-\frac{1}{2}(2n+1)\pi\}\sin\alpha + \cos\alpha$.

22. Max^a., $\pm(a^{2n} + b^{2n})$; min., $2a^nb^n$; max., $-2a^nb^n$.

24. Max., $\cos^2\alpha$, when $\theta = 0$; min., $-\frac{2}{3\sqrt{3}}\sin^3\alpha$, when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\sin\alpha\right)$,

in 1st quadrant; max., $\frac{2}{3\sqrt{3}}\sin^3 \alpha$, when $\theta = \pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\sin\alpha\right)$; min., $-\cos^2\alpha$, when $\theta = \pi$; etc.

25. $h = \sqrt{2S/\pi}$, where $S =$ given area; vertical angle $(2\alpha) = \cos^{-1} 7/9$.

26. The normal is at the point for which $\phi = \tan^{-1}\sqrt{b/a}$.

27. Geometrical proof.

28. $\cot^{-1}\sqrt{2}$. See Ex. 10.

29. $\frac{b}{v} + \frac{a}{uv}\sqrt{v^2 - u^2}$.

31. $10\sqrt{3}$ feet (geometrical proof).

32 On major axis at distances 43·36 from S; on minor axis at distances 16·74 from C.

33. (1) Semi-vertical angle $(\alpha) = \cot^{-1}\sqrt{2}$; (2) $\alpha = \pi/6$.

36. Take $P\left\{r_1, \theta - \frac{\alpha}{2}\right\}$, $Q\left\{r_2, \theta + \frac{\alpha}{2}\right\}$.

(1) Max., $l^2 \cot\frac{\alpha}{2}(1-e^2)$, when $\cos\theta = -\left(\cos\frac{\alpha}{2}\right)/e$; inadmissible if $e < \cos\frac{\alpha}{2}$.

(2) Min., $\frac{1}{2}l^2 \sin\alpha \bigg/ \left(1 + e\cos\frac{\alpha}{2}\right)^2$, when $\theta = 0$.

(3) Max., $\frac{1}{2}l^2 \sin\alpha \bigg/ \left(1 - e\cos\frac{\alpha}{2}\right)^2$, when $\theta = \pi$.

38. $14 - 4\sqrt{7}$, $5\sqrt{7} - 7$, $14 - \sqrt{7}$.

CHAPTER XV.

TANGENTS AND NORMALS.

209. Equation to Tangent to $y = f(x)$.—We have already shown that if the graph of the function $y = f(x)$ be drawn, then $dy/dx = \tan\psi$, where ψ is the inclination of the tangent at the point (x, y), to the axis of x.

We shall now find the equation to the tangent at the point (x, y). It is usual to choose X and Y for the co-ordinates of the variable point on the tangent; in other words, X and Y are the *current* co-ordinates of the tangent, while (x, y) are those of the temporarily fixed point on the curve.

The equation of the tangent is evidently that of a straight line through (x, y) inclined at the angle ψ to Ox, and is therefore

$$Y - y = \tan\psi\,(X - x)\,;$$

i.e.

$$Y - y = \frac{dy}{dx}(X - x)\,. \qquad (1)$$

Ex. $y = x\log x + 1$.

$dy/dx = 1 + \log x$; $\therefore$ tangent is $Y - y = (1 + \log x)(X - x)$

$$= X(1 + \log x) - x - x\log x,$$

and, putting $x\log x + 1$ for y, we get

$$Y = X(1 + \log x) + 1 - x.$$

If $x = 1$, then $y = 1$; and the tangent at the point $(1, 1)$ is $X = Y$.

210. Tangent to $f(x, y) = 0$.—It is not always easy to express y as an explicit function of x, when it is given implicitly; and the same remark applies to dy/dx. We have, however, from

the equation $f(x, y) = 0$, $\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$; and we may therefore write for equation (1) above,

$$Y - y = \left\{-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}\right\}(X - x),$$

or
$$(X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} = 0.$$

Ex. $x^4 - 4xy^3 + y^4 = a^4$ . . . . (1)

We have $\partial f/\partial x = 4(x^3 - y^3)$; $\partial f/\partial y = 4(y^3 - 3xy^2)$.

$\therefore$ the tangent is $(X - x)(x^3 - y^3) + (Y - y)(y^3 - 3xy^2) = 0$,

or $(x^3 - y^3)X + (y^3 - 3xy^2)Y = x^4 - 4xy^3 + y^4 = a^4$, from (1).

Thus, for example $(a, 4a)$ is a point on the curve, and the tangent at this point is

$$(a^3 - 64a^3)X + 16a^2(4a - 3a)Y = a^4,$$

or
$$16Y - 63X = a.$$

211. Equation to Normal.

If $y = f(x)$ be the curve, then since the normal is a straight line through (x, y), perpendicular to the line (1) of Art. 209, its equation is

$$(Y - y)\frac{dy}{dx} + X - x = 0.$$

If $f(x, y) = 0$ be the curve, the normal is

$$(X - x)\frac{\partial f}{\partial y} = (Y - y)\frac{\partial f}{\partial x}$$

212. Subtangent and Subnormal.

Let PT be the tangent and PG the normal to the curve; draw the ordinate PN.

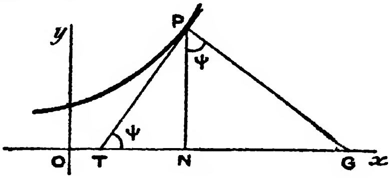

FIG 42.

Then TN is called the *subtangent*, and NG the *subnormal*.

The subtangent $TN = PN \cot\psi = y\dfrac{dx}{dy}$.

The subnormal $NG = PN \tan\psi = y\dfrac{dy}{dx}$.

213. Length of Tangent intercepted between the Axes of Co-ordinates.

Let P be a point (x, y) on the curve $y = f(x)$, *drawn in the figure so that* x, y, *and* y_1 *or* dy/dx, *are all* $+^{ve}$.

Then Tt

$$\begin{aligned}
&= Pt - PT = PM \sec\psi - PN \operatorname{cosec}\psi \\
&= x \sec\psi - y \operatorname{cosec}\psi \\
&= \sec\psi\,(x - y \cot\psi) \\
&= \sqrt{1 + y_1^2}\left(x - \frac{y}{y_1}\right) \\
&= \frac{1}{y_1}\sqrt{1 + y_1^2}\,(xy_1 - y).
\end{aligned}$$

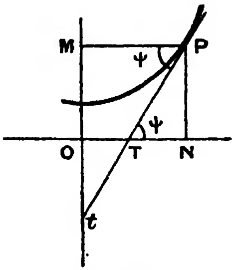

Fig. 43.

NOTE.—If x and y are $+^{ve}$, while y_1 is $-^{ve}$, then ψ is either $-^{ve}$ or an obtuse angle. In either case, $x \sec\psi - y \operatorname{cosec}\psi$ becomes an *arithmetical sum*, $+^{ve}$ in the first case, and $-^{ve}$ in the second, since $\operatorname{cosec}\psi$ is $-^{ve}$ if ψ is $-^{ve}$, while $\sec\psi$ is $-^{ve}$ if ψ is obtuse. This may be also seen by drawing a special figure; or, better, by imagining the tangent at P to rotate about P, when either PT or Pt changes sign on passing through infinity.

214. Tangent and Normal to the Curve $x = f(\theta)$; $y = \phi(\theta)$.

It is sometimes convenient to express x and y in terms of a third variable, called a *parameter;* thus, in the cycloid, we have

$$x = a(\theta - \sin\theta)\,;\ y = a(1 + \cos\theta).$$

To find the tangent, since $\dfrac{dy}{dx} = \dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta}$, we have

$$Y - y = \left(\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}\right)(X - x), \text{ or } (Y - y)\frac{dx}{d\theta} = (X - x)\frac{dy}{d\theta}.$$

Ex. 1. In the cycloid above $dy/d\theta = -a\sin\theta$, $dx/d\theta = a(1-\cos\theta)$.

$\therefore$ the tangent is $(Y-y)(1-\cos\theta) = -(X-x)\sin\theta$,

$$\begin{aligned}\text{or } Y(1-\cos\theta)+X\sin\theta &= y(1-\cos\theta)+x\sin\theta\\ &= a(1-\cos^2\theta)+a\sin\theta(\theta-\sin\theta)\\ &= a\theta\sin\theta.\end{aligned}$$

This may be written

$$2Y\sin^2\frac{\theta}{2}+2X\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2a\theta\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\text{or } Y+X\cot\frac{\theta}{2} = a\theta\cot\frac{\theta}{2}.$$

Otherwise, since $\dfrac{dy}{dx} = \dfrac{-a\sin\theta}{a(1-\cos\theta)} = -\cot\dfrac{\theta}{2}$, the tangent is

$$Y-y = -\cot\frac{\theta}{2}(X-x), \text{ or } Y+X\cot\frac{\theta}{2} = y+x\cot\frac{\theta}{2} = \text{etc.}$$

The normal is $(Y-y)\dfrac{dy}{d\theta}+(X-x)\dfrac{dx}{d\theta} = 0$; that is

$$-(Y-y)\sin\theta+(X-x)(1-\cos\theta) = 0,$$

$$\begin{aligned}\text{or}\quad Y\sin\theta-X(1-\cos\theta) &= y\sin\theta-x(1-\cos\theta)\\ &= a(1+\cos\theta)\sin\theta-a(\theta-\sin\theta)(1-\cos\theta)\\ &= 2a\sin\theta-a\theta(1-\cos\theta),\end{aligned}$$

which reduces to $\quad Y-X\tan\dfrac{\theta}{2} = 2a\left(1-\dfrac{\theta}{2}\tan\dfrac{\theta}{2}\right).$

Ex. 2. *Find the length of the tangent intercepted between the axes.*

In cases like this it is better to work from first principles rather than quote the formula.

Using the above figure, we have

$$Tt = Pt - PT = x\sec\psi - y\operatorname{cosec}\psi.$$

Now, $\tan\psi = -\cot\dfrac{\theta}{2}$ by Ex. 1;

$$\therefore \sec\psi = \pm\operatorname{cosec}\frac{\theta}{2},\ \operatorname{cosec}\psi = \mp\sec\frac{\theta}{2}. \quad \text{[Art. 213, note.]}$$

$$\begin{aligned}\therefore\ Tt &= \pm a\left\{(\theta-\sin\theta)\operatorname{cosec}\frac{\theta}{2}+(1+\cos\theta)\sec\frac{\theta}{2}\right\}\\ &= \pm a\left(\theta\operatorname{cosec}\frac{\theta}{2}-2\cos\frac{\theta}{2}+2\cos\frac{\theta}{2}\right) = \pm a\theta\operatorname{cosec}\frac{\theta}{2}.\end{aligned}$$

EXAMPLES XXXIV.

1. Find the equations to the tangent and normal to the following curves:—

(1) $y^2 = 4ax$. (2) $xy = c^2$.
(3) $x^2 + y^2 = a^2$. (4) $x^3 + y^3 = 3axy$.
(5) $x^n/a^n + y^n/b^n = 1$. (6) $x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$.

2. Find the lengths of the subtangent and subnormal in each of the above curves.

3. Find the length of the perpendicular from the origin to the tangent in each of the above curves.

4. (1) Show that the length of the tangent intercepted between the axes, for the curve $xy = c^2$, is $2r$, where r is the radius vector of P.

(2) Show that, for the circle $x^2 + y^2 = a^2$, it is a^3/xy; and verify geometrically.

(3) Show that, for the parabola $y^2 = 4ax$, it is $\sqrt{x^2 + ax}$; and verify geometrically.

(4) Show that, for the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, it is a, and therefore constant.

5. If p, p' be the lengths of the perpendiculars from the origin on the angent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, then

$$4p^2 + p'^2 = a^2.$$

6. In the same curve, if Y be the foot of the perpendicular from O on the tangent at $P(x, y)$, show that the co-ordinates of Y are $(x^{\frac{1}{3}}y^{\frac{2}{3}}, x^{\frac{2}{3}}y^{\frac{1}{3}})$. Hence show that the locus of the mid-point of PY is a circle; and also deduce it from the result of the previous question.

7. In the curve $3y^2 = (x + 1)^3$, show that the subnormal varies as the square of the subtangent.

8. If p and p' be as in Ex. 5, but for any curve, prove that

$$p = x \sin\psi - y \cos\psi; \quad p' = x \cos\psi + y \sin\psi.$$

Hence show that $p' = dp/d\psi$.

9. In the curve of Ex. 5, show that we may write $x = a\cos^3\theta$, $y = a\sin^3\theta$: and that the equation to the tangent at θ is

$$y + x \tan\theta = a \sin\theta.$$

Hence show that the locus of the intersection of perpendicular tangents is the curve

$$2(x^2 + y^2)^3 = a^2(x^2 - y^2)^2.$$

10. If α and β be the intercepts on the axes of x and y, cut off by the tangent to the curve $x^4 + y^4 = c^4$, prove that $\alpha^{-\frac{4}{3}} + \beta^{-\frac{4}{3}} = c^{-\frac{4}{3}}$.

11. Show that for the curve $x^n + y^n = c^n$, $\alpha^{-n/(n-1)} + \beta^{-n/(n-1)} = c^{-n/(n-1)}$, α and β being as in the last question.

12. Show that for the curve $(x/a)^n+(y/b)^n=1$, $(a/\alpha)^{n/(n-1)}+(b/\beta)^{n/(n-1)}=1$. Consider also the cases in which $n=2$, and $n=\frac{2}{3}$.

13. Find the equation to the tangent to the curve $xy^2=a^2(2a-x)$ at the point where it is cut by the line $x-y=0$.

14. A variable tangent is drawn to the curve $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1$, meeting the axes of x and y in H and K, and the rectangle $OHQK$ is completed. Prove that the locus of Q is the ellipse $x^2/a^2+y^2/b^2=1$. Hence also find the locus of the mid-point of the tangent HK. [See Ex. 12.]

15. In the curve $x^4=a^2(x^2-y^2)$, find the points on the curve for which the lengths of the tangent and normal between the point of contact and Ox are equal.

16. Show that, in the curve $x^m y^n=a^{m+n}$, the equation of the tangent is $mX/x+nY/y=m+n$.

Show also that the portion of the tangent intercepted between the axes is divided in the constant ratio $m:n$, at the point of contact.

17. In the catenary $y=a\cosh\dfrac{x}{a}\equiv\dfrac{a}{2}(e^{x/a}+e^{-x/a})$, if N be the foot of the ordinate, and NV the perpendicular on the tangent, prove that NV is constant and equal to a.

18. In the tractrix $y=a\sin\theta$, $x=a\log\cot\dfrac{\theta}{2}-a\cos\theta$; show that

(1) $\psi=\pi-\theta$;

(2) When $x=0$, dy/dx changes sign through infinity, and that, therefore, y is a maximum at that point;

(3) The subtangent $=-a\cos\theta$, and explain the negative sign;

(4) The length of the tangent between the curve and Ox is constant.

19. If u be a homogeneous function of x and y of the nth degree, prove that the equation to a tangent to the curve $u=c$ is

$$X\frac{\partial u}{\partial x}+Y\frac{\partial u}{\partial y}=nc.$$

What is the value of c when the curve $x^n+y^n=c^n$ touches the ellipse $x^2/a^2+y^2/b^2=1$?

20. The equation to a parabola, referred to two perpendicular tangents as axes, being $(x/h)^{\frac{1}{2}}+(y/k)^{\frac{1}{2}}=1$; find the equation to the tangent at any point.

Hence prove that the area of the triangle formed by two fixed tangents at right angles to each other, and a variable tangent, is proportional to the distance of the point of contact of the variable tangent from the chord of contact of the fixed tangents.

21. Show that the area of the maximum triangle formed by a tangent to the curve $a^2y = x^3$ and the lines $y = 0$, $x = c$, is $27c^4/128a^2$.

22. Prove that a pair of straight lines can be drawn through the origin, each of which touches every curve of the family obtained by giving different values to c in the equation $y = c\cosh\dfrac{x}{c}$.

23. Obtain the equation to the normal at any point of the curve

$$x = \phi(t), \quad y = \chi(t).$$

Find the feet of the normals drawn to the parabola $x = at^2$, $y = 2at$, from the point on the curve whose parameter is t_0.

Prove that two of the normals drawn from the point $(8a, 4a\sqrt{2})$ are coincident.

ANSWERS.

1. (1) $yY = 2a(X + x)$; $yX + 2aY = (x + 2a)y$.

(2) $yX + xY = 2c^2$; $xX - yY = x^2 - y^2$. (3) $xX + yY = a^2$; $xY = yX$.

(4) $(x^2 - ay)X + (y^2 - ax)Y = axy$;
$(x^2 - ay)Y - (y^2 - ax)X = (x - y)(ax + ay + xy)$.

(5) $b^nx^{n-1}X + a^ny^{n-1}Y = a^nb^n$;
$a^ny^{n-1}X - b^nx^{n-1}Y = xy(a^ny^{n-2} - b^nx^{n-2})$.

(6) $Y - X\cot\dfrac{\theta}{2} = a\left(2 - \theta\cot\dfrac{\theta}{2}\right)$; $Y\cot\dfrac{\theta}{2} + X = a\theta$.

2. (1) $2x$; $2a$. (2) $-x$; $-y^3/c^2$. (3) $-y^2/x$; $-x$.

(4) $-y(y^2 - ax)/(x^2 - ay)$; $-y(x^2 - ay)/(y^2 - ax)$.

(5) $\dfrac{x^n - a^n}{x^{n-1}}$; $\dfrac{-b^nx^{n-1}}{a^ny^{n-2}}$. (6) $2a\sin^3\dfrac{\theta}{2}\sec\dfrac{\theta}{2}$; $a\sin\theta$.

3. (1) $x\sqrt{a}/\sqrt{a+x}$. (2) $2c^2/r$. (3) a. (4) $axy/\sqrt{(x^2-ay)^2+(y^2-ax)^2}$.

(5) $\dfrac{a^nb^n}{\sqrt{b^{2n}x^{2n-2}+a^{2n}y^{2n-2}}}$. (6) $a\left(2\sin\dfrac{\theta}{2} - \theta\cos\dfrac{\theta}{2}\right)$.

9. The tangents are $x - yt = at/\sqrt{1+t^2}$; $xt + y = at/\sqrt{1+t^2}$; where $t \equiv \tan\theta$. Eliminate t.

13. $X + Y = 2a$. **14.** $x^2/a^2 + y^2/b^2 = \frac{1}{4}$

15. $(0, 0)$ and $(\pm\sqrt{3}a/2, \pm\sqrt{3}a/4)$.

19. $c^m = a^m + b^m$, where $m \equiv 2n/(2 - n)$.

23. $Y.\chi'(t) + X.\phi'(t) = \chi(t).\chi'(t) + \phi(t).\phi'(t)$; points on the curve which $t = (-t_0 \pm \sqrt{t_0^2 + 8})/2$.

215. Differential of an Arc.

Let P and Q be two indefinitely near points on a given curve, and let their co-ordinates be (x, y), $(x + dx, y + dy)$ respectively; dx and dy being infinitesimals. Let ds be the length of the infinitesimal arc PQ.

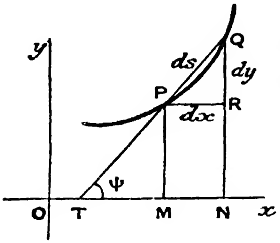

FIG. 44.

Then the error in substituting the arc for the chord is an infinitesimal of the 3rd order; but the error must be compared with PQ, which is of the 1st order; hence it is of the 2nd order. Therefore, by Art. 93, we may write at once

$$ds^2 = dx^2 + dy^2; \text{ or, } (ds/dx)^2 = 1 + (dy/dx)^2.$$

Also $$\cos\psi = dx/ds,\ \sin\psi = dy/ds.$$

216. Polar Coordinates.

Let (r, θ), $(r + dr, \theta + d\theta)$ be the polar co-ordinates of the same points P and Q. Draw $PR \perp$ to OQ.

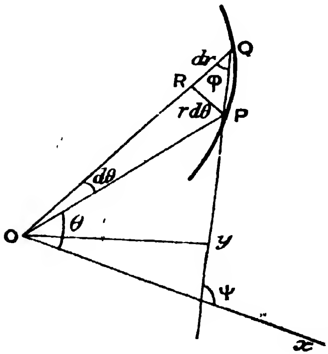

FIG. 45.

Then, since OR and OP differ by an infinitesimal of the 2nd order, we have

$$QR = OQ - OR = OQ - OP = dr,$$

the error being of the 1st order compared with QR.

Similarly, since PR differs from the arc of a circle by an infinitesimal of the 3rd order, the error in putting $PR = rd\theta$ is of the 2nd order compared with PR.

Hence, by Art. 93, we may write at once

$$\sin\phi = r\frac{d\theta}{ds};\ \cos\phi = \frac{dr}{ds};\ \tan\phi = r\frac{d\theta}{dr},$$

ϕ being the inclination of the tangent at P to the radius vector OP.

Also, $ds^2 = r^2 d\theta^2 + dr^2$; or, $(ds/d\theta)^2 = r^2 + (dr/d\theta)^2$.

217. Perpendiculars on Tangent and Normal.

Let $OY = p$ be the perpendicular from O on the tangent. Then

$$p = r \sin\phi = r . r\frac{d\theta}{ds} = r^2\frac{d\theta}{ds}.$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^4}\left(\frac{ds}{d\theta}\right)^2 = \frac{1}{r^4}\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\};$$

$$i.e. \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{r'^2}{r^4}, \text{ if } r' \equiv \frac{dr}{d\theta}.$$

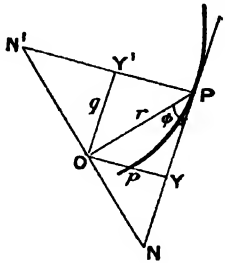

FIG. 46.

This can be put more simply by assuming $u = 1/r$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$.

Hence $$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = u^2 + u'^2, \text{ if } u' \equiv \frac{du}{d\theta}.$$

218. Again, if $OY' = q$ be the perpendicular on the normal, we have

$$q = PY = r\cos\phi = r\frac{dr}{ds}.$$

$$\therefore \frac{1}{q^2} = \frac{1}{r^2}\left(\frac{ds}{dr}\right)^2 = \frac{1}{r^2}\frac{r^2 d\theta^2 + dr^2}{dr^2} = \frac{1}{r^2} + \left(\frac{d\theta}{dr}\right)^2 = \frac{1}{r^2} + \frac{1}{r'^2}.$$

Or, $$q = p\cot\phi = \frac{p}{r}\frac{dr}{d\theta}; \text{ etc.}$$

219. Polar Subtangent and Subnormal.

If NON' be drawn $\perp$ to OP, to meet the tangent and normal respectively in N and N', then ON is called the *polar subtangent* and ON' the *polar subnormal.*

We have $$ON = r\tan\phi = r^2\frac{d\theta}{dr}.$$

$$ON' = r\cot\phi = \frac{dr}{d\theta}.$$

Hence *polar subtangent* $= r^2\dfrac{d\theta}{dr}$, and *polar subnormal* $= \dfrac{dr}{d\theta}$.

220. Second Value for q.

Let PY, QZ be two indefinitely near tangents, and OY, OZ the $\perp^{rs}$ from O. Let $\angle PLx = \psi$ as usual, so that $\angle YTZ = d\psi$.

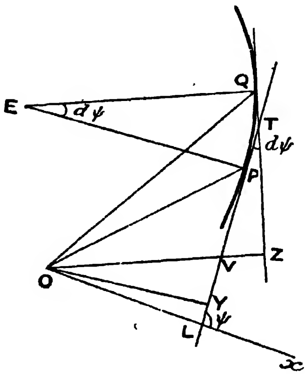

FIG. 47.

Consider the triangle TVZ.

The error in putting $VZ = OZ - OY$ or dp is of the 1st order compared with VZ.

The error in putting $TV = PY$ or q is of the 1st order compared with TV.

The error in making VZ the arc of a circle, centre T, is of the 2nd order compared with VZ.

Hence we may write at once, the error being infinitesimal,

$$VZ/TV = d\psi, \text{ i.e. } dp/q = d\psi.$$

whence $\qquad q = dp/d\psi.$

Cor. Since $PY^2 = OP^2 - OY^2$, i.e. $q^2 = r^2 - p^2$, we have

$$r^2 = p^2 + (dp/d\psi)^2.$$

NOTE.—P, T, Q become ultimately a single point, as also Y, V, Z.

221. Examples Worked.

Ex. 1. *In the curve $4ya = x^2 - 2a^2 \log x$, find ds/dx and ds/dy.*

We have $4y'a = 2x - \dfrac{2a^2}{x}$; i.e. $y' = \dfrac{1}{2}\left(\dfrac{x}{a} - \dfrac{a}{x}\right)$.

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + y'^2 = \frac{1}{4}\left(\frac{x}{a} + \frac{a}{x}\right)^2; \text{ i.e. } \frac{ds}{dx} = \frac{1}{2}\left(\frac{x}{a} + \frac{a}{x}\right).$$

Also $\dfrac{ds}{dy} = \dfrac{ds}{dx}\Big/\dfrac{dy}{dx} = \left(\dfrac{x}{a} + \dfrac{a}{x}\right)\Big/\left(\dfrac{x}{a} - \dfrac{a}{x}\right) = \dfrac{x^2 + a^2}{x^2 - a^2}$.

Cor. Let $x = a \cot\theta$, so that $4y = a\cot^2\theta - 2a \log \cot\theta - 2a \log a$.

Then $\tan\psi = y' = \frac{1}{2}(\cot\theta - \tan\theta) = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta} = \cot 2\theta.$

Hence, neglecting other roots, we have $2\theta = \frac{\pi}{2} - \psi$, or $\theta = \frac{\pi}{4} - \frac{\psi}{2}.$

$$\therefore x = a\cot\left(\frac{\pi}{4} - \frac{\psi}{2}\right) = a\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right).$$

Ex. 2. *In the curve $r^2 = a^2\sin 2\theta$, find ϕ, $ds/d\theta$, p, and the polar subtangent.*

Differentiating in r, $\qquad r = a^2\cos 2\theta \,.\, \frac{d\theta}{dr},$

$$\therefore \tan\phi = r\frac{d\theta}{dr} = \frac{r^2}{a^2\cos 2\theta} = \tan 2\theta, \text{ whence } \phi = 2\theta.$$

Next, $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + \frac{a^4\cos^2 2\theta}{r^2} = a^2\left\{\sin 2\theta + \frac{\cos^2 2\theta}{\sin 2\theta}\right\}$

$$= \frac{a^2}{\sin 2\theta} = \frac{a^4}{r^2}.$$

$$\therefore \frac{ds}{d\theta} = \frac{a^2}{r}.$$

Or, since $r\frac{d\theta}{ds} = \sin\phi = \sin 2\theta = \frac{r^2}{a^2}$, the result follows at once.

To find p, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{r'^2}{r^4} = \frac{1}{a^2\sin 2\theta} + \frac{1}{a^4\sin^2 2\theta}\cdot\frac{a^4\cos^2 2\theta}{r^2}$$

$$= \frac{1}{a^2}\left\{\frac{1}{\sin 2\theta} + \frac{\cos^2 2\theta}{\sin^3 2\theta}\right\} = \frac{1}{a^2\sin^3 2\theta}.$$

$$\therefore p = a(\sin 2\theta)^{\frac{3}{2}}.$$

Or, at once, $p = r\sin\phi = r\sin 2\theta = a(\sin 2\theta)^{\frac{3}{2}}$.

Finally, the polar subtangent $= r\tan\phi = r\tan 2\theta = a(\sin 2\theta)^{\frac{3}{2}}.\sec 2\theta$.

EXAMPLES XXXV.

1. Prove that

(1) In the circle $x^2 + y^2 = a^2$, $ds/dx = a/y$; and verify by geometry;

(2) In the parabola $y^2 = 4ax$, $\frac{ds}{dx} = \sqrt{\frac{a+x}{x}}$;

(3) In the catenary $y = a\cosh\frac{x}{a}$, $\frac{ds}{dx} = \frac{y}{a}$;

(4) In the curve $\sec\frac{x}{a} = e^{y/a}$, $\frac{ds}{dx} = \sec\frac{x}{a}$. Show also that $x = a\psi$.

2. Prove that

(1) In the parabola $\frac{2a}{r} = 1 + \cos\theta$, $\tan\phi = \cot\frac{\theta}{2}$; and verify by geometry;

(2) In the equiangular spiral $r = ae^{\theta\cot\alpha}$, $\tan\phi = \tan\alpha$; whence ϕ is constant and $= \alpha$;

(3) In the circle $r = a\sin\theta$, $\phi = \theta$, and $ds = a\,d\theta$; verify by geometry;

(4) In the cardioide $2r = a(1 - \cos\theta)$, $\phi = \theta/2$, and $ds/d\theta = \sqrt{ar}$.

3. Prove that

(1) In the curve $r\cos 2\theta = a$, the polar subtangent $= \frac{1}{2}a\operatorname{cosec} 2\theta$, and the polar subnormal $= 2a\sec 2\theta\tan 2\theta$;

(2) In the hyperbolic spiral $r\theta = a$, the polar subtangent is constant, and $= a$;

(3) In the curve $r = a\sec^2\frac{\theta}{2}$, polar subtangent $= 2a\operatorname{cosec}\theta$; polar subnormal $= a\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}$;

(4) In the curve $\cos^r\theta = c$, polar subtangent $= \cot\theta$; polar subnormal $= r^2\tan\theta$.

4. Prove that $d\psi = d\theta + d\phi$.

5. Prove that $p = r^2\frac{d\theta}{ds}$, and verify geometrically by equating two values of the area of the triangle OPQ.

6. If $u \equiv \frac{1}{r}$, prove that the polar subtangent $= -\frac{d\theta}{du}$.

Hence, show that if $t \equiv$ polar subtangent, $1/p^2 = 1/r^2 + 1/t^2$, and verify by geometry.

Similarly, if $n \equiv$ polar subnormal, prove that $1/q^2 = 1/r^2 + 1/n^2$.

7. Prove that $q\frac{dq}{dp} = \frac{d^2p}{d\psi^2}$.

8. In the figure of Art. 217, prove that

(1) $YN = r^3\frac{d\theta}{ds}\cdot\frac{d\theta}{dr}$; (2) $Y'N' = \frac{dr}{ds}\cdot\frac{dr}{d\theta}$; (3) $\frac{YN}{Y'N'} = \left(r\frac{d\theta}{dr}\right)^3$.

9. Prove that $x = p\sin\psi + \frac{dp}{d\psi}\cos\psi$; $y = -p\cos\psi + \frac{dp}{d\psi}\sin\psi$.

10. Prove that

(1) In the straight line $r\cos\theta = a$, $p = a$;

(2) In the circle $r = a\cos\theta$, $p = a\cos^2\theta$, and verify by geometry;

(3) In the conic $l/r = 1 + e\cos\theta$, $p = l/(1 + 2e\cos\theta + e^2)^{\frac{1}{2}}$;

(4) In the rectangular hyperbola $r^2\cos 2\theta = a^2$, $p^2 = a^2\cos 2\theta$.

11. In the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, prove that $\dfrac{ds}{dx} = \sqrt{\dfrac{a^2 - e^2x^2}{a^2 - x^2}} = \dfrac{b^2}{py}$.

12. In the curve $2y = a\log\left(\sec\dfrac{x}{a} + \tan\dfrac{x}{a}\right) - a\sin\dfrac{x}{a}$, prove that

$$\frac{ds}{dy} = 1 + 2\cot^2\frac{x}{a}.$$

222. The Pedal Equation.—An invariable relation can always be found between p and r for a given curve; in other words, we can always obtain an equation of the form $f(p, r) = 0$, the only variables being p and r. Such an equation is called the *pedal equation* of a curve.

223. To show that we can (theoretically) obtain this equation, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$r = f(\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which two equations we can eliminate θ and obtain the pedal equation.

NOTE.—Since θ is not involved, the pedal equation is not dependent on the direction of the axes, but only on the position of the origin.

224. Simple Cases.

The student can easily verify, from first principles, the following pedal equations:—

Straight line, $p = a$.
Circle, origin at centre, $p = r$.
Circle, origin on circumference, $ap = r^2$.
Parabola, origin at focus, $p^2 = ar$.

We shall also show in the next article that the equation to a *point*, regarded as the limit of a circle, is $r = a$.

225. Two other Cases.

(1) *Any Circle.*—Let the constants of the circle be $CP = c$ and $OC = l$. Then $p/r = OY/OP = \sin YPO = \cos OPC = (c^2 + r^2 - l^2)/2cr$ [*Misc. Theorems* (6)].

$$\therefore 2cp = c^2 - l^2 + r^2 \text{ [or, at once by Euc. II. 13.]}$$

The general form is $ap = r^2 \pm b^2$, according as O is *within* or *without* the circle.

Cor. 1.—If $c = l$, then $b = 0$, and O is on the circumference. The pedal equation also becomes $ap = r^2$ as before.

Cor. 2.—If $c = 0$, the circle becomes a point, and the equation becomes $r^2 = l^2$, or $r = l$.

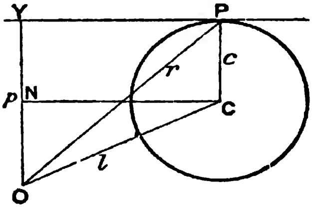

Fig. 48.

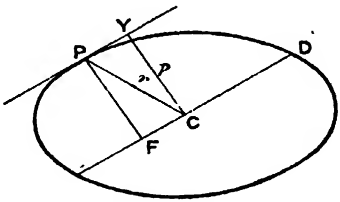

Fig. 49.

(2) *Ellipse, Origin at Centre.*—We know that, if CD be conjugate to CP, and PF be perpendicular to CD,

$$PF.CD = ab, \quad \text{and } CP^2 + CD^2 = a^2 + b^2.$$

$$\therefore p = CY = PF = ab/CD,$$

or, $$p^2CD^2 = a^2b^2; \text{ whence } p^2(a^2 + b^2 - r^2) = a^2b^2,$$

the pedal equation.

Similarly, for the hyperbola, we get $p^2(a^2 - b^2 - r^2) = -a^2b^2$.

226. Pedal Curves.

Def.—The locus of the foot of the perpendicular from the origin to a movable tangent to a given curve, is called the *first positive pedal*, or more simply the *pedal*, of the given curve. Also

the pedal of the first positive pedal is called the ***second positive pedal***; and so on.

Thus, in Fig. 50, the curve YZ is the pedal of the curve PQ. Let the tangents to the two near points P and Q meet in T; then the chord YZ is a chord of the circle on OT as diameter, since Y and Z are right angles.

In the limit, when P, Q, and T coincide, the chord YZ becomes

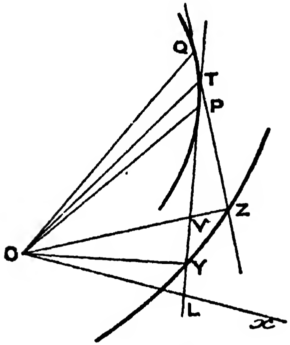

FIG. 50.

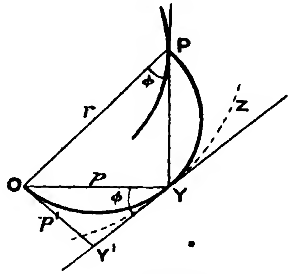

FIG. 51.

a tangent to the circle on OP as diameter, at the point where the circle cuts the tangent PY (Fig. 51).

Hence :—

If a circle described on OP *as diameter meet the tangent to the curve in* Y, *the tangent to the circle at* Y *is also a tangent to the pedal curve at the same point.*

227. Let $OY' = p'$ be the perpendicular from O to the tangent YY' to the first pedal. Then since [Euc. III. 32] $\angle OYY' = \phi$, we have
$$p' = p \sin\phi$$
and
$$r \sin\phi = p; \text{ whence } p^2 = p'r.$$

228. Pedal Equation of Pedal Curve.—On referring to Fig. 51 above, it will be seen that p is the *radius vector* of the point Y of the pedal curve, while p' is the *perpendicular on the*

tangent. Hence we require a relation between p and p', which we shall afterwards change respectively into r and p.

Now, by Art. 227, $p'^2 = p'r$, or $r = p^2/p'$.

But if $f(p, r) = 0$ be the equation to the original curve, and we want a relation between p and p', we must substitute for r its value in terms of p and p'. Hence the relation is $f(p, p^2/p') = 0$. *Now* change p into r and p' into p, and we get for the pedal equation of the pedal curve $f(r, r^2/p) = 0$.

Hence, *the pedal curve can be obtained from the given curve by changing* p *into* r *and* r *into* r^2/p *in the original pedal equation.*

By repeated applications of this rule we can write down the pedal equations of the 1st, 2nd, 3rd, etc., $+^{ve}$ pedals.

229. Example.

Given the parabola $p^2 = ar$, *the pole being at the focus.*

First $+^{ve}$ pedal is $r^2 = a\,.\,\dfrac{r^2}{p}$, or $p = a$, *a straight line,* which is the tangent at the vertex.

Second $+^{ve}$ pedal is $r = a$, *a single point,* viz. the vertex.

Third $+^{ve}$ pedal is $r^2/p = a$, or $r^2 = ap$, *a circle, origin on the circumference,* viz. the circle on SA as diameter. This may be easily verified; thus, if P be a fixed point, and PP' a straight line which turns about P, then the locus of the foot of the perpendicular from the origin O to PP' is a circle on OP as diameter.

Fourth $+^{ve}$ pedal is $r^4/p^2 = ar$, or $r^3 = ap^2$, which we shall find to be the *cardioide.*

Similarly the nth $+^{ve}$ pedal is $r^{n-1} = ap^{n-2}$.

230. We can, moreover, pass back from a given curve to that of which it is the pedal, and which is called the *first negative pedal,* by changing r back to p_1, and r^2/p back to r_1 (using suffixes temporarily).

But if $r = p_1$, and $r^2/p = r_1$, then $p = r^2/r_1 = p_1^2/r_1$.

Hence, suppressing the suffixes, the rule is as follows :—

To obtain the first negative pedal, change r *into* p *and* p *into* p^2/r; which rule is at once deducible from the preceding rule, by interchanging the p and r therein.

231. Example.

From the last example we have

$r^3 = ap^2$, *a cardioide*; and by the rule above,

1st —^{ve} pedal is $p^3 = ap^4/r^2$, or $r^2 = ap$, *a circle.*

2nd —^{ve} pedal is $p^2 = ap^2/r$, or $r = a$, *a point.*

3rd —^{ve} pedal is $p = a$, *a straight line.*

4th —^{ve} pedal is $p^2/r = a$, or $p^2 = ar$, *a parabola*; etc.

nth —^{ve} pedal is $p^{n-2} = ar^{n-3}$.

232. System of Curves $r^m = a^m \cos m\theta$, or $r^m = a^m \sin m\theta$

We shall obtain the pedal equation of this system of curves, and also the polar equations of the pedal curves; first showing, however, that the two equations above represent the same system of curves.

For we have $r^m = a^m \sin m\theta = a^m \cos\left(\frac{\pi}{2} - m\theta\right) = a^m \cos m\left(\frac{\pi}{2m} - \theta\right)$

Hence, if we take for the initial line a line making an angle $\pi/2m$ with Ox, and measure the inclination of r to this line in the *negative* direction (calling this angle $+\theta$), we obtain the first equation.

To find the pedal equation, we have, from the first equation,

$$m \log r = m \log a + \log \cos m\theta.$$

Differentiating,
$$\frac{mr'}{r} = -m \tan m\theta,$$

$$\therefore \frac{r'}{r^2} = -\frac{1}{r}\tan m\theta,$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{r'^2}{r^4} = \frac{1}{r^2}\sec^2 m\theta.$$

$$\therefore p = r\cos m\theta, \quad \text{and } a^m p = r.a^m \cos m\theta = r^{m+1},$$

which is the pedal equation.

The equation of the pedal can be written down by the rule of Art. 228, and is

$$a^m r = r^{2m+2}/p^{m+1},$$

or
$$a^m p^{m+1} = r^{2m+1},$$

or
$$a^{m/(m+1)} p = r^{m/(m+1)+1},$$

which is *of the same form as before,* except that m is changed to $m/(m+1)$. Hence the polar equation of the pedal is

$$r^{\frac{m}{m+1}} = a^{\frac{m}{m+1}} \cos \frac{m}{m+1}\theta.$$

233. This can also be obtained as follows :—

We obtained above—

$$p = r\cos m\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But $$p = r\sin\phi$$

$$\therefore \phi = \frac{\pi}{2} \pm m\theta.$$

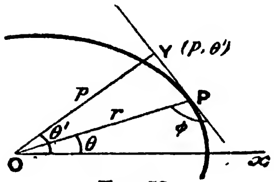

FIG. 52.

In drawing the figure, if we suppose $\pi/2 > m\theta > 0$, so that r is $+^{ve}$, *we must be careful to note that* r *diminishes as* θ *increases;* i.e. $\tan\phi$ *or* $r\frac{d\theta}{dr}$ *is* $-^{ve}$; *hence* ϕ *is obtuse.*†

We must, then, take the upper sign, so that

$$\phi = \frac{\pi}{2} + m\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Hence, if $\angle YOx = \theta'$, $\theta' - \theta = YOP = \phi - \frac{\pi}{2} = m\theta$

$$\therefore \theta' = (m+1)\theta, \text{ and } \theta = \frac{1}{m+1}\theta' \quad . \quad . \quad . \quad (2)$$

Since the pedal is the locus of Y, we must find a relation between p and θ', which we can do by eliminating r and θ from (1), (2) and the equation $r^m = a^m \cos m\theta$.

Thus $$p^m = r^m \cos^m m\theta = a^m \cos^{m+1} m\theta = a^m \cos^{m+1} \frac{m}{m+1}\theta'$$

$$\therefore p^{\frac{m}{m+1}} = a^{\frac{m}{m+1}} \cos \frac{m}{m+1}\theta',$$

and changing p and θ' into r and θ, we get the same as above.

† This caution is not necessary in the curves $r^m = a^m \sin m\theta$, as r increases with θ in this case.

234. The second $+^{ve}$ pedal becomes $r^k = a^k \cos k\theta$ where

$$k = \frac{m}{m+1} \Big/ \left(\frac{m}{m+1} + 1\right) = \frac{m}{2m+1},$$

and so on.

Again, if $\dfrac{m}{m+1} = m'$; $\therefore\ m = \dfrac{m'}{1-m'}$; hence, to find the first $-^{ve}$ pedal, we change m into $\dfrac{m}{1-m}$; and so on.

235. We may observe that the above system of curves is exactly the same as that given by $p^l = kr^n$; for comparing with $a^m p = r^{m+1}$ [Art. 232], we have $p = k^{1/l}.r^{n/l}$,

$$\therefore\ \frac{n}{l} = m + 1, \text{ or } m = \frac{n-l}{l},$$

$$\text{and } a^m = k^{-1/l}, \text{ or } a = k^{1/(l-n)}.$$

This system, therefore, includes the point, line, circle, parabola, etc. [see Art. 229], and is such that *the pedal curves, both* $+^{ve}$ *and* $-^{ve}$, *are curves of the same system.* Moreover, m may be anything, $+^{ve}$ or $-^{ve}$, integral or fractional.

236. Examples.

Ex. 1. If $m = 1$, $r = a\cos\theta$; a *circle*, origin on circumference.

Also $\dfrac{m}{m+1} = \frac{1}{2}$; hence the pedal is $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos\frac{1}{2}\theta$;

or $$r = a\cos^2\frac{\theta}{2} = \frac{a}{2}(1 + \cos\theta), \text{ the } \textit{cardioide}. \quad \text{[Art. 229.]}$$

Ex. 2. If $m = -\frac{1}{2}$, $r^{-\frac{1}{2}} = a^{-\frac{1}{2}} \cos(-\frac{1}{2}\theta)$;

or $$a = r\cos^2\frac{\theta}{2},$$

$$\therefore\ 2a/r = 1 + \cos\theta, \text{ a } \textit{parabola}, \text{ origin at focus.}$$

Since $\dfrac{m}{m+1} = -1$, the pedal is $r^{-1} = a^{-1}\cos(-\theta)$;

or $$a = r\cos\theta, \text{ a } \textit{straight line}.$$

Ex. 3. If $m = -2$, we have $r^{-2} = a^{-2}\cos(-2\theta)$; or $a^2 = r^2 \cos 2\theta$, a *rectangular hyperbola.*

Since $\dfrac{m}{m+1} = 2$, the pedal is $r^2 = a^2 \cos 2\theta$, the *lemniscate of Bernouilli.*

237. Pedal of $\dfrac{x^n}{a^n} + \dfrac{y^n}{b^n} = 1$.

In Ex. 12 of the last set of examples but one (p. 208), we have

$$\left(\frac{a}{\alpha}\right)^{n/(n-1)} + \left(\frac{b}{\beta}\right)^{n/(n-1)} = 1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

α and β being the intercepts which the tangent cuts off from Ox and Oy.† But if (r, θ) be the polar co-ordinates of Y, then $\alpha = r \sec\theta$, $\beta = r \operatorname{cosec}\theta$,

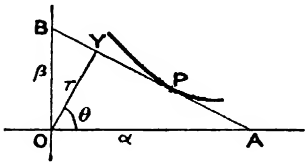

FIG. 53.

$\therefore$ in (1) $(a\cos\theta)^{n/(n-1)} + (b\sin\theta)^{n/(n-1)} = r^{n/(n-1)}$.

Or, changing to cartesians by multiplying throughout by $r^{n/(n-1)}$,

$$(ax)^{n/(n-1)} + (by)^{n/(n-1)} = (x^2 + y^2)^{n/(n-1)}.$$

Ex. 1. If $n = 2$, we have the ellipse, whose pedal is therefore

$$a^2x^2 + b^2y^2 = (x^2 + y^2)^2.$$

Ex. 2. If $n = \frac{1}{2}$, we have $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$, the parabola referred to two perpendicular tangents as axes, and the pedal is

$$\frac{1}{ax} + \frac{1}{by} = \frac{1}{x^2 + y^2}, \text{ or } (x^2 + y^2)(ax + by) = abxy.$$

238. The above method applies to the general equation

$$f(x, y) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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† For the tangent is $\dfrac{Xx^{n-1}}{a^n} + \dfrac{Yy^{n-1}}{b^n} = 1$, whence $\alpha = \dfrac{a^n}{x^{n-1}}$, $\beta = \dfrac{b^n}{y^{n-1}}$.

$$\therefore a/\alpha = (x/a)^{n-1}, \quad b/\beta = (y/b)^{n-1};$$

hence $\quad (a/\alpha)^{n/(n-1)} + (b/\beta)^{n/(n-1)} = (x/a)^n + (y/b)^n = 1.$

We may also adopt the following method :—

The tangent is $$Y - y = y'(X - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The line through the origin $\perp$ to this is

$$y'Y + X = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The result of eliminating x and y from (1), (2), and (3) is the equation to the pedal.

239. Radius of Curvature.

Def.—If the normals to two infinitely near points, P and Q, on a curve meet in E, then E is called the *centre of curvature*, and EP the *radius of curvature*, its length being denoted by ρ.

240. To prove that $\rho = ds/d\psi$.

In the figure of Art. 220, since PQ may be regarded as the arc of a circle, centre E, we have $\angle\, QEP = QP/EP$; and since $TQEP$ is concyclic, $\angle\, QEP = \angle\, T = d\psi$.

$$\therefore\ d\psi = ds/\rho, \text{ or } \rho = ds/d\psi.$$

241. Intrinsic Equation to a Curve.—This is the equation connecting the quantities s and ψ, s being the length of arc measured from a fixed point on the curve to the variable point, and ψ the inclination of the tangent at the latter point to the tangent at the fixed point.

Ex. 1. The equation to the circle is $s = a\psi$, a being the radius. And $\rho = ds/d\psi = a$; hence the radius of curvature of a circle is always the radius of the circle.

Ex. 2. The equation to the catenary can be shown to be $s = a \tan \psi$. Hence $\rho = a \sec^2 \psi$.

Ex. 3. The equation to the cycloid can be shown to be $s = a \sin \psi$. Hence $\rho = a \cos \psi$.

242. To prove that $\rho = r\dfrac{dr}{dp}$.

In the figure of Art. 220, since $ds = dr \sec \phi$,

and $d\psi = \angle VTZ = \dfrac{VZ}{TZ} = \dfrac{dp}{PY}$ (ultimately) $= \dfrac{dp}{r} \cdot \sec\phi$;

$$\therefore \rho = \frac{ds}{d\psi} = r\frac{dr}{dp}.$$

Ex. 1. *In the circle* $r^2 = ap$, $\rho = r\dfrac{dr}{dp} = \dfrac{a}{2}$, constant.

Ex. 2. *In the parabola* $p^2 = ar$; $\rho = r\dfrac{dr}{dp} = \dfrac{2pr}{a} = 2\sqrt{\dfrac{r^3}{a}}$.

243. To prove that $\rho = p + d^2p/d\psi^2$.

In Fig. 47, Art. 220, since $PY = q = dp/d\psi$, we have

$$r^2 = p^2 + (dp/d\psi)^2.$$

$$\therefore \text{ differentiating in } p,\ r\frac{dr}{dp} = p + \frac{dp}{d\psi}\cdot\frac{d}{dp}\left(\frac{dp}{d\psi}\right)$$

$$= p + \frac{dp}{d\psi}\cdot\frac{d}{d\psi}\left(\frac{dp}{d\psi}\right)\cdot\frac{d\psi}{dp};$$

$$\text{i.e. } \rho = p + \frac{d^2p}{d\psi^2}.$$

For a further discussion of curvature see Ch. XIX.

EXAMPLES XXXVI.

1. Prove that the pedal equation of—

(1) The equiangular spiral $r = a\epsilon^{\theta\cot\alpha}$ is $p = r\sin\alpha$.

(2) The hyperbolic spiral $r\theta = a$ is $1/p^2 = 1/r^2 + 1/a^2$.

(3) The parabola $2a/r = 1 + \cos\theta$ is $p^2 = ar$.

(4) The circle $r = a\cos\theta$ is $ap = r^2$.

2. Write down the 1st, 2nd, and nth $+^{ve}$ and $-^{ve}$ pedals of—

(1) $r/p = a$. (2) $p + r = a$. (3) $a^2r = p^3$. (4) $a^kp^k = r^{h+k}$.

3. Show that the pedal of the equiangular spiral $p = r\sin\alpha$ is the same spiral.

Show that the polar equation of the pedal is $r = a\sin\alpha\, e^{\left(\theta+\frac{\pi}{2}-\alpha\right)\cot\alpha}$;

and hence show that it is an equal spiral turned through a negative angle, whose value is $\frac{\pi}{2} - \alpha + \tan\alpha \,.\, \log\sin\alpha$. [See question 1 (1).]

4. Prove by first principles, not quoting the results of Arts. 232, 233, that in the curve $r = a\sin^3\frac{\theta}{3}$,

(1) $\phi = \theta/3$, and *not* $\pi - \theta/3$; (2) the pedal equation is $ap^3 = r^4$; and find the polar equation of its pedal.

5. Find the locus of the feet of perpendiculars from the origin on the tangents to the rectangular hyperbola $r^2\sin 2\theta = a^2$.

6. Answer the same question for the lemniscate $r^2 = a^2\sin 2\theta$.

7. Show that the pedal equation of the cardioide $2r = a(1 + \cos\theta)$ is $ap^2 = r^3$; and that the polar equation of its pedal is $r^{\frac{1}{3}} = a^{\frac{1}{3}}\cos\frac{1}{3}\theta$.

8. In the curve $r^m = a^m\sin m\theta$, prove that $\phi = m\theta$; and that the pedal equation is $a^m p = r^{m+1}$.

9. Show that the pedal equation of the curve $2r + a = a\theta^2$ is $r^2 = pr + ap$· and that the curve is the pedal of the curve $p = r - a$.

10. In the curve $r = a\sin^3\frac{\theta}{3}$, show that $\left(\frac{dr}{d\theta}\right)^2 = r^{\frac{4}{3}}(a^{\frac{2}{3}} - r^{\frac{2}{3}})$.

Hence show that the pedal equation is $ap^3 = r^4$.

11. In the curve $r^m = a^m\cos m\theta$, show that $\left(\frac{dr}{d\theta}\right)^2 = \frac{a^{2m} - r^{2m}}{r^{2m-2}}$.

Hence show that the pedal equation is $a^m p = r^{m+1}$.

12. Show (1) geometrically, (2) analytically, that the pedal equation to the ellipse, origin at focus, is $p^2(2a - r) = b^2 r$.

Find the pedal equation of the pedal, and show that it is a circle.

13. Show that the pedal equation of the spiral of Archimedes, $r = a\theta$, is $= p^2(a^2 + r^2)$; and that the latter is the pedal of the curve $p^2 = r^2 - a^2$. Prove that in the last-mentioned curve the normal touches a fixed rcle.†

14. Show that the pedal equation of the limaçon, $r = a + b\cos\theta$, is $= p^2(2ar - a^2 + b^2)$; and that this curve is the pedal of the circle

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† This curve, $p^2 = r^2 - a^2$, is called the ***involute of the circle***, since it is e curve traced by the end of a string which is *unwound* from a circle, the ring being kept stretched, as in unwinding a reel of cotton. This property llows from the statement in the question above.

$r^2 = 2ap - a^2 + b^2$, a being its radius, and b the distance of its centre from the origin.

15. Prove by elementary geometry the converse of the last question, that the pedal of the circle about any point is $r = a + b\cos\theta$.

16. Prove that the pedal equation of the cardioide $2r = a(1 + \sin\theta)$ is $ap^2 = r^3$.

17. Find ρ in terms of r in the following curves:—

(1) The parabola $p^2 = ar$. (2) The circle $ap = r^2 + b^2$.

(3) The logarithmic, or equiangular, spiral $p = r\sin\alpha$.

(4) The cardioide $r^3 = ap^2$.

18. The intrinsic equation to the involute of a circle is $2s = a\psi^2$. Show that $\rho = a\psi$; and also, from Question 13, that $\rho = p$, the origin being at the centre of the circle. Interpret geometrically the deduction that $p = a\psi$.

19. In the lemniscate, $r^2 = a^2\sin 2\theta$, prove that

(1) $\psi = 3\theta$; (2) $ds = a^2 d\theta/r$; and therefore (3) $\rho = a^2/3r$.

20. In the curve $r = a\sin^3\dfrac{\theta}{3}$, prove that

(1) $\psi = \frac{4}{3}\theta$; (2) $ds = (ar^2)^{\frac{1}{3}}\,d\theta$; and therefore (3) $\rho = \frac{3}{4}(ar^2)^{\frac{1}{3}}$.

21 In the curve $r^m = a^m\sin m\theta$, prove that

(1) $\psi = (m+1)\theta$; (2) $ds = \dfrac{a^m}{r^{m-1}}d\theta$; and (3) $\rho = \dfrac{a^m}{(m+1)r^{m-1}}$.

22. Prove that $\sin^2\phi\dfrac{d\phi}{d\theta} + r\dfrac{d^2r}{ds^2} = 0$.

23. Prove geometrically that $p = \dfrac{ds}{d\psi} - \dfrac{dq}{d\psi}$; and hence that

$$\rho = p + \frac{d^2p}{d\psi^2}.$$

24. Show, by the methods of Arts. 237 and 238, that the equation to the pedal of $xy = 2c^2$ is $(x^2 + y^2)^2 = 8c^2xy$.

25. Show, by the same methods, that the pedal of the parabola $y^2 = 4ax$, with respect to the origin, is the cissoid $ay^2 + x(x^2 + y^2) = 0$.

26. A point P moves in a curve such that the product of the distances of P from two fixed points, A, B, in the plane of the curve, is constant. If lines through A perpendicular to AP, and through B perpendicular to

BP meet the tangent at P in points Q and R respectively, prove PQ is equal to PR. [*See Ans.*]

Answers.

2. The nth $+^{\text{ve}}$ pedals are

(1) $\dfrac{r}{p} = a$; (2) $\dfrac{r^n}{p^n}(p + r) = a$; (3) $p^{2n-3}a^2 = r^{2n-1}$; (4) $a^k p^{nk+k} = r^{(n+1)k+k}$

The nth $-^{\text{ve}}$ pedals are

(1) $\dfrac{r}{p} = a$; (2) $\dfrac{p^n}{r^n}(p + r) = a$; (3) $r^{2n+1}a^2 = p^{2n+3}$; (4) $a^k r^{(n-1)k-k} = p^{nk-k}$.

4. $\theta' = \dfrac{4}{3}\theta - \dfrac{\pi}{2}$ [see Art. 233]; $\quad \therefore \dfrac{\theta}{3} = \dfrac{1}{4}\left(\theta' + \dfrac{\pi}{2}\right)$.

Also $p = r \sin\phi = r \sin\dfrac{\theta}{3} = a \sin^4\dfrac{\theta}{3} = a \sin^4\dfrac{1}{4}\left(\theta' + \dfrac{\pi}{2}\right)$.

Changing to r and θ, we get $r^{\frac{1}{4}} = a^{\frac{1}{4}} \sin\dfrac{1}{4}\left(\theta + \dfrac{\pi}{2}\right)$.

5. If $2\theta < \dfrac{\pi}{2}$, r *diminishes* as θ increases; $\therefore \phi$ is obtuse and $= \pi - 2\theta$; $\theta' = \dfrac{\pi}{2} - \theta$, and we shall obtain for the pedal $r^2 = a^2 \sin 2\theta$.

6. $r^{\frac{2}{3}} = a^{\frac{2}{3}} \sin\dfrac{2}{3}\left(\theta + \dfrac{\pi}{2}\right)$.

17. (1) $2r^{\frac{3}{2}}a^{-\frac{1}{2}}$; (2) $\frac{1}{2}a$; (3) $r \operatorname{cosec}\alpha$; (4) $\frac{2}{3}\sqrt{ar}$.

26. Let equation be $rr' = c$. Then, if ϕ, ϕ' be the inclinations of the tangent to AP and BP, we have

$PQ = r\sec\phi = r\dfrac{ds}{dr}$; $PR = r'\sec\phi' = -r'\dfrac{ds}{dr'}$, ds being the same for both.

$$\therefore \frac{PQ}{PR} = -\frac{r}{r'}\cdot\frac{dr'}{dr}. \quad \text{But } \log r + \log r' = \log c; \ \therefore \frac{dr}{r} + \frac{dr'}{r'} = 0.$$

$$\therefore \frac{PQ}{PR} = 1.$$

CHAPTER XVI.

ASYMPTOTES.

244. When a curve, corresponding to a given equation, has one or more branches extending to an infinite distance from the origin, it is possible to find some more elementary curve, *i.e.* one whose equation is simpler, to which the given curve approaches, and with which it ultimately coincides. The latter curve (viz. the more elementary one) is called a *curvilinear asymptote*, or is said to be *asymptotic* to the given curve.

We shall, however, only consider the cases in which the asymptotic curve is a straight line, in which case it is called a *rectilinear asymptote*, or simply an *asymptote*.

Assuming that a given curve has an asymptote, we may thus define it :—

Def.—*An asymptote* is a straight line, at a finite distance from the origin, to which a curve approaches indefinitely near as we recede along it to an infinite distance.

An asymptote is also defined as a tangent whose point of contact is at infinity; and again, as a straight line which meets the curve in two points at infinity.

245. Simple Examples.—When y can be expressed as an explicit function of x, and *vice versâ*, asymptotes parallel to the axes, if any exist, can be seen by inspection.

Ex. 1. $xy = ay - bx$.

Here $y = \dfrac{bx}{a - x}$. If now x either increase up to a or diminish down to

a, y will increase indefinitely. Hence $x = a$ is an asymptote, as may be easily seen by a figure. Since y changes from $+^{ve}$ to $-^{ve}$ as it passes through infinity (a and b being $+^{ve}$), the curve is above the axis of x to the left of the asymptote, and below to the right.

Similarly $x = \dfrac{ay}{y+b}$, and $y + b = 0$ is an asymptote.

The curve, of course, is a rectangular hyperbola.

Ex. 2. $(x^2 - a^2)(y^2 - b^2) = a^2b^2$.

$$\therefore\ y^2 = b^2 + \frac{a^2b^2}{x^2 - a^2} = \frac{b^2x^2}{x^2 - a^2}; \quad \therefore\ x = \pm a \text{ are two asymptotes.}$$

Similarly $x^2 = \dfrac{a^2y^2}{y^2 - b^2}$; $\therefore\ y = \pm b$ are two asymptotes.

246. General Statement.—Suppose $f(x, y) = 0$ to be the given curve; then, if y can be expanded in the form

$$y = \mu x + \beta + \frac{\gamma}{x^a} + \frac{\delta}{x^b} \cdots \qquad (1)$$

a, b, etc., being any *positive* quantities, it follows that when x is increased indefinitely the equation $f(x, y) = 0$ approaches the form

$$y = \mu x + \beta. \qquad (2)$$

And if (x, y_1) be a point on (2), the difference of the ordinates in (1) and (2) for the same value of x

$$= y - y_1 = \frac{\gamma}{x^a} + \frac{\delta}{x^b} \cdots,$$

which evidently vanishes when x becomes infinite.

Hence (2) is the asymptote of (1), and therefore of $f(x, y) = 0$. We shall, however, only be concerned with functions which can be expanded in the form

$$y = \mu x + \beta + \frac{\gamma}{x} + \frac{\delta}{x^2} \cdots.$$

We shall first consider oblique asymptotes, *i.e.* asymptotes *not* parallel to the axes.

247. Oblique Asymptotes—Direct Expansion.—When y is an explicit function of x, or when implicit contains no power higher than the square, it can be expanded in descending powers of x directly.

Ex. 1. $y^3 = x^2(x-a) = x^3\left(1-\dfrac{a}{x}\right).$

$$\therefore y = x\left(1-\frac{a}{x}\right)^{\frac{1}{3}} = x\left(1-\frac{a}{3x}-\frac{a^2}{9x^2}\cdots\right) = x-\frac{a}{3}-\frac{a^2}{9x}\cdots$$

Hence $y = x - \dfrac{a}{3}$ is an asymptote.

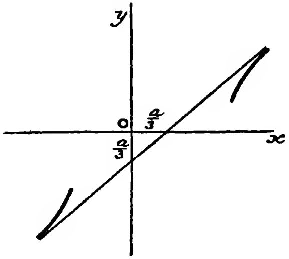

Fig. 54.

Moreover, if a is $+^{ve}$, then when x is $+^{ve}$ the "y" of the curve is *less* than $x - \dfrac{a}{3}$ by $\dfrac{a^2}{9x}$ (since the terms beyond this are negligible), hence the curve is *below* the asymptote in the right hand portion of the figure. But when x is $-^{ve}$, then the "y" of the curve is *greater* than $x - \dfrac{a}{3}$, and the curve is *above* the asymptote in the left hand portion of the figure.

Ex. 2. $y^2 = \dfrac{x^2(x+a)(x-2a)}{(x-a)(x+2a)} = \dfrac{x^2(x^2-ax-2a^2)}{x^2+ax-2a^2} = \dfrac{x^2\left(1-\dfrac{a}{x}-\dfrac{2a^2}{x^2}\right)}{1+\dfrac{a}{x}-\dfrac{2a^2}{x^2}}.$

$$\therefore y = \pm x\left(1-\frac{a}{x}-\frac{2a^2}{x^2}\right)^{\frac{1}{2}}\left(1+\frac{a}{x}-\frac{2a^2}{x^2}\right)^{-\frac{1}{2}}$$

$$= \pm x\left(1-\frac{a}{2x}\cdots\right)\left(1-\frac{a}{2x}\cdots\right) = \pm x\left(1-\frac{a}{x}\cdots\right) = \pm(x-a).$$

The two asymptotes are, therefore, $y = x - a$, and $y = -x + a$. $x - a = 0$ and $x + 2a = 0$ are, of course, two other asymptotes.

248. Method for Implicit Functions.—The method which follows is similar to that given in Arts. 123–125 (*q.v.*).

We shall first give a simple example to illustrate the method, and then take the general case. The methods *actually adopted in practice* will be given afterwards.

Ex. *Expand y in descending powers of x when*

$$x^3 + y^3 + 2y^2 - x^2 + 1 = 0, \quad \ldots \quad (1)$$

and find the equation to the asymptote.

Assume $y = \mu x + \beta + \frac{\gamma}{x} \ldots$

Substituting this value of y in (1), we get

$$x^3 + \left(\mu x + \beta + \frac{\gamma}{x} \ldots\right)^3 + 2\left(\mu x + \beta + \frac{\gamma}{x} \ldots\right)^2 - x^2 + 1 = 0,$$

or $(1 + \mu^3)x^3 + (3\mu^2\beta + 2\mu^2 - 1)x^2 + (3\mu\beta^2 + 3\mu^2\gamma + 4\mu\beta)x \ldots = 0,$ which must therefore be an identity in x.

Equating to zero the coefficients of x^3, x^2, etc., we shall have

$$\begin{aligned} 1 + \mu^3 &= 0, \quad \text{whence } \mu = -1, \\ 3\mu^2\beta + 2\mu^2 - 1 &= 0, \quad \text{whence } \beta = -\tfrac{1}{3}, \\ 3\mu\beta^2 + 3\mu^2\gamma + 4\mu\beta &= 0, \quad \text{whence } \gamma = -\tfrac{1}{3}; \text{ etc., etc.} \end{aligned}$$

$$\therefore y = -x - \frac{1}{3} - \frac{1}{3x} \ldots$$

The asymptote is $y = -x - \frac{1}{3}$, or $3(x + y) + 1 = 0$, and the curve is *below* the asymptote to the right, and *above* to the left.

249. General Case.

Let $f(x, y) = 0$ be the curve, and arrange $f(x, y)$ into homogeneous groups of the nth, $(n-1)$th, $(n-2)$th, etc. degrees; *i.e.* let

$$f(x, y) \equiv x^n\phi(y/x) + x^{n-1}\psi(y/x) + x^{n-2}\chi(y/x) + \ldots = 0,$$

which may be also written

$$u_n + u_{n-1} + u_{n-2} + \ldots = 0.$$

Assume $y = \mu x + \beta + \frac{\gamma}{x} + \ldots$; substitute for y this value, and expand by Taylor's Theorem, afterwards arranging in descending powers of x.

We have

$$\begin{aligned} f(x, y) &\equiv x^n\phi\left(\mu + \frac{\beta}{x} + \frac{\gamma}{x^2} \ldots\right) + x^{n-1}\psi\left(\mu + \frac{\beta}{x} + \frac{\gamma}{x^2} \ldots\right) + x^{n-2}\chi\left(\mu + \frac{\beta}{x} \ldots\right) \\ &= x^n\left\{\phi(\mu) + \left(\frac{\beta}{x} + \frac{\gamma}{x^2} \ldots\right)\phi'(\mu) + \frac{1}{2}\left(\frac{\beta}{x} + \ldots\right)^2 \phi''(\mu) \ldots\right\} \\ &\quad + x^{n-1}\left\{\psi(\mu) + \left(\frac{\beta}{x} + \ldots\right)\psi'(\mu) + \ldots\right\} + x^{n-2}\left\{\chi(\mu) \ldots\right\} + \text{etc.} \\ &= x^n\phi(\mu) + x^{n-1}\{\psi(\mu) + \beta\phi'(\mu)\} \\ &\quad + x^{n-2}\{\chi(\mu) + \beta\psi'(\mu) + \gamma\phi'(\mu) + \tfrac{1}{2}\beta^2\phi''(\mu)\} + \ldots = 0. \end{aligned}$$

Hence, since this is an identity as far as x is concerned, we have, equating to zero the coefficients of x^n, x^{n-1}, ...,

$$\begin{aligned}&\phi(\mu) = 0, \text{ whence } \mu\,;\\ &\psi(\mu) + \beta\phi'(\mu) = 0, \text{ or } \beta = -\psi(\mu)/\phi'(\mu)\,;\\ &\chi(\mu) + \beta\psi'(\mu) + \gamma\phi'(\mu) + \tfrac{1}{2}\beta^2\phi''(\mu) = 0,\end{aligned}$$

giving γ ; etc.

If $\mu_1, \mu_2, \ldots \mu_n$ be the n roots of $\phi(\mu) = 0$, the asymptotes are

$$y = \mu_1 x - \frac{\psi(\mu_1)}{\phi'(\mu_1)}, \quad y = \mu_2 x - \frac{\psi(\mu_2)}{\phi'(\mu_2)}; \text{ etc.}$$

250. The equation to the straight lines through the origin *parallel* to the asymptotes, is

$$(y - \mu_1 x)(y - \mu_2 x) \ldots (y - \mu_n x) = 0 \quad . \quad . \quad . \quad (1)$$

But we can show that this is the same as equating to zero the terms of highest degree in $f(x, y)$. For, doing this latter thing, we have

$$x^n\phi(y/x) = 0,$$

$$\text{i.e. } \phi(y/x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the roots of which are $y/x = \mu_1$, $y/x = \mu_2$, etc.

i.e. $y - \mu_1 x = 0$, $y - \mu_2 x = 0$, etc.

∴ equations (1) and (2) are identical, which proves the statement.

Hence the important conclusion :—

The terms of highest degree in x *and* y *equated to zero, give the straight lines through the origin parallel to the asymptotes.*

Cor. It follows from the preceding and from Art. 256 (*q.v.*) that a curve of the nth degree cannot have more than n asymptotes.

NOTE 1.—We do not generally require to go further than the coefficient of x^{n-1}, to find μ and β.

NOTE 2.—We can, of course, obtain the asymptotes equally well by putting $x = \mu y + \beta + \dfrac{\gamma}{y} \ldots$

251. Examples.

Ex. 1. *Find the asymptotes of the curve in Art.* 248.

Writing the curve in the form

$$(x + y)(x^2 - xy + y^2) + 2y^2 - x^2 + 1 = 0,$$

the lines $(x+y)(x^2-xy+y^2)=0$ are parallel to the asymptotes; the second factor, however, gives imaginary roots.

Hence there is only one real asymptote, which is parallel to $y+x=0$.

Put $y+x=\beta$, or $y=\beta-x$;

then $$\beta\{x^2-x(\beta-x)+(\beta-x)^2\}+2(\beta-x)^2-x^2+1=0;$$

or $$x^2(3\beta+1)+\ldots=0,$$

omitting terms below x^2.

Hence $3\beta+1=0$, and $\beta=-\frac{1}{3}$.

$$\therefore \text{ the asymptote is } y+x=-\tfrac{1}{3}; \text{ or } 3(x+y)+1=0.$$

Ex. 2. $x^4-y^4-3x^3-xy^2-2x+1=0$.

Since $x^4-y^4=(x^2+y^2)(x+y)(x-y)=0$ gives lines through the origin parallel to the asymptotes, it follows that there are only two real asymptotes, viz. those parallel to $x\pm y=0$.

(1) Put $y+x=\beta$; or $y=\beta-x$.

$$\therefore x^4-(\beta-x)^4-3x^3-x(\beta-x)^2\ldots=0.$$

The coefficient of x^4 vanishes naturally; equating to zero the coefficient of x^3, we have

$$4\beta-3-1=0, \text{ or } \beta=1.$$

$$\therefore \text{ asymptote is } x+y=1.$$

(2) Put $y-x=\beta$.

$$\therefore x^4-(\beta+x)^4-3x^3-x(\beta+x)^2\ldots=0.$$

$$\therefore -4\beta-3-1=0; \text{ and } \beta=-1.$$

$$\therefore \text{ asymptote is } y-x=-1, \text{ or } x-y=1.$$

252. We should have obtained the same result by putting

$$x=\beta-y \text{ in (1), and } x=y-\beta \text{ in (2).}$$

Thus in (2) we have

$$(y-\beta)^4+y^4-3(y-\beta)^3-(y-\beta)y^2\ldots=0.$$

Coefficient of $y^3=-4\beta-3-1=0$; and $\beta=-1$.

$$\therefore \text{ asymptote is } x-y=1, \text{ as before.}$$

253. Second Method.—Reverting again to the equation

$$f(x,y)\equiv x^n\phi(y/x)+x^{n-1}\psi(y/x)+x^{n-2}\chi(y/x)+\ldots=0,$$

let $y-\mu x$ be a factor of the terms of highest degree, $x^n\phi(y/x)$ so that $y-\mu x=\beta$ is an asymptote, β being at present unknown. Also let $$x^n\phi(y/x)=(y-\mu x).x^{n-1}\phi_1(y/x).$$

Then the equation to the curve may be written

$$(y - \mu x).x^{n-1}\phi_1(y/x) + x^{n-1}\psi(y/x) + x^{n-2}\chi(y/x) \dots = 0,$$

or
$$y - \mu x = -\frac{\psi\left(\frac{y}{x}\right) + \frac{1}{x}\chi\left(\frac{y}{x}\right) \dots}{\phi_1(y/x)} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, when y and x increase indefinitely—their ratio, however, being finite—the right-hand side of (1) has a finite limit, and this limit will be β, provided that (x, y) is on the branch approaching the asymptote.

In this case, since $y - \mu x =$ a finite quantity, x and y, being large,

$$\therefore y/x = \mu + \text{an infinitesimal.}$$

Hence, ultimately,* we may regard μ as the ratio of y to x; † and since the term $\frac{1}{x}\chi\left(\frac{y}{x}\right)$ in (1) is negligible, we may obtain β by putting $y = \mu x$ in the fraction above, neglecting the terms of lower degree.

254. Examples.

Ex. 1. $(x - y)(x - 2y)(x^2 + 4y^2) + 2x^3 - xy^2 + x + 1 = 0.$

(1) $x - y = \dfrac{xy^2 - 2x^3 \dots}{(x - 2y)(x^2 + 4y^2)}.$

Put $y = x$ on the right-hand side, and neglect lower powers of x; the asymptote is therefore

$$x - y = \underset{x=\infty}{lt} \frac{x^3 - 2x^3}{(x - 2x)(x^2 + 4x^2)} = \frac{1}{5}.$$

(2) Similarly, the other asymptote is

$$x - 2y = \underset{x=2y=\infty}{lt} \frac{xy^2 - 2x^3}{(x - y)(x^2 + 4y^2)} = \underset{y=\infty}{lt} \frac{2y^3 - 16y^3}{y(4y^2 + 4y^2)} = -\frac{7}{4}.$$

The remaining two asymptotes are imaginary.

Ex. 2. $(y - x + 1)(x^2 + y^2) + x - 2y = 1.$

$y - x = 0$ is parallel to the asymptote.

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† This is equivalent to stating that as P moves along the curve in the direction of the asymptote, OP becomes more and more nearly parallel to the asymptote.

We may in this case put

$$y - x + 1 = \underset{y=x=\infty}{lt} \frac{2y - x}{x^2 + y^2} = 0,$$

since the numerator is of lower degree than the denominator.

In the same way we can show that in the curve

$$(ax + by + c)u_{n-1} + u_{n-2} + \dots = 0,$$

u_{n-1}, u_{n-2} *being homogeneous groups of the* $(n - 1)$*th and* $(n - 2)$*th degrees,* $ax + by + c = 0$ *is an asymptote.*

Ex. 3. $(2x - y + 1)(x^4 + y^4) - x^3 + 3y^3 - x - 1 = 0.$

By the preceding rule, $2x - y + 1 = 0$ is an asymptote.

Ex. 4. $(x - y - 1)(x + y - 1)(x - 2y - 1) + 3x - 2y - 1 = 0.$

The asymptotes are, by the same rule,

$$x - y - 1 = 0, \quad x + y - 1 = 0, \quad x - 2y - 1 = 0.$$

Ex. 5. $(y - x + 1)(x^2 + y^2) + x^2 - 2y^2 = 1.$

Here $y - x + 1 = 0$ is *not* an asymptote, for we have

$$y - x + 1 = \underset{y=x=\infty}{lt} \frac{2y^2 - x^2}{x^2 + y^2} = \frac{1}{2},$$

the numerator being of the same degree as the denominator.

The asymptote is therefore $y - x + \frac{1}{2} = 0.$

Note.—To find the next term of the expansion, put $y = x - \frac{1}{2} + \frac{\beta}{x}$ in the equation to the curve, and equate coefficients of the highest power of x which remains.

EXAMPLES XXXVII.

1. Find by inspection the asymptotes of the following curves:—

(1) $xy = a^2$. (2) $y = \log x$.

(3) $y = e^{x-a}$. (4) $y = \tan x$.

(5) $y = \dfrac{x}{x-1}$. (6) $y^2 = a^2\dfrac{x^2 - a^2}{x^2 + a^2}$.

(7) $y^2(x - a) = a^2(x + a)$. (8) $y = \dfrac{ay + b}{x - c}$.

(9) $e^x y = x$.

2. Find by direct expansion of x in terms of y, or y in terms of x, the asymptotes of the following curves:—

(1) $y = \dfrac{x^2}{x-a}$.   (2) $y = \dfrac{x(x-a)}{x+a}$.

(3) $x^2 - y^2 = a^2$.   (4) $xy^2 = x^3 - a^3$.

(5) $y^2 = \dfrac{x^2(x+a)(x+2a)}{(x-a)(x-2a)}$.   (6) $\dfrac{(a+b)x}{x^2+ab} = \dfrac{y^2-x^2}{y^2+x^2}$.

(7) $(y-a)^2 = x^2 - 2a^2$.   (8) $(y-a)^2(x-a) = x^3 + a^3$.

(9) $x^3 - y^3 = (x-1)^2$.   (10) $(y-x)^2 = 4x^2 - a^2$.

(11) $(y-x)^2 - 2y = x^2 + \dfrac{1}{x^2}$.   (12) $(y-x)^2 - 2ay = \dfrac{x^2(x^2+a^2)}{x^2-a^2}$.

3. Adopt either or both of the methods of Arts. 249 and 253 to find the asymptotes of:—

(1) $x^2 - y^2 = ax + by + c$.

(2) $(x-2y)(x-3y) = 2a(x-y) + b^2$.

(3) $(x-y)(x^2-2xy+2y^2) = 2x^2 - 3y$.

(4) $x^3 + y^3 = 3axy$.

(5) $x^2(a-x) = y^2(b-y)$.

(6) $(x^2-y^2)(x+2y) = y^2 - y + 1$.

(7) $(x^2-a^2)(y^2-b^2) = 2x^4 - y^4 + a(x^3+2y^3)$.

(8) $(x-2y+a)(x-y+2a)(x+y+a) = b^2(x+y)$.

(9) $(x-2y+a)(x-y+2a)(x+y+a) = b(x^2+y^2)$.

(10) $(x-2y+a)(x-y+2a)(x+y+a) = 4y^2(x-y)$.

4. Find the asymptotes of the following curves, stating on which side of the asymptote each branch lies:—

(1) $y = \dfrac{x(x-1)}{x+1}$.   (2) $y^2 = x^2\dfrac{x-a}{x+a}$.

(3) $y^2 = x^2\dfrac{x^3-a^3}{x^3+a^3}$.   (4) $x^3 - y^3 + ax^2 + axy + b^3 = 0$.

(5) $(x+2y)(x^2+y^2) = 2x^2 - 3xy + y^2 + x + y + 1$.

ANSWERS.

1. (1) $x = 0$; $y = 0$. (2) $x = 0$. (3) $y = 0$. (4) $x = (2n+1)\frac{\pi}{2}$.

(5) $x = 1$; $y = 1$. (6) $y = \pm a$. (7) $x = a$; $y = \pm a$.

(8) $x = a + c$; $y = 0$. (9) $y = 0$, since $\underset{x=\infty}{lt}\ (x/e^x) = 0$.

2. (1) $y = x + a$; $x = a$. (2) $y = x - 2a$; $x + a = 0$. (3) $x \pm y = 0$.

(4) $y = \pm x$; $x = 0$. (5) $y = \pm(x + 3a)$; $x = a$; $x = 2a$.

(6) $\pm y = x + a + b$; $x = a$; $x = b$. (7) $y = a \pm x$.

(8) $2(x - y) + 3a = 0$; $2(x + y) - a = 0$. (9) $3(x - y) = 1$.

(10) $y + x = 0$; $y - 3x = 0$. (11) $y = 2(x + 1)$; $y = 0$; $x^2 = 0$.

(12) $y = 2(x + a)$; $y = 0$; $x = \pm a$.

3. (1) $2(x + y) = a - b$; $2(x - y) = a + b$.

(2) $x - 2y + 2a = 0$; $x - 3y = 4a$. (3) $x - y = 2$.

(4) $x + y + a = 0$. (5) $3(x - y) = a - b$.

(6) $6(x - y) = 1$; $2(x + y) + 1 = 0$; $3(x + 2y) = 1$.

(7) $2(x - y) + a = 0$; $6(x + y) = a$.

(8) $x - 2y + a = 0$; $x - y + 2a = 0$; $x + y + a = 0$.

(9) $3(x - 2y + a) = 5b$; $x - y + 2a + b = 0$; $3(x + y + a) = b$.

(10) $3(x - y) + 2a = 0$; $5(x - 3y) + 9a = 0$; $15(x + 2y) + 23a = 0$.

4. (1) $y = x - 2$; above to the right, below to the left; $x + 1 = 0$; to the left below Ox, to the right above.

(2) $y = x - a$; above to the right, below to the left; $x + a = 0$; to the left above and below Ox.
$y = -x + a$; below to the right, above to the left.

(3) $y = x$; below to the right, below to the left; $x + a = 0$; to the left above and below Ox.
$y + x = 0$; above to the right, above to the left.

(4) $y = x + \frac{2a}{3}$; below to the right, above to the left.

(5) $x + 2y = 3$; below to the right, above to the left.

255. Exceptional Cases.—We shall now consider the equations of Art. 249, viz.—

$$\phi(\mu) = 0 \quad \ldots \quad (1)$$
$$\psi(\mu) + \beta\phi'(\mu) = 0 \quad \ldots \quad (2),$$
$$\chi(\mu) + \beta\psi'(\mu) + \gamma\phi'(\mu) + \tfrac{1}{2}\beta^2\phi''(\mu) = 0 \quad \ldots \quad (3)$$

The following cases may arise :—

I. Suppose that $\psi(\mu) = 0$, when $\phi(\mu) = 0$.

This means either (1) that the same value of μ makes both $\phi(\mu)$ and $\psi(\mu)$ vanish—*i.e.* that $\phi(\mu)$ and $\psi(\mu)$, and therefore u_n *and* u_{n-1} [Art. 249], *have a common factor*,—or (2) that *the terms giving* $\psi(\mu)$, *viz.* u_{n-1}, *are absent.* In either case $\beta = 0$, and $y = \mu x$ *is the asymptote.*

Ex. 1. $x^3 - y^3 + x^2 - y^2 = x + 1$.

Ex. 2. $x^3 - y^3 = x - 2y + 1$.

In both cases $x - y = 0$ is the asymptote.

II. Suppose that both $\psi(\mu) = 0$ and $\phi'(\mu) = 0$, when $\phi(\mu) = 0$.

If $\phi(\mu)$ and $\phi'(\mu)$ simultaneously vanish when $\mu = \mu_1$ say, then it is known, by the Theory of Equations, that $\phi(\mu)$ has a square factor $(\mu - \mu_1)^2$ †

Hence u_n *has the square factor* $(y - \mu_1 x)^2$; and since $\psi(\mu_1) = 0$,

$\therefore$ u_{n-1} *has a factor* $y - \mu_1 x$; *or*, u_{n-1} *may be absent.*

In either case β cannot be found from (2) above.

But turning to (3), and putting $\phi'(\mu_1) = 0$, we have

$$\chi(\mu_1) + \beta\psi'(\mu_1) + \tfrac{1}{2}\beta^2\phi''(\mu_1) = 0,$$

which is a quadratic in β ; so that there are two values of β, giving *two parallel asymptotes.*

If $\psi(\mu)$ is absent, $\psi'(\mu)$ will be absent also ; and the two values of β will be equal and opposite.

The same thing will happen if $\psi'(\mu) = 0$, or $\psi(\mu)$ has the square factor $(\mu - \mu_1)^2$; *i.e.* u_{n-1} has the square factor $(y - \mu_1 x)^2$.

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† For if $\phi(\mu_1) = 0$, $\mu - \mu_1$ is a factor of $\phi(\mu)$.
Put $\phi(\mu) = (\mu - \mu_1)f(\mu)$, and differentiate in μ;

$$\therefore \phi'(\mu) = f(\mu) + (\mu - \mu_1)f'(\mu).$$

But $\because$ $\phi'(\mu_1) = 0$; $\therefore$ $\mu - \mu_1$ is a factor of $\phi'(\mu)$;
$\therefore$ $\mu - \mu_1$ is a factor of $f(\mu)$; $\therefore$ $(\mu - \mu_1)^2$ is a factor of $\phi(\mu)$.

Ex. 1. $(x-y)^2(x^2+y^2)+a(x^3-y^3)+a^2x^2+a^4=0.$

Put $y-x=\beta$, or $y=x+\beta$.

$$\therefore\ \beta^2\{x^2+(x+\beta)^2\}+a(-\beta)\{x^2+x(x+\beta)+(x+\beta)^2\}+a^2x^2+a^4=0.$$

$$\therefore\ \text{Coefficient of } x^2 = 2\beta^2-3a\beta+a^2=0.$$

$$\therefore\ \beta=a, \quad \text{or } a/2.$$

The asymptotes are therefore $y-x=a$, and $y-x=a/2$.

Otherwise:—$\beta^2(x^2+y^2)-a\beta(x^2+xy+y^2)+a^2x^2\ldots=0$; and putting $y=x$ and dividing by x^2, $2\beta^2-3a\beta+a^2=0$; etc.

Ex. 2. $(y+2x)^2(x^2+y^2)-2x^2+xy-4y^2=x-1.$

Put $y+2x=\beta$, or $y=-2x+\beta$.

$$\therefore\ \beta^2\{x^2+4x^2\ldots\}-2x^2+x(-2x\ldots)-16x^2\ldots=0.$$

$$\therefore\ 5\beta^2-20=0;\quad \beta=\pm 2.$$

The asymptotes are therefore $y+2x=2$, and $y+2x=-2$.

Otherwise:—$\beta^2(x^2+y^2)-2x^2+xy-4y^2\ldots=0$; put $y=-2x$ and divide by x^2, $\therefore\ 5\beta^2-20=0$; etc.

Ex. 3. $(y-x)^2(x^2+y^2)+(y-x)^2(x+y)=x^2+7y^2+1.$

Putting $y-x=\beta$, we have $\beta^2(x^2+x^2\ldots)+\beta^2(\ldots\ldots)=x^2+7(x^2\ldots)\ldots$

$$\therefore\ 2\beta^2=8;\quad \text{and } \beta=\pm 2.$$

The asymptotes are therefore $y-x=\pm 2$.

Otherwise:—$\beta^2(x^2+y^2)\ldots=x^2+7y^2\ldots$; $2\beta^2=8$; etc.

For the case in which $\phi'(\mu)=0$, while $\psi(\mu)$ is finite (so that $\beta=\infty$), the reader is referred to other text-books.

EXAMPLES XXXVIII.

Find the asymptotes of the following curves:—

1. $x^4-y^4=a^2x^2+b^2y^2+a^2b^2.$
2. $(x+y)(x+2y)(x+3y)=x+y+1.$
3. $x^3+y^3+a(x^2-y^2)=a^3(x+a).$
4. $(x+2y)(x^2-y^2-2x+y)=x-y.$
5. $x^3(a^2-x^2)=y^3(b^2-y^2).$

6. $x^4 - y^4 + a(x^3 - y^3) = a^2(x^2 + ay + a^2)$.

7. $(x - 2y)(x^3 - y^3) + x^2(x - y) = 2xy(y - 1)$.

8. $y^2(y - a)^2 = (x^2 + a^2)(x - a)^2$.

9. $y^2\dfrac{y - a}{x - a} = x^2\dfrac{x - b}{y - b}$.

10. $(x + y)^2(x^2 + y^2) - (x + y)(2x^2 - y^2) = 2x^2 + xy - 1$

11. $(x - 2y)^2(2x^2 + y^2) - (x - 2y)(x^2 + 2y^2) + y^2 = 1$.

12. $(3x - y)^2(3x^2 + y^2) = (y - x)(3x - y) - 2x + 3$.

13. $(3x - y)^2(3x^2 + y^2 - 2x) = (y - x)(3x + y) + 5x - 2$.

14. $(x - 2y)^2(x^2 + y^2) + y^2 - x - y + 1 = 0$.

15. $(x - 2y)^2(x^2 + y^2) - (x - 2y)(x^2 - y^2) + y^2 = 0$.

16. $(x + y)^2(x - y) + 2(x^2 - y^2) = 1 - 2x$.

17. $(x + 2y)^2(x - y) + x^2 - 4y^2 = 2x - 1$.

18. $(3x - 2y)^2(x^2 + 2xy + 6y^2) + 5(x^2 - 2y^2) = 1$

19. $(x - y)^2(x^3 + y^3) + x^4 - y^4 = x^2 - 2y - 3$.

20. $(x - y + 1)^2(x + y + 2) + x - 3y = 4$.

21. $(x^2 - 3xy + 2y^2)^2 + x(x - 2y) = 3x$.

ANSWERS.

1. $x \pm y = 0$. 2. $x + y = 0$; $x + 2y = 0$; $x + 3y = 0$.
3. $x + y = 0$. 4. $x + 2y = 0$; $2(x - y) = 1$; $2(x + y) = 3$.
5. $x = y$. 6. $x = y$; $2(x + y) + a = 0$. 7. $x = 2y$; $3(x - y) + 2 = 0$.
8. $x = y$; $x + y = a$. 9. $x = y$; $2(x + y) = a + b$.
10. $x + y = 1$; $2(x + y) + 1 = 0$. 11. $3x - 6y = 1$ (coincident).
12. $y = 3x \pm 1$. 13. $y = 3x \pm 1$. 14. Imaginary.
15. Imaginary. 16. $x = y$; $x + y + 1 = 0$ (coincident).
17. Two imaginary; $3(x - y) = 1$. 18. $3(3x - 2y) = \pm 1$.
19. $x = y$; $x - y + 2 = 0$. 20. $x = y$; $x = y - 2$; $x + y + 2 = 0$.
21. $x = 2y$; $x - y = \pm 1$.

256. Asymptotes Parallel to the Coordinate Axes.

We have seen that when $y - \mu x$ is a factor of u_n, $y = \mu x + \beta$ is an asymptote.

I. Let $\mu = 0$, so that the asymptote is parallel to Ox.

Then y is a factor of u_n; hence the term x^n must be missing.

Arrange $f(x, y)$ in descending powers of x thus :—

$$(ay + b)x^{n-1} + (cy^2 + dy + e)x^{n-2} \ldots = 0 \quad . \quad . \quad (1)$$

Now put $y = \beta + \dfrac{\gamma}{x} \ldots$, and only retain the highest powers of x;

$$\therefore\ (a\beta + b)x^{n-1} + \text{lower powers of } x = 0.$$

Hence $$a\beta + b = 0, \text{ or } \beta = -\frac{b}{a};$$

$$\therefore\ ay + b = 0 \text{ is the asymptote.}$$

That is, "coefficient of $x^{n-1} = 0$" gives the asymptote.

Next, suppose u_n contains y^2 as a factor. Then the terms x^n, $x^{n-1}y$ must be absent; $\therefore$ in (1) $a = 0$.

This, however, makes $\beta = \infty$, unless $b = 0$ also.

Supposing this to be so, (1) becomes

$$(cy^2 + dy + e)x^{n-2} + (fy^3 \ldots)x^{n-3} \ldots = 0.$$

Now put $y = \beta + \dfrac{\gamma}{x} \ldots$ as before, and we get $c\beta^2 + d\beta + e = 0$.

Hence, putting y for β, the asymptotes are given by

$$cy^2 + dy + e = 0.$$

Or, "coefficient of $x^{n-2} = 0$" gives the asymptotes; and so on.

II. Let $\mu = \infty$. In this case let x be expanded in terms of y thus :—

$$x = \mu' y + \beta' + \frac{\gamma'}{y} \ldots$$

Then $\mu' = 0$,† which is analogous to the preceding case.

Hence the rule :—*If the terms of highest degree in* $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ *are absent, there are asymptotes parallel to the axis of* $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$, *which are found by equating to zero the complete* coefficient of the highest power *of* $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ *present.*

† For $y = \mu x$ is parallel to the asymptote; and when $\mu = \infty$, this becomes $x = 0$.

257. Examples.

Ex. 1. $xy(x^2 + y^2) + 2x^3 - xy^2 + x^2 = 1.$

There are two real asymptotes, parallel to $x = 0$ and $y = 0$; and two imaginary ones.

Coefficient of highest power of y is x;

$$\therefore x = 0 \text{ is an asymptote.}$$

Coefficient of highest power of x is $y + 2$;

$$\therefore y + 2 = 0 \text{ is the other asymptote.}$$

Ex. 2. $xy^2(x - y) - 5x^2y = y^3 - 6x^2.$

Arranging in descending powers of y,

$$-(x + 1)y^3 + x^2y^2 - \dots = 0;$$

$$\therefore x + 1 = 0 \text{ is an asymptote.}$$

Arranging in descending powers of x,

$$(y^2 - 5y + 6)x^2 - \dots = 0;$$

$$\therefore y^2 - 5y + 6 = 0, \text{ i.e. } y = 2 \text{ and } y = 3, \text{ are two asymptotes.}$$

Also $x - y = \underset{x=y=\infty}{lt} \dfrac{5x^2y + y^3}{xy^2} = 6$, is the remaining asymptote.

Ex. 3. $x^2y^2 = a^2(x^2 - y^2).$

The asymptotes are $x^2 + a^2 = 0$, imaginary; and

$$y^2 - a^2 = 0, \quad \text{or } y = \pm a.$$

EXAMPLES XXXIX.—GENERAL.

1. Find the asymptotes of :—

(1) $y^2(2a - x) = x^3.$

(2) $x^2y = y - 2a.$

(3) $ay(x + 2a) = x^2y + a^3.$

(4) $xy(y - x) = a(3x^2 + 2y^2).$

(5) $xy^2 + 2x^2y = a(x^2 + y^2).$

(6) $(a + x)(b^2 + x^2) = (x + c)y^2.$

(7) $a^2/x^2 + b^2/y^2 = 1.$

(8) $xy^2(x - y) - 5x^2y = y^3 - 6x^2 + x.$

(9) $x^3(y - 2a) + x^2(y^2 - 3a^2) - a^3y + a^3x = 0.$

(10) $xy^3 - 7yx^3 - 6x^4 + 4a(x^3 + y^3) = a^4.$

(11) $xy^3 - x^3y - ay^3 - ax^3 = 0.$

(12) $y^2(x - 2a) = x^2(x - a).$

(13) $x^3(y - a) = y(y + a)^2(y - 2a)$.

(14) $(y - a)^2 = (x^4 + a^4)/(x^2 - a^2)$.

(15) $(x - y - 1)^2(x + 2y)^2 + x^2 - 4y^2 + 3x = 0$.

(16) $(2x^2 - y^2)^2 + ax(x^2 + y^2) = (y^2 + ax + a^2)^2$.

(17) $(x^2 - 2xy - y^2)(x^2 + y^2) - 3(x^3 + y^3) = x^2 + 1$.

(18) $x^2(y - 1)^2 = (y - 2x)^2(1 + y)^2 - y^2 + 2x$.

(19) $[y/(x + a)]^3 = (x - a)/(x + 2a)$; and show that the oblique asymptote cuts the curve at an angle $\tan^{-1} 8$.

2. Show that, if we include asymptotes which are at an infinite distance from the origin, every curve of the nth degree has n asymptotes, real or imaginary.

Hence show that every cubic has either one or three real asymptotes.

ANSWERS.

(1) $x = 2a$. (2) $x = \pm 1$; $y = 0$. (3) $y = 0$; $x + a = 0$; $x - 2a = 0$.

(4) $x = 2a$; $y + 3a = 0$; $y - x = 5a$.

(5) $x = a$; $2y = a$; $2(y + 2x) + 5a = 0$. (6) $x + c = 0$; $2(x \pm y) + a - c = 0$.

(7) $x = \pm a$; $y = \pm b$. (8) $y = 2$; $y = 3$; $x + 1 = 0$; $x - y = 6$.

(9) $y = 2a$; $x = 0$; $x + y + 2a = 0$.

(10) $x + 4a = 0$; $y + x = 0$; $5(y + 2x) - 28a = 0$; $5(y - 3x) + 28a = 0$.

(11) $x = a$; $y + a = 0$; $x + y = 0$; $y - x = a$. (12) $x = 2a$; $2y = \pm(2x + a)$.

(13) $y = a$; $3(x - y) = a$. (14) $x = \pm a$; $y \pm x = a$.

(15) $x + 2y = 0$; $(x - y - 1)\sqrt{3} = \pm 1$. (16) $x \pm y = 0$.

(17) $x \pm y(\sqrt{2} \mp 1) \pm \dfrac{3\sqrt{2}}{8}(3 \mp \sqrt{2}) = 0$.

(18) $y + 3 = 0$; $3y + 1 = 0$; $y - x = 2$; $9(y - 3x) + 26 = 0$.

(19) $x + 2a = 0$; $y = x$.

258. Alternative Method—Condition for Infinite Roots.

Suppose that in the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0 \quad . \quad . \quad . \quad . \quad (1)$$

$a_0 = 0$. Then the equation becomes apparently of the $(n - 1)$th degree.

But if *we still suppose* (1) *to be of the nth degree* we shall show that one value of x is infinite.

For let $x = 1/v$; then (1) becomes, after clearing of fractions,

$$a_n v^n + a_{n-1} v^{n-1} + \ldots + a_1 v + a_0 = 0.$$

Now let $a_0 = 0$, *the equation being still of the nth degree if we do not divide down by* v; then one value of v is evidently zero. But $x = 1/v$; $\therefore$ one value of x is infinite.

Similarly, if in addition $a_1 = 0$, then two values of x are infinite.

259. Defining an asymptote as a straight line which meets a curve in two points at infinity, suppose $y = \mu x + \beta$ to be an asymptote. Then, if we substitute for y in $f(x, y) = 0$, the resulting equation for finding x will be, as in Art. 249 (except that $\gamma = \delta = \ldots = 0$),

$$x^n\phi(\mu) + x^{n-1}[\psi(\mu) + \beta\phi'(\mu)] + x^{n-2}[\chi(\mu) + \beta\psi'(\mu) + \tfrac{1}{2}\beta^2\phi''(\mu)]\ldots = 0. \quad (1)$$

Two roots will be infinite if

$$\phi(\mu) = 0 \text{ and } \psi(\mu) + \beta\phi'(\mu) = 0, \quad . \quad . \quad . \quad . \quad (2)$$

which are the same equations as before.

260. The exceptional cases may be discussed as before.

With regard to Case II. of Art. 255, it appears that if the value of μ which makes the coefficient of x^n vanish, also makes that of x^{n-1} vanish, then *for any value of* β the line $y = \mu x + \beta$ cuts the curve in two points at infinity; and the two lines given by equating the coefficient of x^{n-2} to zero, will meet the curve each in three points at infinity; which, as we shall see (Art. 273), corresponds to a point of inflexion.

Similarly, if $\psi(\mu)$ and $\chi(\mu)$ *are absent*, then from (2) $\beta = 0$; and the coefficient of x^{n-2} in (1) vanishes. Hence, if μ_1, μ_2, etc., be the roots of $\phi(\mu) = 0$, $y = \mu_1 x$, $y = \mu_2 x$, etc., are the asymptotes, and these will meet the curve, each in three points at infinity; which again correspond to points of inflexion.

261. Asymptotes in Polar Coordinates.

Let P be a point on a curve *near* the asymptote QN_1.

Draw the radius vector OP, the tangent PN, and the polar subtangent ON.

Then as P moves up to infinity, the tangent becomes the

asymptote QN_1 (Art. 244), OP becomes parallel to QN_1, and ON becomes ON_1 which is perpendicular to QN_1.

Hence, if $p = r\cos(\theta - a)$ be the polar equation of the asymptote, p is the limiting value of ON or $r^2\dfrac{d\theta}{dr}$ when $r = \infty$ †, and $a = \theta_1 - \dfrac{\pi}{2}$, where θ_1 is the limiting value of θ when $r = \infty$.

In Fig. 55, θ increases with r, so that $d\theta/dr$ is $+^{ve}$; hence *the polar subtangent is $+^{ve}$, and is drawn to the right.*

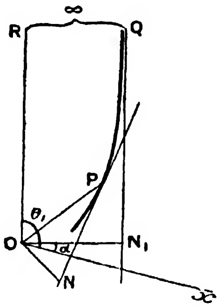

Fig 55.

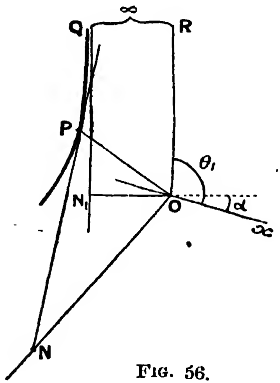

Fig. 56.

In Fig. 56, θ diminishes as r increases, so that *the polar subtangent is $-^{ve}$ and is drawn to the left.*

Hence the rule for finding and drawing the asymptote :—

Find $\underset{r=\infty}{lt}\ r^2\dfrac{d\theta}{dr} = t$ say, and $\underset{r=\infty}{lt}\,\theta = \theta_1$ say.

Draw OR in the direction given by θ_1; draw ON_1 perpendicular to OR and equal to t, to the right if t is $+^{ve}$, to the left if $-^{ve}$.

Finally, draw N_1Q parallel to OR; then N_1Q is the asymptote.

---

† It is also the limit of the perpendicular on the tangent, but it is simpler to take ON.

Since "p" $= t$, and "a" $= \theta_1 - \frac{\pi}{2}$, the equation is

$$t = r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right) = -r\sin(\theta - \theta_1).$$

Note.—In Fig. 56, a is still equal to $\theta_1 - \frac{\pi}{2}$, but ON_1 or t is drawn in the opposite direction since it is $-^{ve}$.

262. Examples.

Ex. 1. $r = a\dfrac{\theta - a}{\theta + a}$. $\therefore r = \infty$ when $\theta = -a$

To find $r^2\dfrac{d\theta}{dr}$, $\qquad r = a - \dfrac{2aa}{\theta + a}$,

$$\therefore 1 = \frac{2aa}{(\theta + a)^2}\frac{d\theta}{dr}.$$

$$\therefore r^2\frac{d\theta}{dr} = a^2\frac{(\theta - a)^2}{(\theta + a)^2}\frac{(\theta + a)^2}{2aa} = \frac{a(\theta - a)^2}{2a} = 2aa \text{ when } \theta = -a.$$

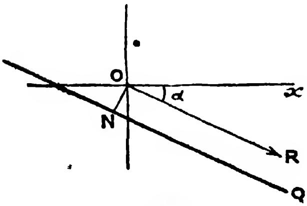

Fig. 57.

Since $2aa$ is $+^{ve}$, ON is drawn to the right of OR, and NQ is the asymptote.

The equation is obtained by referring to the diagrammatic figure (Fig. 55) above.

Since "p" $= 2aa$, $\theta_1 = -a$,

and $\qquad$ "a" $= \theta_1 - \dfrac{\pi}{2} = -a - \dfrac{\pi}{2}$,

the equation is

$$2aa = r\cos\left(\theta + a + \frac{\pi}{2}\right) = -r\sin(\theta + a).$$

Ex. 2. $r(3\theta - \pi) = a\sin\theta$; $\therefore r = \infty$ when $\theta = \dfrac{\pi}{3}$.

Differentiating in r, $3\theta - \pi + 3r\dfrac{d\theta}{dr} = a\cos\theta\dfrac{d\theta}{dr}$;

$$\therefore \frac{d\theta}{dr} = \frac{3\theta - \pi}{a\cos\theta - 3r} = \frac{(3\theta - \pi)^2}{(3\theta - \pi)a\cos\theta - 3a\sin\theta};$$

$$\therefore \underset{\theta = \pi/3}{lt}\left(r^2\frac{d\theta}{dr}\right) = lt\frac{a^2\sin^2\theta}{(3\theta - \pi)a\cos\theta - 3a\sin\theta} = -\frac{a}{2\sqrt{3}}.$$

$\therefore$ equation to asymptote is $-\dfrac{a}{2\sqrt{3}} = r\cos\left(\theta - \theta_1 + \dfrac{\pi}{2}\right)$

$$= r\cos\left(\theta - \frac{\pi}{3} + \frac{\pi}{2}\right) = -r\sin\left(\theta - \frac{\pi}{3}\right),$$

or $$2\sqrt{3}\,r\sin\left(\theta - \frac{\pi}{3}\right) = a.$$

In Cartesians this becomes $a = \sqrt{3}(y - x\sqrt{3})$.

Ex. 3. $r = \dfrac{a\sin^2\theta}{\cos\theta} = a(\sec\theta - \cos\theta) = \infty$ when $\theta = \pm\dfrac{\pi}{2}$.

Also $1 = a\left(\dfrac{\sin\theta}{\cos^2\theta} + \sin\theta\right)\dfrac{d\theta}{dr} = \dfrac{a\sin\theta(1 + \cos^2\theta)}{\cos^2\theta}\cdot\dfrac{d\theta}{dr}$.

$$\therefore r^2\frac{d\theta}{dr} = \frac{a^2\sin^4\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{a\sin\theta(1 + \cos^2\theta)} = \frac{a\sin^3\theta}{1 + \cos^2\theta}$$

$$= a \text{ when } \theta = \frac{\pi}{2}\text{; and } = -a \text{ when } \theta = -\frac{\pi}{2}.$$

The asymptotes are

$$(1)\quad a = r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right) = r\cos\theta$$

$$(2)\quad -a = r\cos(\theta + \pi), \quad \text{or } a = r\cos\theta, \text{ as before.}$$

Hence there is only one asymptote, viz. $x = a$.

Ex. 4. $(a - r)\tan 3\theta = c$; or $r = a - c\cot 3\theta = \infty$ when $\theta = 0$, $\pi/3$, or $2\pi/3$, etc.

Also $3(a - r)\sec^2 3\theta\dfrac{d\theta}{dr} - \tan 3\theta = 0$, or $3c\sec^2 3\theta\dfrac{d\theta}{dr} = \tan^2 3\theta$;

$$\therefore \frac{d\theta}{dr} = \frac{\sin^2 3\theta}{3c}.$$

$$\therefore r^2\frac{d\theta}{dr} = (a - c\cot 3\theta)^2\frac{\sin^2 3\theta}{3c}$$

$$= \frac{(a\sin 3\theta - c\cos 3\theta)^2}{3c} = \frac{c}{3}$$

when $\theta = 0$, $\pi/3$, or $2\pi/3$, etc.

There are six asymptotes forming a regular hexagon of which O is the centre.

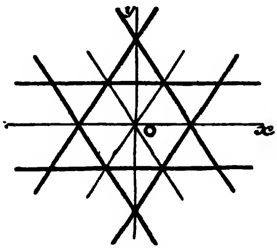

Fig. 58.

263. Circular Asymptotes.—When for an infinite value of θ, r is finite, the curve is said to have a *circular asymptote*, since r is ultimately constant.

Ex. 1. $r = \dfrac{\theta}{\theta + 1} = 1$ when $\theta = \infty$. Hence the asymptotic circle has 1 for its radius.

To find the rectilinear asymptote, $r = \infty$ when $\theta = -1$.

Also $1 = \dfrac{1}{(\theta + 1)^2}\dfrac{d\theta}{dr}$; $\therefore\ r^2\dfrac{d\theta}{dr} = \theta^2 = 1$ when $\theta = -1$.

The asymptote is $1 = r\cos\left(\theta - \theta_1 + \dfrac{\pi}{2}\right) = -r\sin(\theta + 1)$.

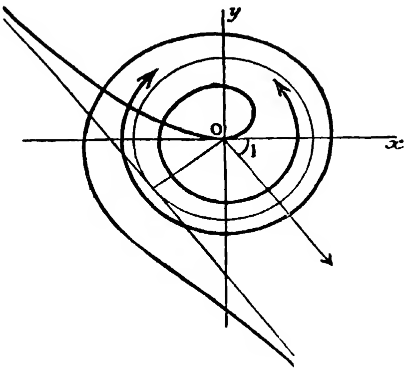

FIG. 59.

Again r is $+^{ve}$ when θ is $+^{ve}$, and when θ is $-^{ve}$ and < -1; r is $-^{ve}$ when θ lies between 0 and -1.

The curve is as in Fig. 59.

EXAMPLES XL.

Find the rectilinear and circular asymptotes of:—

1. $r\theta = a$.
2. $r\sin\theta = a\theta$.
3. $r\cos\theta = a\theta$.
4. $r\tan\theta = a\theta$.

5. $r(\sin\theta - \sin\alpha) = a\cos\alpha$.

6. $r\sin(\theta - \alpha)\sin(\theta - \beta) = a\cos(\alpha - \beta)$.

7. $r\{\tan(\theta + \alpha) + \tan(\theta - 3\alpha)\} = a$.

8. $r\sin(2\theta - \alpha) = a\cos\theta$.

9. $r\cos(3\theta - \alpha) = a\sin\theta$.

10. $r\tan(3\theta - \alpha) = a\sin\left(\theta + \frac{\alpha}{2}\right)$.

11. $r\cos 3(\theta + \alpha) = a\sin(\theta + \alpha)$.

12. $r\sin\frac{\theta}{2} = a$.

13. $r\sin\frac{\theta}{3} = a$.

14. $r\tan\frac{\theta}{3} = a\cos\theta$.

15. $r\theta^2 = r + a\theta^2$.

16. $(r - 1)(\theta - 1) = 1$.

17. $(r - a)(\theta - \alpha) = b$.

18. $(r - 2)\theta = 3\sin\theta$.

19. $r\theta\sin 2\theta = a$.

Answers.

1. $y = a$. 2. $y = \pm na\pi$. 3. $x = \pm\frac{2n+1}{2}a\pi$. 4. $y = \pm na\pi$.

5. $a = r\sin(\theta \pm \alpha)$. 6. $r\sin(\theta - \alpha) = a\cot(\alpha - \beta)$; $r\sin(\theta - \beta) = a\cot(\beta - \alpha)$.

7. $2r\sin(\theta - \alpha) = \pm a\cos^2 2\alpha$; $2r\cos(\theta - \alpha) = \pm a\sin^2 2\alpha$.

8. $2r\sin\left(\theta - \frac{\alpha}{2}\right) = a\cos\frac{\alpha}{2}$; $2r\cos\left(\theta - \frac{\alpha}{2}\right) = -a\sin\frac{\alpha}{2}$.

9. $a = 3r\left\{\cos\theta - \sin\theta\cot\left(\gamma + \frac{\alpha}{3}\right)\right\}$, where $\gamma = \frac{\pi}{6}, \frac{3\pi}{6}$, or $\frac{5\pi}{6}$.

10. $a\sin\left(\gamma + \frac{5\alpha}{6}\right) = 3r\sin\left(\theta - \frac{\alpha}{3} - \gamma\right)$, where $\gamma = 0, \frac{\pi}{3}$, or $\frac{2\pi}{3}$.

11. $a = 6r\sin\left(\frac{\pi}{6} - \alpha - \theta\right)$; $-a = 3r\sin\left(\frac{\pi}{2} - \alpha - \theta\right)$; $a = 6r\sin\left(\frac{5\pi}{6} - \alpha - \theta\right)$

12. $y = \pm 2a$. 13. $y = 3a$. 14. $y = 3a$.

15. $a = 2r\sin(\theta - 1)$; $a = -2r\sin(\theta + 1)$; also circular asymptote $r = a$.

16. $r\sin(\theta - 1) = 1$; circular asymptote $r = 1$.

17. $r\sin(\theta - \alpha) = b$; circular asymptote $r = a$.

18. Circular asymptote $r = 2$.

19. $x = (-1)^n\frac{a}{(2n+1)\pi}$; $y = (-1)^n\frac{a}{2n\pi}$

CHAPTER XVII.

FORM OF THE CURVE AT THE ORIGIN—SINGULAR POINTS.†

264. Tangents at the Origin.

Let the equation of a curve of the nth degree be written in ascending powers of x and y, thus :—

$$a + (bx + cy) + (dx^2 + exy + fy^2) + (gx^3 + \,..\,) + \ldots = 0. \quad (1)$$

To find where the line $y = mx$ (through the origin) meets (1), we have, on substituting mx for y,

$$a + (b + cm)x + (d + em + fm^2)x^2 + (g \ldots)x^3 + \ldots = 0. \quad . \quad (2)$$

This gives n values, real or imaginary, of x, corresponding to the n points of intersection of $y = mx$ with the curve.

If $a = 0$, then one of the values of x is zero, and one of the points of intersection is the origin; it also appears that if a *gradually* approach zero, the point in question will gradually approach the origin at the same time, whatever be the value of m.‡

Again, if m be varied (*i.e.* if the line be rotated) until $b + cm = 0$ —so that the line $y = mx$ becomes $bx + cy = 0$—then (2) becomes

$$(d + em + fm^2)x^2 + (g + \ldots)x^3 + \ldots = 0 \quad . \quad . \quad (3)$$

and *two* values of x are zero.

† The term *singular point* includes multiple points, conjugate points, points of inflexion, etc.

‡ Unless, perhaps, when $m = \infty$, in which case we should put $x = m'y$ (so that $m' = 0$), and discuss the values of y instead of x, when the objection would no longer hold.

Hence the line $bx + cy = 0$ meets the curve

$$bx + cy + dx^2 + exy + fy^2 + \dots = 0 \quad . \quad . \quad . \quad (4)$$

in *two* points at the origin; and, as above, it appears that if m *gradually* approach this value, the second point will gradually approach the origin, and therefore the secant through the origin and this point will become the tangent at the origin.

Hence $bx + cy = 0$ is the tangent at the origin to the curve (4).

Ex. 1. $x^4 - 2x^2y^2 + x^2 + x - 3y = 0$; tangent at origin, $x = 3y$.

Ex. 2. $y^2 = 4ax$; tangent at origin, $x = 0$.

Next, let $a = b = c = 0$. Then (1) becomes

$$(dx^2 + exy + fy^2) + (gx^3 \dots) + \dots = 0 \quad . \quad . \quad . \quad (5)$$

and (3) gives the points of intersection of $y = mx$ with the curve, showing that there are two points at the origin, which fact however does not correspond to tangency, since in this case $y = mx$ is *any line*. [See next Art.]

If now m vary until it satisfies the equation

$$d + em + fm^2 = 0, \text{ or } dx^2 + exy + fy^2 = 0 \quad . \quad . \quad . \quad (6)$$

we shall have a third point approaching the origin, and therefore the secant through the origin and this point will become a tangent.

But there are *two* values of m which satisfy (6); hence there are *two* tangents at the origin, and their equation is

$$dx^2 + exy + fy^2 = 0.$$

Similarly, if $d = e = f = 0$, there will be three tangents at the origin; and so on.

Hence, *the terms of lowest degree equated to zero give the tangents at the origin.*

Ex. $x^5 + y^5 + x^3y + x^2y^2 - 6xy^3 = 0$.

Tangents at origin are given by $x^3y + x^2y^2 - 6xy^3 = 0$;

i.e. by $xy(x - 2y)(x + 3y) = 0$.

265. Multiple Points.

Def.—A point through which two or more branches of a curve pass is called a *multiple point*, the terms *double point*, *triple point*, *r-ple point*, corresponding to two, three, and r branches respectively.

We have seen that when the terms of lowest degree are those of the second, there are two points at the origin in which *any* line $y = mx$ meets the curve; also that *two* tangents can be drawn at the origin. In this case two branches of the curve evidently pass through the origin, which is therefore a double point.

Similarly, if the terms of the third degree are the lowest, we have a triple point; and so on.

266. Further Explanation.

To make the preceding remarks more clear, let us again consider (1), which represents some curve not passing through the origin. If we suppose *any* of the constants $a, b, c \ldots$ to vary imperceptibly, then the shape of the curve will do likewise. Let us now imagine a to gradually approach zero; the curve will alter in shape, and will also move until one of its branches passes through the origin. At this moment $a = 0$; and, however we vary the other constants, the curve will always pass through the origin while $a = 0$. Next, turning to (2), supposing a still zero, we may note that there are *two* ways in which $b + cm$ may vanish, (i) by making m approach $-\frac{b}{c}$, (ii) by making b and c both vanish.

(i) In the first case *the curve does not alter in shape*, but the line $y = mx$ turns about O, and as OM diminishes P evidently travels along the curve towards O; the final direction of OP must, therefore, be that of the tangent at O, viz. OP' in Fig. 60.

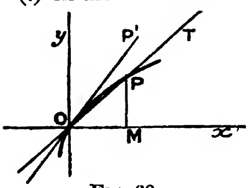

FIG. 60.

(ii) Suppose b and c to be gradually vanishing. Then *the curve is altering in shape*; and since when b and c are both zero x has another of its values equal to zero, we may assume by the law of continuity that when b and c are nearly zero this second value of x will be nearly zero, for any value of m; *i.e.* for any direction of OP, OM will be small. This shows that there must be a second branch of the curve approaching the

origin, as may be seen in Fig. 61. Hence ultimately we shall have two branches passing through the origin, forming a double point, as in Fig. 62.

Again, let OP' be the line $y = mx$, meeting this second branch at the origin and at the point P'. If we no longer vary the constants of the curve, but vary m, the curve in Fig. 62 will maintain its shape while

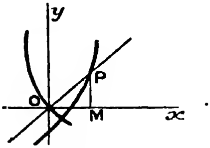

Fig. 61.

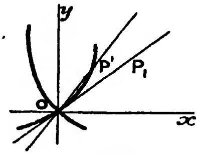

Fig. 62.

OP' turns about O. The point P' will travel towards O, and OP' will ultimately cut the curve in three coincident points at the origin, two belonging to one branch (to which OP_1 is now a tangent) and the third to the other branch.

267. Nodes—Conjugate Points—Cusps.

Consider the curve (5), in which there is a double point at the origin.

Then the tangents at the origin are

$$dx^2 + exy + fy^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Three cases may arise :—

(1) *If the roots of* (7) *are real*, we have two real branches passing through the origin, and there is said to be a *node* at that point.

(2) *If the roots are imaginary*, then, although the origin is evidently a point on the curve, there are no real branches passing through the origin. In fact, the equation for finding the points of intersection of $y = mx$ with the curve is

$$(d + em + fm^2)x^2 + (g + \dots)x^3 \dots = 0,$$

giving $x^2 = 0$, and

$$(d + em + fm^2) + (g + \dots)x + \dots = 0.$$

But by hypothesis the first term cannot vanish, *i.e.* must always

remain finite; therefore, for no value of m can the equation be satisfied when x is indefinitely small [see Art. 264, footnote], which shows that no line through the origin can be found to cut the curve in the vicinity of the origin, except at the origin itself.

The origin is therefore called a *conjugate* or *isolated point.*

Ex. $x^5 - 2xy^3 + x^2 + y^2 = 0.$

Put $y = mx$; $\therefore\ x^3 - 2m^3x^2 + 1 + m^2 = 0.$

If possible suppose x to be small, then, neglecting x^3, we have

$$x^2 = \frac{1 + m^2}{2m^3},$$

which cannot be made indefinitely small for finite values of m; although it vanishes when $m = \infty$. However, putting $x = m'y$, we have

$$m'^5y^3 - 2m'y^2 + m'^2 + 1 = 0;$$

or, neglecting y^3,
$$y^2 = \frac{m'^2 + 1}{2m'},$$

which obviously does not become indefinitely small when

$$m' = 0, \text{ i.e. } m = \infty.$$

Hence there are no points within the vicinity of the origin, except the origin itself, which is therefore a conjugate point.

(3) *If the roots are equal*, the tangents at the origin coincide and there is said to be a *cusp* at that point.

A cusp is, in fact, formed when two branches of a curve touch one another.

FIG. 63.

FIG. 64.

FIG. 65.

FIG. 66.

FIG. 67.

Cusps may be *single* or *double*; Figs. 63 and 64 represent single cusps since the curve extends only in *one* direction of the common tangent; Figs. 65, 66, and 67 represent double cusps, where the curve extends in both directions of the common tangent.

Again, cusps (single or double) are of *two species.*

A cusp of the *first species*, or a *keratoid cusp*, is that in which the branches are on *opposite* sides of the common tangent, as in Figs. 63 and 65.

A cusp of the *second species*, or a *rhamphoid cusp*, is that in which both branches are on the *same* side of the common tangent, as in Figs. 64 and 66.

In Fig. 67 we have a combination of both species. The point of contact has been called *a point of oscul-inflexion.*

268. We now give some simple examples of cusps.

Ex. 1. $y = x^{\frac{3}{2}} + x^2$.

The tangent at the origin is $y = 0$, or the axis of x.

If x is $-^{\text{ve}}$, y is imaginary; while if x is $+^{\text{ve}}$, y is double-valued. If we neglect x^2 in comparison with $x^{\frac{3}{2}}$, then near the origin y has equal and opposite values. Hence there is a single keratoid cusp.

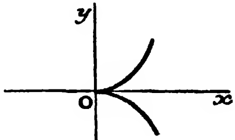

FIG. 68.

Ex. 2. $y = x + x^{\frac{3}{2}} + x^2$.

The tangent at the origin is $y = x$. If x is $-^{\text{ve}}$, y is imaginary; and if $+^{\text{ve}}$, y is double-valued.

Neglecting x^2, we have, near the origin,

$$y = x \pm x^{\frac{3}{2}}.$$

If P be a point on $y = x$, then if we take PR and PS each equal to the value of $x^{\frac{3}{2}}$ when $x = OM$, R and S will be two points on the curve, and we have a keratoid cusp.

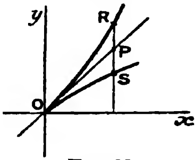

FIG. 69.

Ex. 3. $y = x^2 + x^{\frac{5}{2}} + x^3$.

The tangent at O is $y = 0$.

Near the origin $y = x^2$, which is a small part of a parabola, denoted by the dotted line OP.

The next approximation is $y = x^2 \pm x^{\frac{5}{2}}$.

At P draw PR and PS each equal to the value of $x^{\frac{5}{2}}$ at that point; then R and S are on the curve, one on each side of the parabola, but *both above* Ox, since $x^2 > x^{\frac{5}{2}}$ when x is small, so that y is $+^{\text{ve}}$ in both cases. Hence we have a rhamphoid cusp.

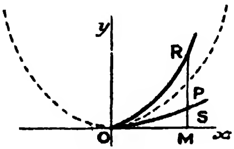

FIG. 70.

Ex. 4. $y = x + x^2 + x^{\frac{5}{2}} + \ldots$

We can show, by reasoning similar to that in Ex. 3, that there is a rhamphoid cusp, the tangent being $y = x$ instead of the axis of x.

Ex. 5. $(y - x^2)^2 = x^6 - x^7$.

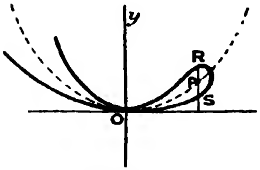

FIG. 71.

We have $y - x^2 = \pm x^3(1 - x)^{\frac{1}{2}}$
$= \pm x^3(1 - \frac{1}{2}x \ldots)$.

Near the origin, $y = x^2 \pm x^3$.

In this case x may be either $+^{ve}$ or $-^{ve}$; and we have, therefore, a double cusp, the curve being on both sides of the auxiliary parabola $y = x^2$; but both branches are above Ox, since $x^2 > x^3$ when x is small.

Ex. 6. $y^2 - x^2y + x^5 = 0$.

Solve for y in terms of x. Then

$$y = \frac{x^2}{2} \pm \frac{x^2}{2}(1 - 4x)^{\frac{1}{2}} = \frac{x^2}{2} \pm \frac{x^2}{2}(1 - 2x + \ldots).$$

Hence, near the origin, we have two approximations:—

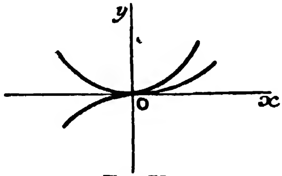

FIG. 72.

(1) $y = \frac{x^2}{2} + \frac{x^2}{2}(1 - 2x) = x^2 - x^3 = x^2$ nearly, and the curve is in the form of a parabola.

(2) $y = \frac{x^2}{2} - \frac{x^2}{2}(1 - 2x) = x^3$, a cubical parabola, such that y changes sign with x.

Hence the curve is as in the figure, and we have a point of oscul-inflexion.

Ex. 7. $(y - x^2)^2 + x^4 = x^5$.

In this case there is a pair of coincident tangents at the origin, viz. $y^2 = 0$; but, neglecting x^5, we have the sum of two squares equal to zero, which shows that, *near the origin*, no values of x and y can be found to satisfy the equation, except the origin itself.

Hence there is a conjugate point at the origin; as may also be seen by writing the equation in the form $y - x^2 = x^2\sqrt{x - 1}$, $\sqrt{x - 1}$ being imaginary if x is small.

269. Loops.—A *loop* occurs when a curve, after passing through a point, bends round and again passes through the same point.

Ex. 1. $ay^2 = x^2(a - x)$.

Here $y = 0$ when $x = 0$ and when $x = a$.

Between these values y has two equal and opposite finite values.

When $x > a$, y is imaginary.

Hence there is evidence of a loop.

The tangents at the origin are $x \pm y = 0$; and we can show that dy/dx is infinite at $x = a$.

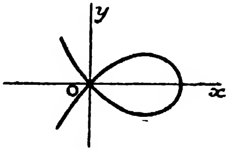

FIG. 73.

Ex. 2. $x^4 - 2xy^2 + y^3 = 0$.

The origin is a triple point. Hence any line through O will meet the curve in one other point only.

To find this point, put $y = mx$, and divide by x^3; then

$$x = m^2(2 - m), \quad \therefore y = m^3(2 - m).$$

By giving m different values we can find x and y.

Since x and y are zero when $m = 0$ and $m = 2$, and are both finite as m increases from 0 to 2, it follows that the point (x, y) moves from the origin and back again between the tangents $y = 0$ and $y = 2x$. Hence there is a loop.

Ex. 3. $(y - x^2)^2 = x^6 - x^7$.

This is the curve discussed in Ex. 5 of the last article (*q.v.*).

We have $y = x^2 \pm x^3\sqrt{1 - x}$.

Now, y has coincident values when $x = 0$ and $x = 1$; and is imaginary if $x > 1$, while between $x = 0$ and $x = 1$, y has finite values. Hence there is a loop as in Fig. 71.

It will be noted that, for the point P, $y = x^2$; hence

$$PR = SP = x^3\sqrt{1 - x}.$$

270. Examination of a Given Curve for Singular Points.—In order to examine as to whether any singularity occurs at a given point, transfer the origin to that point; the preceding methods can then be adopted

In other works will be found a full method of discovering whether any multiple point occurs in a given curve.

The following method for finding double points, if any, is based on that referred to above :—

If $f(x, y) = 0$ be the curve, transfer the origin to the point

(h, k), so that the equation becomes $f(x+h, y+k) = 0$. Expand in ascending powers of x and y, and equate to zero the absolute term, and the coefficients of x and y respectively. This gives *three* equations for finding h and k, and unless the values as found from two of the equations satisfy the third, there is no double point on the curve.

Ex. $x(2x^2 - 5ax + 4a^2) = ay(2a - y)$.

Transferring the origin to (h, k), we have

$$(x+h)\{2(x+h)^2 - 5a(x+h) + 4a^2\} = a(y+k)\{2a - (y+k)\} \quad \text{(A)}$$

Equating to zero the absolute term, and the coefficients of x and y; we shall get

$$h(2h^2 - 5ah + 4a^2) = ak(2a - k) \quad . \quad . \quad . \quad . \quad (1)$$

$$3h^2 - 5ah + 2a^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$a - k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3), we have

$$h = a, \text{ or } \tfrac{2}{3}a; \ k = a.$$

By substitution in (1), we find that $h = k = a$ satisfies.

Hence there is probably a double point at (a, a).

Putting $h = k = a$ in (A), it becomes

$$(x+a)\{2(x+a)^2 - 5a(x+a) + 4a^2\} = a(y+a)(a-y),$$

which reduces to $\quad (x+a)(2x^2 - ax + a^2) + a(y^2 - a^2) = 0,$

or $\quad 2x^3 + a(x^2 + y^2) = 0.$

Hence there is a conjugate point at the new origin, *i.e.* at the point (a, a).

NOTE.—It can be shown that if the curve be $f(x, y) = 0$, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ will give the equations for finding (h, k).

Thus in the curve above given, viz.—

$$x(2x^2 - 5ax + 4a^2) - ay(2a - y) = 0, \text{ we shall get}$$

$$\partial f/\partial x = 6x^2 - 10ax + 4a^2 = 0,$$

$$\partial f/\partial y = -2a + 2y = 0,$$

leading to equations (2) and (3) above.

EXAMPLES XLI.

1. Find the tangents at the origin in the following curves:—

(1) $\dfrac{a}{x} + \dfrac{b}{y} = 1.$ (2) $x^3y - 4x^2 - y^3 + y^2 = 0.$

(3) $x^3y + x^2 - y^3 + y^2 = 0$. (4) $x(x - y^2)(y - x^2) = y(x - y)^2$.

(5) $(x - y)^2 = x^3 + y^3 - y^2$.

(6) $y^2(a + x)(a + 2x) = x^2(a - x)(a - 2x)$.

2. Find the nature of the cusps, if any, in the following curves, and draw a figure :—

(1) $y = x^{\frac{3}{2}}$. (2) $y = x^{\frac{1}{3}}$.

(3) $x^3 + y^2 = 0$. (4) $y^2 = x^3(1 - x)$.

(5) $y = 2x^2 + 3x^{\frac{5}{2}}$. (6) $x = y^2 + y^{\frac{5}{2}} + y^3$.

(7) $(y - x)^2 = x^3 + x^4$. (8) $x + y = y^2(2 + 3\sqrt{y})$.

(9) $x^5 + (y - 2x + x^2)^2 = 0$.

3. Show that the origin is a conjugate point in the following :—

(1) $x^2 = y^2(y^2 - a^2)$. (2) $y = x\sqrt{x - 1}$.

(3) $y^2 = x^2(x^2 - a^2)/(x^2 + a^2)$. (4) $x^2 + y^4 = y^5$.

(5) $(y - x)^2 + x^4 = x^5$. (6) $y^2 = x^2 + x^3\sqrt{x - 2}$.

(7) $\dfrac{x^2(x - a)}{x + a} + \dfrac{y^4(y + a)}{y - a} = 0$. (8) $3x^2(x + a) = ay(2x - y)$.

4. Show that a loop exists in each of the following curves :—

(1) $y^2 = x^2(1 - x)$. (2) $y^2 = (x - a)^2(2a - x)$.

(3) $y^4 = a^2(y^2 - x^2)$. (4) $(y - 2)^2y = x^2$.

(5) $x^3 + y^3 = 3axy$. (6) $x^2y = (y - x)(y - 2x)$.

5. Show that there is a double point at the point (1, 1), and a loop in the curve $x - 2y + x^2 + y^2 = x^3$, by transferring the origin to that point.

6. Show that in the curve $x^3 + x^2 + y^2 - x - 4y + 3 = 0$ there is a node at the point $(-1, 2)$, and a loop.

7. Show that the curve $y = x + (x - a)^{\frac{3}{2}}$ has a cusp at the point (a, a).

8. Examine the curve $y = 1 + x + x^2 + (x - 1)^{\frac{5}{2}}$ at the point (1, 3).

9. Examine the curve $xy(x - 2a)(y - 2a) = a^4$ at the point (a, a).

10. Show that the curve $r = a\sec^3\theta$ has a conjugate point at the origin.

11. In the curve $(x^2 + y^2)^2 = ax(x^2 - 3y^2)$, show that the tangents at the origin are equally inclined to each other, and that there are three loops. Find the equation in polar coordinates.

12. Examine the curve $y = x^2 + x^3\sqrt{2 - x}$ at the origin; and show that there is a loop.

13. Show that the curve $c(a^2y - x^3)(b^2y - x^3) = x^7$ has a double cusp of the second species at the origin, the branches being in the first and third quadrants respectively.

14. Find whether there are any singular points on the curves:—

(1) $ay(y-2a) = x(x-a)^2 - a^3$. (2) $2y(y-4) + x(x^2 + 2x - 4) = 0$.

ANSWERS.

1. (1) $ay + bx = 0$. (2) $y = \pm 2x$. (3) Imaginary.
(4) $y^2(y - 2x) = 0$. (5) Imaginary. (6) $x = \pm y$.

2. (1) 1st species. (2) None. (3) 1st sp. (4) 1st sp. (5) 2nd sp.
(6) 2nd sp. (7) 1st sp. (8) 2nd sp. (9) 2nd sp.

8. Transformed equation is $y = 3x + x^2 + x^{\frac{5}{2}}$; cusp, 2nd species.

9. Conjugate point. **11.** $r = a \cos 3\theta$. **12.** Double cusp, 2nd species.

14. (1) Node at (a, a), and loop. (2) Node at $(-2, 2)$, and loop.

271. Points of Inflexion—Stationary Tangents.—In Art. 115 (*q.v.*) we have shown that if d^2y/dx^2 change sign on passing through zero, a point of inflexion will occur.

The same will occur when d^2y/dx^2 changes sign on passing through infinity.

The tangent at a point of inflexion is called a *stationary tangent*, for the reason that as we pass through a point of inflexion the tangent becomes temporarily stationary.

We now give further examples.

272. Ex. 1. $ay^2 = x(a^2 - x^2)$ . . . . (1)

Differentiating twice, we have

$$2ayy_1 = a^2 - 3x^2 \quad \text{. . . . .} \quad (2)$$

$$2ay_1^2 + 2ayy_2 = -6x \quad \text{. . . . .} \quad (3)$$

Multiply (3) by $2ay^2$,

$$\therefore\ 4a^2y^2y_1^2 + 4a^2y^3y_2 = -12ay^2x.$$

$\therefore$ from (1) and (2), squaring the latter,

$$a^4 - 6a^2x^2 + 9x^4 + 4a^2y^3y_2 = -12x^2(a^2 - x^2);$$

$$\therefore\ 4a^2y^3y_2 = 3x^4 - 6x^2a^2 - a^4;$$

$$\therefore\ y_2 = \frac{3x^4 - 6x^2a^2 - a^4}{4a^{\frac{1}{2}}x^{\frac{3}{2}}(a^2 - x^2)^{\frac{3}{2}}}.$$

$\therefore$ for a point of inflexion, $3x^4 - 6x^2a^2 - a^4 = 0$,

$$\text{or}\quad 3(x^2 - a^2)^2 = 4a^4.$$

$$\therefore\ x^2 = a^2 \pm \frac{2}{\sqrt{3}}a^2 = \frac{\pm 2 + \sqrt{3}}{\sqrt{3}}a^2.$$

Since $2 > \sqrt{3}$, we cannot take the lower sign:

$$\therefore\ x = \pm a\sqrt{\frac{2 + \sqrt{3}}{\sqrt{3}}} = \pm 1{\cdot}5a \text{ nearly.}$$

It is now necessary to test (a) whether these values of x correspond to real points on the curve; (b) whether y_2 changes sign as x passes through these values.

Since $y = \sqrt{x(a^2 - x^2)/a}$, y will be imaginary if $x > +a$ (supposing a +^{ve}), but real if $x < -a$. Hence we can only take $x = -1{\cdot}5a$.

There exists, then, subject to (b), one real point of inflexion for $y_2 = 0$, viz. that at which $x = -1{\cdot}5a$.

To find y, we have

$$ay^2 = -x(x^2 - a^2) = 1{\cdot}5a \cdot \frac{2a^2}{\sqrt{3}} = \sqrt{3}a^3 \text{ nearly.}$$

$$\therefore\ y = \pm \sqrt[4]{3}\,.\,a = \pm 1{\cdot}3a \text{ nearly.}$$

Since none of the factors in y_2, except $3x^2 - 6x^2a^2 - a^4$, vanish for this value of x, they will not change sign as x passes through this value. We need only, therefore, consider this factor.

Now $3x^2 - 6x^2a^2 - a^4 = 3(x^2 - a^2)^2 - 4a^4$, and when x increases algebraically through the value $-1{\cdot}5a$, $3(x^2 - a^2)^2$ diminishes, hence the expression diminishes through zero, and therefore changes sign. Hence there is a point of inflexion.

Again $y_2 = \infty$ if $x = 0$, or $\pm a$, but since the denominator cannot change sign as x passes through these values, there is no point of inflexion corresponding to them.

Ex. 2. $y = x^{\frac{1}{3}}\{(x-1)^{\frac{5}{3}} + 2\}$.

We have $y_1 = \frac{1}{3}x^{-\frac{2}{3}}\{(x-1)^{\frac{5}{3}} + 2\} + \frac{5}{3}x^{\frac{1}{3}}(x-1)^{\frac{2}{3}}$,

$$y_2 = -\tfrac{2}{9}x^{-\frac{5}{3}}\{(x-1)^{\frac{5}{3}} + 2\} + \tfrac{10}{9}x^{-\frac{2}{3}}(x-1)^{\frac{2}{3}} + \tfrac{10}{9}x^{\frac{1}{3}}(x-1)^{-\frac{1}{3}}$$

$$= \frac{2}{9x^{\frac{5}{3}}(x-1)^{\frac{1}{3}}}[-(x-1)^2 - 2(x-1)^{\frac{1}{3}} + 5x(x-1) + 5x^2]$$

$$= \frac{2}{9x^{\frac{5}{3}}(x-1)^{\frac{1}{3}}}[9x^2 - 3x - 1 - 2(x-1)^{\frac{1}{3}}].$$

Ignoring the numerator which is complicated, we observe that $y_2 = \infty$ when $x = 0$ and $x = 1$; moreover, in both cases, y_2 changes sign as x passes through these values. Hence there are points of inflexion where $x = 0$ and $x = 1$ respectively.

Again, when $x = 0$, $dy/dx = \infty$; but when $x = 1$, $dy/dx = \frac{2}{3}$.

Hence y_2 may be infinite whether y_1 is infinite or not, *i.e.* whether the tangent be parallel to Oy or not.

In the curve $y = x^{\frac{1}{3}}(x-1)^{\frac{5}{3}}$, $y_1 = 0$, while $y_2 = \infty$, at the point where $x = 1$.

273. Geometrical View.—From the nature of a point of inflexion P on a continuous curve, it is evident that if a straight line QPR be drawn near to the tangent at the point, it will cut the curve in three points. When the line approaches the tangent by rotating about P, the points Q and R approach and ultimately coincide with P.

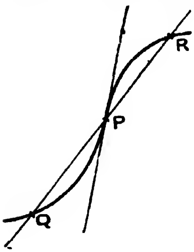

FIG. 74.

Hence, *the tangent at a point of inflexion meets the curve in three coincident points.*

274. Points of Inflexion at the Origin.—We are now enabled to determine briefly whether the origin is a point of inflexion. The general case may be deduced from the following examples.

Ex. 1. $a^2y = x^3 + 2x^2y + ay^2$.

Here $y = 0$ is the tangent at the origin. To find the points in which $y = 0$ meets the curve: put $y = 0$ in the equation, and we get $x^3 = 0$, giving three coincident points at the origin. Hence $y = 0$ is a stationary tangent, and the origin a point of inflexion.

Ex. 2. $y - 2x = x^3 + y^3$.

Here $y = 2x$ is the tangent at the origin. To find the other points in which the tangent cuts the curve, put $y = 2x$, and we get $0 = x^3 + 8x^3$. Hence the origin is a point of inflexion.

Ex. 3. $x(y - 2x) = x^2y - y^3$.

Here, (i) putting $x = 0$, we have $y^3 = 0$;

(ii) putting $y = 2x$, we have $0 = 2x^3 - 8x^3 = -6x^3$.

Apparently, in both cases we have a point of inflexion. *It must be remembered, however, that the origin is a double point*, so that one point belongs to the branch which is cut obliquely and the other two to the branch which is touched by the line. This may be seen by drawing a straight line PQR parallel to the tangent ($x = 0$, say), and supposing it to approach Oy, when the three points will ultimately coincide, but there will not be a point of inflexion. Similarly for the other tangent. By putting $y = mx$ in the equation to the curve, and making m vary from 2 to ∞, it will be seen that the part of the curve near the origin is as in the figure.

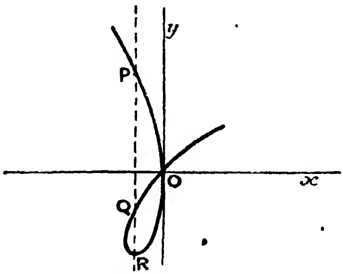

FIG. 75.

Ex. 4. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Since $x = \pm y$ both meet the curve in four points at the origin, one of which belongs to the branch cut obliquely by either tangent, there will be two points of inflexion, one for each tangent.

275. When a straight line meets a curve in four coincident points, there is said to be a *point of undulation*, and we can show that the tangent is not crossed by the curve, but is indistinguishable from an ordinary tangent. [See Art. 207 (B).]

Ex. In $y - x = x^4 + y^4$, there is a point of undulation at the origin.

276. Polar Coordinates.—The condition for a point of inflexion is that the perpendicular (p) on the tangent is either a maximum or a minimum.

Now $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$, and if p is a max. or min., then $\frac{1}{p^2}$ will be a min. or max. accordingly.

Hence $\frac{d}{d\theta}\left(\frac{1}{p^2}\right) = 0$, and changes sign. It may also be infinite or discontinuous; but if it changes sign as θ passes through this value a point of inflexion will generally occur.

Ignoring the latter cases, which are less common, we have

$$2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} = 0,$$

$$\therefore \text{ either } u + \frac{d^2u}{d\theta^2} = 0, \text{ or } \frac{du}{d\theta} = 0.$$

(1) $u + \frac{d^2u}{d\theta^2} = 0$ is the *general condition* which, as we shall show, must hold good always.

(2) $\frac{du}{d\theta} = 0$. This makes the polar subtangent, $-\frac{d\theta}{du}$, infinite.

Hence $\phi = \pi/2$, which is not necessarily true for a point of inflexion, though it is one condition for a turning value of p.†

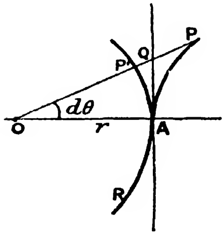

FIG. 76.

Let the adjoining figure represent the case in which $du/d\theta = 0$. Then at A we might have either an *apse*, as in the curve $P'AR$, or a point of inflexion, as in PAR.

A further condition is therefore required, viz. that $OP - OQ$ changes sign as P passes through A.

For convenience, let $v = \frac{1}{OP} - \frac{1}{OQ} = \frac{OQ - OP}{OP.OQ}$, so that v will have to change sign. Also let $OA = r$, $1/OA = u$.

---

† It is also a condition for a turning value of u, and therefore of r.

Now, v or $\frac{1}{OP} - \frac{1}{OQ} = u + du\dagger - \frac{\cos d\theta}{r} = u + du - u\left(1 - \frac{d\theta^2}{2}\dots\right)$

$$= du + u.\frac{d\theta^2}{2}\dots$$

But if $u = f(\theta)$, $u + du = f(\theta + d\theta) = f(\theta) + f'(\theta).d\theta + f''(\theta)\frac{d\theta^2}{2}\dots$

$$= u + \frac{du}{d\theta}d\theta + \frac{d^2u}{d\theta^2}\frac{d\theta^2}{2}\dots$$

$$\therefore du = \frac{d^2u}{d\theta^2}\frac{d\theta^2}{2}, \text{ noting that } \frac{du}{d\theta} = 0,$$

and neglecting higher orders.

Hence $v = \frac{d\theta^2}{2}\left(u + \frac{d^2u}{d\theta^2}\right)$; and since $\frac{d\theta^2}{2}$ is always $+^{\text{ve}}$, it follows that $u + \frac{d^2u}{d\theta^2}$ must change sign, as in (1).

277. Examples.

Ex. 1. $r(\theta^2 - 3) = 2$.

Here $2u = \theta^2 - 3$;

$2u_1 = 2\theta$; $2u_2 = 2$.

$\therefore u + u_2 = \frac{1}{2}\theta^2 - \frac{1}{2} = \frac{1}{2}(\theta + 1)(\theta - 1) = 0$ if $\theta = \pm 1$;

and since $u + u_2$ changes sign as θ passes through these values, there will be a point of inflexion in each case.

Ex. 2. $r = a + b\cos\theta$ (the Limaçon)

Here $u = \frac{1}{a + bc}$, where $c \equiv \cos\theta$;

$$\therefore u_1 = -\frac{1}{(a + bc)^2}(-bs) = \frac{bs}{(a + bc)^2}.$$

$$u_2 = b.\frac{c(a + bc)^2 - 2(a + bc)(-bs)s}{(a + bc)^4} = \frac{b}{(a + bc)^3}\{c(a + bc) + 2bs^2\}$$

$$= \frac{b(2b + ac - bc^2)}{(a + bc)^3};$$

---

† Strictly, $\frac{1}{OP} = \frac{1}{r + dr} = u + du + Kdu^2$, as may be easily shown.

$$\therefore u + u_2 = \frac{a^2 + 2abc + b^2c^2 + 2b^2 + abc - b^2c^2}{(a + bc)^3} = \frac{a^2 + 2b^2 + 3abc}{(a + bc)^3}$$

$$= 0 \text{ if } a^2 + 2b^2 + 3abc = 0, \quad \text{or } \cos\theta = -\frac{a^2 + 2b^2}{3ab}.$$

Since $\cos\theta < 1$ numerically, and assuming a and b to be $+^{\text{ve}}$, we must have

$$a^2 + 2b^2 < 3ab;$$
$$i.e.\ a^2 - 3ab + 2b^2 < 0;$$
$$i.e.\ (a - b)(a - 2b) < 0.$$

$\therefore$ a must lie between b and $2b$ for real points of inflexion.

EXAMPLES XLII.

1. Find the points of inflexion in the following curves:—

(1) $(a^2 + x^2)y = ax^2$. (2) $y(x - 1) = x^3$.

(3) $y(x + a) = x^3 + 2a^3$. (4) $xy = \log x$.

(5) $xy^2 = a^2(a - x)$. (6) $a^2y^2 = x^2(a^2 - x^2)$.

(7) $y^2 = (x - 1)^2(x - 2)$.

2. State whether or not the following curves have points of inflexion at the origin:—

(1) $x = y^3$. (2) $y = 2x + x^3 + x^4$.

(3) $xy = y^3 - x^3$. (4) $x^2 = y^2(a^2 + y^2)$.

(5) $x(1 + y - x) = y^3$. (6) $a^2y^2 = x^4 + y^4$.

(7) $y^2 = 4x^2 + x^3$. (8) $(x - y)(x + 2y) = x^3 + y^3$.

(9) $(ax - by)(a + x + 2y) = x^3$. (10) $y = \sin x$.

(11) $(1 + x) = e^y(1 - x)$.

3. Find the points of inflexion in the following curves:—

(1) $r^2\theta = a^2$. (2) $r(1 - 3\theta) = \theta$.

(3) $r^2 = a^2(2 + \theta^2)$. (4) $(r - a)\cos\theta = a$ [Conchoid].

(5) $r = a + b\cos\theta$ [Limaçon].

4. Show that the curve $x - y = x^3 - 2$ has a point of inflexion where it is met by the line $x - y + 2 = 0$, and find the point.

5. Show that the curve $(x + y)^3 = y(x - y + 2)$ has a point of inflexion at the origin, and at the point $(-1, 1)$.

6. Show that the curve $(x - y)(x - 2y + 3)(2x - 3) = (x + 2y - 3)^3$ has three points of inflexion which are collinear. [See Answer.]

7. Show that the curve $x^3 - y^3 = x$ has three points of inflexion on the x-axis.

8. Show that the curve $xy(x - y) = x^3 - y^5$ has a point of inflexion at the origin.

9. Show that in the *trochoid*, whose equation is given by

$$x = a(1 - m\cos\phi); \quad y = a(\phi - m\sin\phi),$$

there is a point of inflexion where $\cos\phi = -m$.

10. Find the points of inflexion in the curves—

(1) $y^3 = x^2(a - x)$. (2) $y^3 = ax(a - x)$. (3) $x^3 - y^3 = y(y + 1)$.

Show that in the last curve the three points of inflexion are collinear.

11. Show that the curve $x^3/a^3 + y^3/b^3 = 1$ has three points of inflexion, one of which is at infinity (Art. 260); and that these are collinear.

Answers.

1. (1) $x = \pm a/\sqrt{3}$. (2) Origin. (3) $x = -2a$. (4) $x = e^{\frac{3}{2}}$.
(5) $x = 3a/4$. (6) $x = 0$; $x = \pm\sqrt{\frac{3}{2}}a$. (7) $x = \frac{7}{3}$.

2. (1) Yes. (2) Yes. (3) No. (4) Two. (5) Yes. (6) Two.
(7) No. (8) No. (9) Yes. (10) Yes. (11) Yes.

3. (1) $\theta = \pm\frac{1}{2}$. (2) $\theta = 1$. (3) $\theta = \pm 1$.

(4) $r = \frac{a}{2}(3 + \sqrt{3})$; $\theta = \cos^{-1}(\sqrt{3} - 1)$.

(5) $r = 2(a^2 - b^2)/3a$; $\theta = \cos^{-1}\{-(a^2 + 2b^2)/3ab\}$. Since $a^2 + 2b^2 < 3ab$, *i.e.* $(a - 2b)(a - b) < 0$; $\therefore$ a must lie between b and $2b$ for real points of inflexion.

6. The line $x - y = 0$ meets the curve in three coincident points given by $x - y = 0$ and $x + 2y - 3 = 0$; similarly for $x - 2y + 3 = 0$ and $2x - 3 = 0$. There are three points of inflexion, therefore, which lie on the line $x + 2y - 3 = 0$.

10. (1) $x = a$. (2) $x = 0$, or a. (3) $y = -\frac{1}{2}$, 0, or 1; $x = -\frac{1}{2}.3^{\frac{1}{3}}$, 0, or $3^{\frac{1}{3}}$.

CHAPTER XVIII.

CURVE TRACING.

278. We shall now give a few examples on the tracing of comparatively simple curves, first in cartesians and then in polars, in each case ending with a summary of the methods adopted.

279. Cartesian Coordinates.

Ex. 1. *Trace* $a^2y^2 = x^2(a^2 - x^2)$.

(*a*) Since no odd powers occur, the curve is doubly symmetrical, *i.e.* symmetrical with respect to both axes. We need only, therefore, take the first quadrant.

(*b*) Since $ay = x\sqrt{a^2 - x^2}$, x cannot be $> a$.
Similarly, solving for x we have

$$x^4 - a^2x^2 + a^2y^2 = 0, \text{ or } x^2 = \frac{a^2 \pm a\sqrt{a^2 - 4y^2}}{2},$$

whence y cannot be $> a/2$.

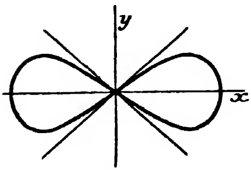

FIG. 77.

(*c*) There are no asymptotes.

(*d*) The tangents at O are $x^2 = y^2$. By Art. 274, there are two points of inflexion at O. Hence the curve is as in the figure.

(*e*) If we wish to draw the curve accurately to scale we may plot points by giving x different values and finding the corresponding values of y.

Ex. 2. *Trace* $y^3 + axy + bx^2 = 0$.

For purposes of tracing let $a = 2$, $b = 3$, say.

(*a*) Tangents at O are $x = 0$, $ay + bx = 0$. Draw these.

(*b*) Put $y = mx$; we may do this when the dimensions of the terms do not differ by more than two.

We get $$x = -\frac{ma+b}{m^3}; \quad y = -\frac{ma+b}{m^2}.$$

By giving different values to m, which will determine the directions of the line $y = mx$, we can find the points in which the curve is cut by this line. Moreover, the values of m must be sufficiently close together to enable us to trace the curve from point to point.

TABLE OF VALUES.

$m = \tan\theta$	x	y
0	$-\infty$	$-\infty$
1	-5	-5
2	$-\frac{7}{8}$	$-\frac{7}{4}$
∞	0	0
-2	$-\frac{1}{8}$	$+\frac{1}{4}$
$-$[illegible]	0	0
-1	$+1$	-1

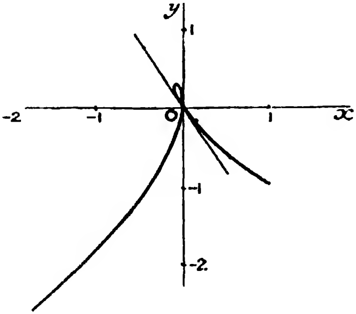

Fig. 78.

(*c*) Solving for x in terms of y, we have
$$x = \frac{-a \pm \sqrt{a^2 - 4by}}{2b} y.$$

Hence $y \ngtr a^2/4b$; *i.e.* $\ngtr \frac{1}{3}$.

Ex. 3. *Trace* $xy^2 = (x-a)^3$ . . . . (1)

(*a*) Curve symmetrical with respect to Ox.

(*b*) If $x > a$, y is real;
$x = a$, $y = 0$;
x between a and 0, y is imaginary;
$x < 0$, y is real.

Hence no part of the curve lies between $x = 0$ and $x = a$.

(*c*) Find the asymptotes. We shall obtain by expansion
$$y = \pm\left(x - \frac{3a}{2} + \frac{3a^2}{8x}\dots\right).$$

The asymptotes are therefore $y = \pm\left(x - \frac{3a}{2}\right)$; and the term $\frac{3a^2}{8x}$ *shows on which side of the asymptote the curve lies* (Art. 247). But the curve is of the third degree; hence there are three asymptotes (Ex. XXXIX., No. 2).

The third asymptote is evidently $x = 0$.

(*d*) Find where the asymptotes cut the curve.

Putting $y = x - \dfrac{3a}{2}$ in (1), we have

$$x\left(x^2 - 3ax + \frac{9a^2}{4}\right) = x^3 - 3ax^2 + 3a^2x - a^3;$$

$$\therefore\ \frac{3a^2x}{4} = a^3, \quad \text{and } x = \frac{4a}{3}.$$

(*e*) Next examine the curve where $x = a$, $y = 0$, at O' in the figure. Transferring the origin, (1) becomes $(x + a)y^2 = x^3$.

The tangents at the new origin are $y^2 = 0$. Also y is imaginary when x is negative, and has equal and opposite values when x is positive. Hence there is a cusp.

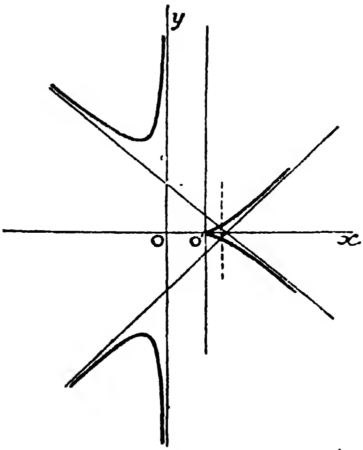

FIG. 79.

(*f*) If necessary, examine for points of inflexion by putting $y_2 = 0$.

This is of secondary importance in the present case, as we can see that there is not likely to be one, from other considerations.

We shall get from (1), $y_2 = \dfrac{3a^2}{4x^{\frac{5}{2}}(x - a)^{\frac{1}{2}}}$, which never vanishes, but becomes infinite when $x = 0$ and $x = a$; it is imaginary, however, for adjacent values on one side of each, and so cannot change sign. Hence there is no point of inflexion.

(*g*) Connect the various portions of the curve, and test the curve by noting whether any line can possibly cut it in more than three points.

Since O' is a double point, no line through O' can cut the curve in more than one other point.

Ex. 4. *Trace* $x(x - y)^2 = x + y$.

(*a*) If we change the sign of x and y, the equation is not changed. Hence *the curve is symmetrical in opposite quadrants.*

(*b*) The asymptotes are $x = 0$, $x - y = \pm \sqrt{2}$.

By solving for y and expanding, we get

$$y = x \pm \sqrt{2} + \frac{1}{2x} \cdots,$$

which gives the position of the curve with respect to the asymptotes.

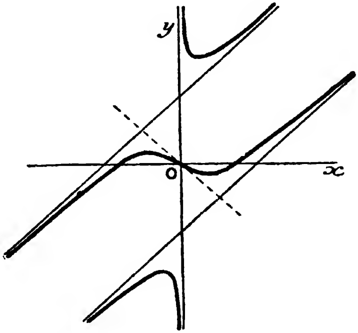

Fig. 80.

(c) The tangent at O is $x + y = 0$.

(d) There is a point of inflexion at O [Art. 274], as also follows from (a).

(e) To find where the curve cuts Ox, put $y = 0$; $\therefore x = 0$, or ± 1.

Ex. 5. *Trace* $ay^2 = x(a^2 - x^2)$.

(a) Curve symmetrical about Ox.

(b) If $x > a$, y is imaginary;
x between a and 0, y real;
x between 0 and $-a$, y imaginary;
$x < -a$, y real.

Since y is finite for finite values of x there is a loop between $x = 0$ and $x = a$.

(c) $dy/dx = \infty$ when $x = +a$, 0, and $-a$.

(d) No asymptotes at a finite distance from O; but $x = 0$ is parallel to the asymptotes. Hence the direction of the curve at infinity is parallel to Oy. This occurs when $x = -\infty$.

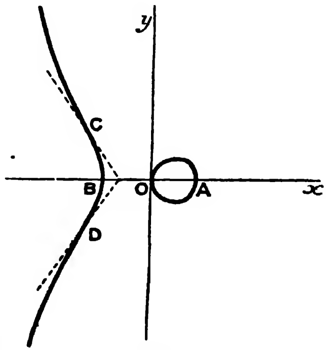

Fig. 81.

(*e*) There is probably a point of inflexion between $x = -a$ and $x = -\infty$. It will be found to occur where $x = -\frac{3}{2}a$.

Ex. 6. *Trace* $ay^2 - 2axy + x^3 = 0$ . . . . (1)

(*a*) Tangents at O are $y = 0$ and $y = 2x$.

(*b*) Solving for y, we have

$$y = \frac{ax \pm \sqrt{a^2x^2 - ax^3}}{a} = x \pm x\sqrt{\frac{a-x}{a}}.$$

Hence we may trace the two *auxiliary* loci,

$$y = x \text{ and } y = \pm x\sqrt{\frac{a-x}{a}}.$$

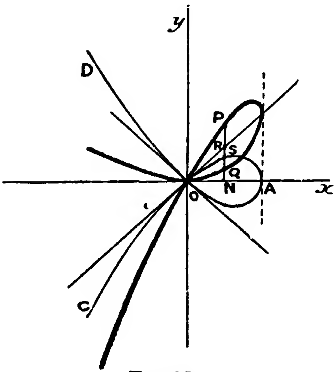

FIG. 82.

The second equation is

$$ay^2 = x^2(a - x);$$

hence the tangents at O are $y^2 = x^2$.

The curve may be easily shown to be as in the figure ($COSAD$).

Now let R be a point on $y = x$; draw $PRSQN$ perpendicular to Ox. Take $RP = RQ = SN$; then P and Q are points on the locus (1).

By thus measuring vertical distances from the line OR, equal to the corresponding ordinates of the auxiliary curve, we can easily trace the locus, which is represented by the dark line.

280. Summary of Methods for tracing Curves in Cartesians.

(1) Note the degree, and whether the curve is symmetrical.

(2) If possible, solve for y or x, and note when either is imaginary.

(3) Find the tangents at the origin; and, when possible, the sign of y for different values of x.

(4) Find the asymptotes, and the position of the curve with respect to them; also where they cut the curve, if possible.

(5) Find where the curve cuts the axis, and plot points. Find dy/dx.

(6) Put $y = mx$, when the dimensions of terms do not differ by more than two.

(7) Transfer the origin to a convenient point.

(8) Use auxiliary curves, by solving for y and separating terms.

(9) When the asymptotes are at infinity, note the direction of the curve at infinity by equating to zero the terms of highest degree.

(10) Turn to polars, if convenient.

(11) Examine for nodes, cusps, conjugate points, loops, points of inflexion, maxima and minima, etc.

(12) Connect branches, and test the curve by noting whether any line can possibly cut it in more than n points, n being the degree of the equation.

EXAMPLES XLIII.

Trace the following curves.—

1. $x^2y = a^2(2a - y)$.

2. $x^2y = a^2(y - 2a)$.

3. $x(x - a)^2 = y^2(x - 2a)$.

4. $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$.

5. $(x - 1)(x - 3)(y - 1)(y - 3) = 9$. Transfer the origin to the point $(2, 2)$.

6. $y^4 = a^2(x^2 - y^2)$. Put $y = mx$.

7. $y^4 = a^2(x^2 + y^2)$. Show that the curve has a conjugate point; show also, by putting $d^2x/dy^2 = 0$, that there are points of inflexion at the points $\left(\pm\dfrac{\sqrt{3}}{2}a,\ \pm\dfrac{\sqrt{3}}{\sqrt{2}}a\right)$.

8. $x^3 + y^3 = 3axy$. [Folium of Descartes.] [See Art. 61, Ex. 5.]

9. $y^2 = \dfrac{a^5}{(x + a)^3}$. Show that there is no point of inflexion.

10. $y^2(2a - x) = x^3$. [Cissoid of Diocles.] There is a cusp at O.

11. $xy^2 = 4a^2(2a - x)$. There is a point of inflexion at $\left(\dfrac{3}{2}a,\ \dfrac{2}{\sqrt{3}}a\right)$.

12. $ay^2 = x(x-a)^2$. The curve has a loop; at infinity the curve is parallel to Oy; no asymptotes.

13. $x^4 + y^4 = 2a^2(x^2 - y^2)$.

14. $x^4 + y^4 = a^2xy$. The curve is symmetrical about the lines $x \pm y = 0$; y is a max. at $(\frac{1}{2}.3^{\frac{1}{8}}a,\ \frac{1}{2}.3^{\frac{3}{8}}a)$.

15. $y^3 = x^2(x-a)$. There is a cusp at O, and a point of inflexion at $(a, 0)$, the tangent being vertical; and the asymptote cuts the curve where $x = a/9$.

16. $x^3/a^3 + y^3/b^3 = 1$. [See Ex. XLII., No. 11.]

17. $y^3 = ax(a-x)$. Two points of inflexion at $(0, 0)$ and $(a, 0)$; at infinity, curve parallel to Ox.

18. $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$. Put $x = a\cos^3\theta$; $y = b\sin^3\theta$.

19. $y(a-x)^2 = x^3$.

20. $y(a-x)(2a-x) = x^3$.

21. $2x(x^2+y^2) = a(2x^2+y^2)$.

22. $4y^3 + xy + 2x^2 = 0$.

23. $y^3 + y^2 = 4x^2$.

24. $x^3 + y^3 = x - 2y$.

25. $x^2(x^2+y^2) = a^2y^2$.

26. $ay(x+2a) = x^2y + a^3$.

27. $y = \sqrt{a^2 - x^2} + x$. Use auxiliary curves.

28. $(y - x^3)^2 = 1 - x^2$.

29. $a^2y^2 + x^2(x-a)(x-2a) = 0$.

30. $a^2y^2 = x^2(x-a)(x-2a)$.

31. $xy^2 + 2yx^2 = a(x^2+y^2)$.

32. $y^2(x+a) = x^2(x-a)$.

33. $y^3(x+2a) = (x+a)^3(x-a)$.

281. Polar Coordinates.

Ex. 1. *Trace the curve* $r = a\cos 4\theta$.

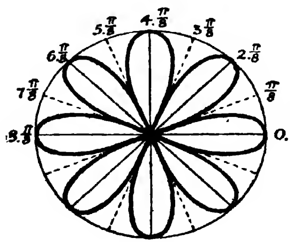

FIG. 83.

By giving θ different values the curve can be easily traced by plotting points. It will be seen that the values of r occur periodically, and oscillate between $+a$ and $-a$; hence the curve is inscribable in a circle of radius a.

TABLE OF VALUES.

θ	0	$\frac{\pi}{8}$	$\frac{2\pi}{8}$	$\frac{3\pi}{8}$	$\frac{4\pi}{8}$	$\frac{5\pi}{8}$	$\frac{6\pi}{8}$	$\frac{7\pi}{8}$	$\frac{8\pi}{8}$ ...
r	a	0	$-a$	0	a	0	$-a$	0	a ...

Note that the values of θ which make $r = 0$ give the directions of the

tangents at the origin, as may be seen by making θ approach these values [see Fig. 60, Art. 266].

Ex. 2. *Trace* $r^2 = a^2 \cos \frac{4\theta}{5}$.

If $\frac{4\theta}{5} = \frac{\pi}{2}$, then $\theta = \frac{5}{4} \cdot \frac{\pi}{2} = \alpha$ say.

Describe a circle, and let the numbers 1, 2, 3, etc., denote the positions of radii for which $\theta = \alpha, 2\alpha, 3\alpha$, etc. There will be sixteen positions in all.

Noting that r^2 cannot be –ve, we have the following table of values :—

θ	0	α	2α	3α	4α	5α	6α	7α	8α...
r^2	a^2	0	$-a^2$	0	a^2	0	$-a^2$	0	a^2...

Again, when r is real it has equal and opposite values; this, however, does not duplicate the curve, as the whole curve could be described by keeping to the +ve value of r only, the reason being that the numerator in $4\theta/5$ is even.

In the curve $r^2 = a^2 \cos \frac{3\theta}{5}$, the numerator of the angle being odd, the curve will be duplicated.

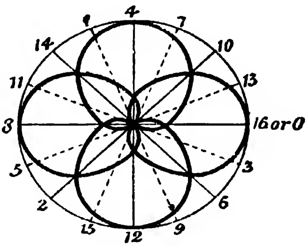

FIG. 84.

Ex. 3. *Trace* $r \cos 2\theta = a \sin 3\theta$.

Here $r = \infty$ if $\cos 2\theta = 0$. To find the asymptotes, if any, we may adopt the ordinary method, or turn to cartesians.

Choosing the latter, we have

$$r(\cos^2\theta - \sin^2\theta) = a \sin\theta(3 - 4\sin^2\theta);$$

$$\therefore \text{ multiplying by } r,\ x^2 - y^2 = ay\left(3 - \frac{4y^2}{x^2 + y^2}\right),$$

$$\text{or} \quad x^4 - y^4 = ay(3x^2 - y^2).$$

The asymptotes will be found to be

$$x + y = -\frac{a}{2}, \quad \text{and } x - y = \frac{a}{2}.$$

The tangents at the origin are given by

$$\sin 3\theta = 0; \quad \therefore \theta = 0,\ \pi/3, \text{ or } 2\pi/3.$$

Using the cartesian equation, since x occurs in even powers, the curve is symmetrical about Oy.

Also if $x = 0$; $y = a$, and $dy/dx = 0$.

To find where the asymptote $x - y = a/2$ cuts the curve, we can show, by putting $x = y + \frac{a}{2}$, that $y = 0{\cdot}14a$, or $-0{\cdot}3a$.

Hence the branch A crosses its asymptote, and there must evidently be a point of inflexion. To find this is troublesome, however, but we may

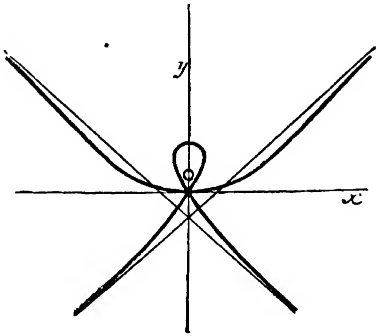

FIG. 85

either solve for x in terms of y, and put $d^2x/dy^2 = 0$, or use polars and put $u + u_2 = 0$. The latter gives ultimately a quadratic in $\cos 2\theta$, the admissible root being $\frac{\sqrt{3}-1}{2}$, or $0{\cdot}36$, whence $\theta =$ about $35°$.

Ex. 4. *Trace* $r = a\frac{\theta - \alpha}{\theta + \alpha}$.

By Ex. 1, Art. 262, the asymptote is $2a\alpha = -r\sin(\theta + \alpha)$.

There is also a circular asymptote, since $r = a$ when $\theta = \infty$.

Again, since $r = 0$ when $\theta = \alpha$, we have the direction of the tangent at the origin.

For purposes of tracing, let $\alpha = \pi/6 = \frac{1}{2}$ nearly; we have then the following table of values :—

θ	∞	α	0	$-\alpha$	$-\infty$	π	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\pi$...
r	a	0	$-a$	∞	a	$\frac{5a}{7}$	$\frac{a}{2}$	$2a$	$\frac{7a}{5}$...

To test for a point of inflexion, use the formula $u + u_2 = 0$. We shall have $(\theta + a)(\theta - a)^2 + 4a = 0$.

Since the left hand side is $+^{ve}$ when $\theta = -a$, and $-^{ve}$ when $\theta = -\infty$;

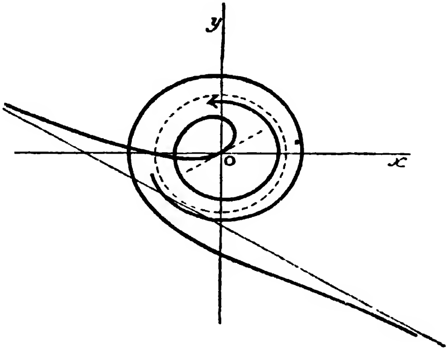

Fig. 86

it must pass through zero for some value of θ between $-a$ and $-\infty$; hence there is a root between those values, giving a point of inflexion.

282. Summary of Methods for Tracing Curves in Polar Coordinates.

(1) Write out a table of values of θ and r, and thence note how r varies with θ.

(2) Find the tangents at the origin by noting for what values of θ, r is equal to 0.

(3) Find the asymptotes, by polars, or by turning to cartesians.

(4) Find the circular asymptotes, if any.

(5) Turn to cartesians if convenient, and adopt methods given before.

(6) Find points of inflexion, if any.

EXAMPLES XLIV.

Trace the following curves:—

1. $r = a\cos 3\theta$. **2.** $r = a\cos 5\theta$. **3.** $r^2 = a^2\cos\theta$.

4. $r^2 = a^2 \cos 3\theta$. **5.** $r^2 = a^2 \cos 4\theta$. **6.** $r = a \cos \frac{2\theta}{3}$.

7. $r = a \cos \frac{3\theta}{2}$. **8.** $r^2 = a^2 \cos \frac{\theta}{3}$. **9.** $r^2 = a^2 \cos \frac{2\theta}{3}$.

10. $r \cos 3\theta = a$. **11.** $r \cos 4\theta = a$. **12.** $r \cos \frac{\theta}{2} = a$.

13. $r \cos \frac{\theta}{3} = a$. **14.** $r^2 \cos \theta = a^2$. **15.** $r^2 \cos 3\theta = a^2$.

16. $r = a \tan \theta$. **17.** $r = a \tan^2 \theta$. **18.** $r = a + b \cos \theta$.

19. $r = a + b \cos 2\theta$. **20.** $r = 1 + 2 \sin^2 \theta$. **21.** $r \cos \theta = 2a \sin^2 \theta$.

22. $r \cos \theta = a \cos 2\theta$. **23.** $r \cos \theta = 1 - \sin \theta$.

24. $r = a \tan \frac{\theta}{2}$. Find the equation in cartesians; transfer the origin to the point $(0, a)$; again, express in polars, and show that the equation becomes $r \sin \theta = a \cos 2\theta$.

25. $r \cos \theta = 1 + 2 \sin \theta$.

26. $r \cos \theta = 2 + \sin \theta$. Find the angle at which the curve cuts Ox. Change to cartesians, and show that there is a conjugate point at the origin.

27. $r^2 \sin \theta = a^2 \cos 2\theta$; the curve is symmetrical about its asymptote, Ox.

28. $r \cos 3\theta = a \sin 2\theta$; there is double point at the origin; also three asymptotes equally inclined to each other; symmetrical about Oy.

29. $r \cos \theta \cos 3\theta = a$.

30. $r = a\theta \sin \theta$.

31. $r\theta = a \sin \theta$. Show that when $\theta = 0$, $r\frac{d\theta}{dr} = 0$, and is $-^{ve}$ when θ is small; hence there is a cusp at the point $(a, 0)$; curve symmetrical.

32. $r^{\frac{2}{3}} \cos \frac{2}{3}\theta = a^{\frac{2}{3}}$. **33.** $r = \theta(\theta - 1)$.

34. $r = \dfrac{\theta}{\theta - 1}$. **35.** $(\theta^2 - 4)r = a\theta$.

CHAPTER XIX.

CURVATURE—CONTACT—ENVELOPES—EVOLUTES.

283. The curvature of a curve at any point on it may be roughly described as the degree of bending about that point.

A sharp bend denotes great, and a gradual one small, curvature; while a straight line, which does not bend at all, has of course no curvature.

A circle bends uniformly at all points, or, in other words, has the same curvature throughout; we may therefore choose the circle as the best means of obtaining a precise definition of this term.

284. Angle of Contingence.

Def.—If P and Q be two points on a continuous curve, and the tangent at P roll on the curve until it becomes the tangent at Q, then the *deflection* of the tangent, *i.e.* the angle through which it has turned, is called the *angle of contingence* of the arc PQ.

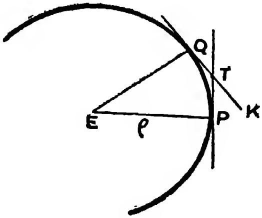

Fig. 87.

If PQ be the arc of a circle of radius ρ, then the angle of contingence is PTK, and this $= PEQ$.

If equal arcs, *however small or large*, be taken on the same circle, the deflection of the tangent will be the same for all. Hence the deflection is proportional to the arc.

285. Definition of Curvature—measured by $1/\rho$.

Def.—The *curvature* at a given point of a curve is the deflection per unit arc from that point, if the curvature is uniform; if variable, the deflection per unit arc, supposing the curvature to be uniform and the same as at that point.

For the circle, this becomes $\dfrac{\angle PEQ}{\text{arc } PQ}$ or $\dfrac{1}{\rho}$, if we use circular measure.

For a curve of variable curvature, the obvious method is to take an infinitesimal arc at the point, so that the curvature is practically the same throughout. If $PQ(=ds)$ be this arc, and we draw the normals PE, QE [see also Fig. 47, Art. 220], then regarding PQ as part of a circle, E will be the centre of this circle, and its radius corresponds with what was defined as the radius of curvature ρ.

The deflection is $d\psi$, hence the curvature $= d\psi/ds = 1/\rho$. [Art. 240.]

286. Expression for ρ in Cartesians.

Since $\tan\psi = dy/dx$, $\sec\psi = ds/dx$; we have, differentiating the former with respect to x,

$$\sec^2\psi \cdot \frac{d\psi}{dx} = \frac{d^2y}{dx^2};$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = \sec^3\psi \Big/ \frac{d^2y}{dx^2}$$

$$= \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{d^2y/dx^2}, \text{ or } \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}.$$

Ex. 1. In the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, we have, differentiating twice,

$$b^2x + a^2yy_1 = 0,$$

$$b^2 + a^2y_1^2 + a^2yy_2 = 0;$$

$$\therefore y_1 = -\frac{b^2x}{a^2y};\quad y_2 = \frac{-(b^2 + a^2y_1^2)}{a^2y} = -\left(b^2 + \frac{b^4x^2}{a^2y^2}\right)\Big/a^2y$$

$$= -\frac{b^2}{a^4y^3}(a^2y^2 + b^2x^2) = -\frac{b^4}{a^2y^3}.$$

$$\therefore \rho = -\left(1 + \frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}} \Big/ \frac{b^4}{a^2y^3} = -\frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

But $a^4y^2 + b^4x^2 = a^2(a^2b^2 - b^2x^2) + b^4x^2 = a^4b^2 - (a^2 - b^2)b^2x^2$
$= a^2b^2(a^2 - e^2x^2)$.

$\therefore$ neglecting the negative sign, we have (numerically)

$$\rho = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab} = \frac{CD^3}{ab},$$

where CD is conjugate to CP.

Ex. 2. For the cycloid, $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$.

$$y_1 = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2},$$

$$y_2 = \frac{dy_1}{d\theta}\Big/\frac{dx}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}\Big/(1 + \cos\theta)a = \frac{1}{4a}\sec^4\frac{\theta}{2}.$$

$$\therefore \rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \sec^3\frac{\theta}{2}\cdot\frac{4a}{\sec^4\frac{\theta}{2}} = 4a\cos\frac{\theta}{2}.$$

Ex. 3. *Find the radius of curvature of the folium* $y(x^2 + y^2) = a(y^2 - x^2)$, *at the point where* $y = a$.

When $y = a$, $x = 0$. Differentiating, we have

$$y_1(x^2 + y^2) + 2y(x + yy_1) = 2a(yy_1 - x).$$

Putting $x = 0$, we have at once $y_1 = 0$ for the point $(0, a)$.

Differentiating again, and omitting terms which contain y_1, we have for the point $(0, a)$

$$y_2(x^2 + y^2) + 2y(1 + yy_2) = 2a(yy_2 - 1);$$

$$\text{or } a^2y_2 + 2a(1 + ay_2) = 2a(ay_2 - 1); \quad \therefore y_2 = -\frac{4}{a}.$$

Hence $$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = -\frac{a}{4}.$$

287. Expression in Polar Coordinates.

Since $\psi = \phi + \theta$; [See Fig. 45, Art. 216.]

$$\therefore \frac{d\psi}{d\theta} = \frac{d\phi}{d\theta} + 1 = 1 + \phi' \text{ say} \quad . \quad . \quad . \quad . \quad (1)$$

Now, $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r}{r'}$, where $r' = \dfrac{dr}{d\theta}$;

$$\therefore\ \sec^2\phi.\phi' = \frac{r'^2 - rr''}{r'^2}.$$

But $\sec^2\phi = 1 + \tan^2\phi = 1 + \dfrac{r^2}{r'^2} = \dfrac{r^2 + r'^2}{r'^2}$ . . . . (2)

Adding, $\sec^2\phi(1 + \phi') = \dfrac{2r'^2 + r^2 - rr''}{r'^2}$.

$\therefore$ by (1) and (2), $\dfrac{r^2 + r'^2}{r'^2}.\psi' = \dfrac{2r'^2 + r^2 - rr''}{r'^2}$.

$$\therefore\ \psi' = \frac{2r'^2 + r^2 - rr''}{r^2 + r'^2}.$$

But $s' = (r^2 + r'^2)^{\frac{1}{2}}$. [Art. 216.]

$$\therefore \text{ by division, } \rho = \frac{s'}{\psi'} = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{2r'^2 + r^2 - rr''}.$$

288. Alternative Method.

We have shown (Art. 242) that $\rho = r\dfrac{dr}{dp}$.

Using, as above, dashes to denote differentiation in θ, we have

$$\rho = r\frac{r'}{p'}.$$

Now $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{r'^2}{r^4} = \dfrac{r^2 + r'^2}{r^4}$;

$$\therefore\ -\frac{2p'}{p^3} = \frac{(2rr' + 2r'r'')r - 4r'(r^2 + r'^2)}{r^5} = \frac{2rr'r'' - 2r'r^2 - 4r'^3}{r^5};$$

$$\therefore\ p' = \frac{2r'^3 + r'r^2 - rr'r''}{r^5}.p^3 = \frac{r'}{r^5}.(2r'^2 + r^2 - rr'').\frac{r^6}{(r^2 + r'^2)^{\frac{3}{2}}};$$

$$\therefore\ \rho = \frac{rr'}{p'} = rr'.\frac{(r^2 + r'^2)^{\frac{3}{2}}}{rr'(2r'^2 + r^2 - rr'')} = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{2r'^2 + r^2 - rr''}.$$

Ex. In the system of curves $r^m = a^m \cos m\theta$, we have

$$m \log r = \log a^m + \log \cos m\theta.$$

By differentiation,

$$\frac{m}{r}\cdot r' = -m\tan m\theta, \quad \text{or } \frac{r'}{r} = -\tan m\theta \quad . \quad . \quad . \quad (1)$$

Differentiating again,

$$\frac{rr'' - r'^2}{r^2} = -m\sec^2 m\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But from (1) $\qquad \dfrac{r^2 + r'^2}{r^2} = \sec^2 m\theta;$

hence $\quad 2r'^2 + r^2 - rr'' = (r^2 + r'^2) - (rr'' - r'^2)$

$$= r^2\sec^2 m\theta + mr^2\sec^2 m\theta = (m+1)r^2\sec^2 m\theta.$$

$$\therefore \rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{2r'^2 + r^2 - rr''} = \frac{r^3\sec^3 m\theta}{(m+1)r^2\sec^2 m\theta} = \frac{r}{(m+1)\cos m\theta}$$

$$= \frac{a^m}{(m+1)r^{m-1}}.$$

We get the same result thus:—

By Art. 232, the pedal equation is $a^m p = r^{m+1}$;

$$\therefore a^m\frac{dp}{dr} = (m+1)r^m.$$

$$\therefore \rho = r\frac{dr}{dp} = \frac{a^m}{(m+1)r^{m-1}}.$$

289. To prove that $\rho = p + \dfrac{d^2p}{d\psi^2}$ (second proof).

A previous proof of this was given in Art. 243.

The following is noteworthy. We have

$$dp/d\psi = PY = OM \text{ [Art. 220]}.$$

$\therefore$ taking the differential, $\dfrac{d^2p}{d\psi^2}\cdot d\psi$

$= NL = ENd\psi = EMd\psi$ (ultimately).

$$\therefore \rho = EM + MP = OY + EM = p + \frac{d^2p}{d\psi^2}.$$

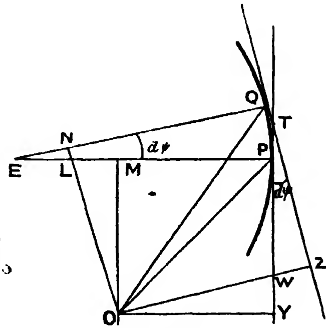

Fig. 88.

Note.—All the increments in the figure are +^{ve}; it will be a good exercise to draw the figure for different cases in which, for instance, (1) M comes to the left of E; (2) the arc PQ is convex to O: and adopt the convention of sign. In (1) we shall obtain the same result as before; but in (2) we shall obtain $\rho = -\left(p + \dfrac{d^2p}{d\psi^2}\right)$.

290. Another Form for ρ.

Since $\dfrac{dx}{ds} = \cos\psi$;

$$\therefore \frac{d^2x}{ds^2}\cdot\frac{ds}{d\psi} = -\sin\psi;\ \textit{i.e.}\ \rho\frac{d^2x}{ds^2} = -\sin\psi.$$

Similarly, $\dfrac{dy}{ds} = \sin\psi$;

$$\therefore \frac{d^2y}{ds^2}\cdot\frac{ds}{d\psi} = \cos\psi;\ \textit{i.e.}\ \rho\frac{d^2y}{ds^2} = \cos\psi.$$

Squaring and adding, we have

$$\rho^2\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\} = 1,$$

$$\text{or}\quad \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

291. Value of ρ at the Origin—Newtonian Method.

In the case of a curve which touches the x-axis at the origin, we have a very simple expression for ρ.

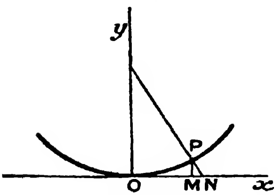

FIG. 89.

In the figure let E be the centre of curvature, and treat the curve as if it were a circle. Draw the ordinate PM, P being indefinitely near to O.

Then by Euc. III. 36,

$$NO^2 = NP\,(2\rho + NP).$$

Now $NO = x +$ a quantity infinitesimal compared with x,

$NP = MP$ or $y +$,, ,, ,, ,, y,

$2\rho + NP = 2\rho +$,, ,, ,, ,, 2ρ,

$\therefore$ we commit an infinitesimal error in writing

$$x^2 = y\,.\,2\rho,$$

whence $\rho = x^2/2y$ ultimately, *i.e.* $\underset{x=y=0}{\mathit{lt}}\ (x^2/2y)$. [See Art. 93.]

Similarly, if the curve touch Oy at the origin, $\rho = \mathit{lt}(y^2/2x)$.

292. Examples.

The equation to a curve touching Ox at O is of the form

$$y = ax^2 + 2hxy + by^2 + cx^3 + \dots, \quad . \quad . \quad . \quad . \quad (1)$$

since by putting $y = 0$ we have $x^2 = 0$, implying tangency.

Let $\rho_1 = x^2/2y$, *before* the limit is reached, so that $lt\,\rho_1 = \rho$.

Then from (1), $1 = a\dfrac{x^2}{y} + 2hx + by + cx\dfrac{x^2}{y} + \dots$

$$= 2a\rho_1 + 2hx + by + 2cx\rho_1 + \dots$$

But x and y are infinitesimal; hence in the limit,

$$1 = 2a\rho, \quad \text{or } \rho = 1/2a.$$

293. This can also be obtained from the ordinary formula; for noting that $y' = 0$ at the origin, we have, from (1),

$$y' = 2ax + 2hy + 2hxy' + 2byy' + \dots$$
$$y'' = 2a + 2hy' + 2hy' + 2hxy'' + 2by'^2 + 2byy'' \dots$$
$$= 2a \text{ since } x, y, y', \text{ ultimately vanish.}$$

$$\therefore \rho = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} = \frac{1}{y''} = \frac{1}{2a}.$$

Ex. In the parabola $y^2 = 4ax$, $\rho = lt\left(\dfrac{y^2}{2x}\right) = 2a$.

294. If the curve pass obliquely through the origin, we can find the value of ρ, thus:—

The equation to the curve will be

$$0 = ax + by + cx^2 + dxy + ey^2 + \dots \quad . \quad . \quad . \quad . \quad (1)$$

$\therefore$ by differentiation,

$$0 = a + by' + 2cx + dy + dxy' + 2eyy' \dots$$
$$0 = by'' + 2c + 2dy' + dxy'' + 2ey'^2 + 2eyy'' \dots$$

$\therefore$ at O, $\quad y' = -\dfrac{a}{b}$; and

$$0 = by'' + 2c + 2dy' + 2ey'^2$$
$$= by'' + 2c - \frac{2ad}{b} + \frac{2a^2e}{b^2};$$

or $\quad y'' = \dfrac{2}{b^3}(abd - a^2e - b^2c).$

$$\therefore \rho = \left(1 + \frac{a^2}{b^2}\right)^{\frac{3}{2}} \Big/ \frac{2}{b^3}(abd - a^2e - b^2c) = \frac{(a^2 + b^2)^{\frac{3}{2}}}{2(abd - a^2e - b^2c)}.$$

None of the terms in (1) beyond those of the second degree are wanted, for if these be differentiated twice, they are bound to contain either x or y as a factor, and will therefore disappear in the limit.

295. Equation of Circle of Curvature.

Let $E(h, k)$ be the centre of curvature, and $EP = \rho$ the radius of curvature.

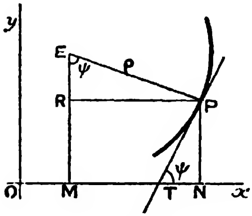

FIG. 90.

We have drawn the figure so that x, y, y_1, y_2 *are all* $+^{ve}$. [Art. 116.]

Noting that $\tan\psi = y_1$, so that

$$\sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}; \quad \cos\psi = \frac{1}{\sqrt{1+y_1^2}},$$

we have

$$h = OM = ON - PR = x - \rho\sin\psi = x - \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \cdot \frac{y_1}{(1+y_1^2)^{\frac{1}{2}}}$$

$$= x - \frac{y_1(1+y_1^2)}{y_2}.$$

Similarly, $\quad k = EM = y + \rho\cos\psi = y + \dfrac{1+y_1^2}{y_2}.$

The equation to the circle of curvature is therefore

$$(X-h)^2 + (Y-k)^2 = \rho^2,$$

where h, k, ρ have the values given above.

Ex. In the parabola $y^2 = 4ax$, we have

$$y = 2\sqrt{a}.\sqrt{x}; \quad y_1 = \sqrt{\frac{a}{x}}; \quad y_2 = -\frac{1}{2}\sqrt{\frac{a}{x^3}}.$$

$$\therefore \rho = \left(1+\frac{a}{x}\right)^{\frac{3}{2}} \Big/ \left(-\frac{1}{2}\sqrt{\frac{a}{x^3}}\right) = -2\frac{(x+a)^{\frac{3}{2}}}{\sqrt{a}}.$$

$$\sin\psi = \sqrt{\frac{a}{x}} \Big/ \sqrt{1+\frac{a}{x}} = \frac{\sqrt{a}}{\sqrt{a+x}}; \quad \cos\psi = 1\Big/\sqrt{1+\frac{a}{x}} = \frac{\sqrt{x}}{\sqrt{a+x}}.$$

$\therefore h = x - \rho\sin\psi = x + 2(x+a) = 3x + 2a;$

$k = y + \rho\cos\psi = y - 2\sqrt{x}(x+a)/\sqrt{a} = y - y(x+a)/a = -xy/a.$

∴ the equation to the circle of curvature is

$$(X - 3x - 2a)^2 + (Y + xy/a)^2 = 4(x + a)^3/a;$$

or

$$X^2 + Y^2 - 2(3x + 2a)X + \frac{2xy}{a}Y = 3x^2.$$

296. Contact.—Suppose two curves to cut one another in the points P, Q, R, S ... ; and let either or both curves alter in position or shape so that the following changes take place :—

(1) Let Q move up to, and coincide with, P. Then the curves are said to have *contact of the first order* at P.

(2) Let R now move up to, and coincide with, P and Q. Then the contact is of the *second order*; and so on, the contact being of the nth order when $n + 1$ coincident points are common to both curves.

297. Algebraical Conditions.

Let $y = f(x)$, $y = \phi(x)$, be the curves. Then for the points P, Q, R, ..., x and y are the same for both curves.

(1) If $Q(x + \delta x, y + \delta y)$ be indefinitely near to P; then, since y and $y + \delta y$ are the same for both curves, their difference, δy, will also be the same. Similarly, δx is the same. Therefore, in the limit, dy/dx is the same.

(2) If R move up indefinitely near to P and Q; then, by similar reasoning, since two consecutive values of $\delta y/\delta x$ are the same for both curves, their difference, $\delta.\frac{\delta y}{\delta x}$, will also be the same. Hence, in the limit, $\frac{d}{dx}.\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ will be the same; and so on.

Generally, *for contact of* n*th order*, d^ny/dx^n *is the same for both curves.*

298. Circle of Curvature.—If P, Q, R, S ..., be any given points on a curve, then, generally, no circle can be made to pass through more than three of them at a time. †

† It is quite true that a circle will frequently cut a curve in more than three points, but the converse is not necessarily true. Hence, if we select

It follows, then, that a circle cannot, as a rule, have contact of a higher order than the *second*, with a given curve, though it may do so in special cases, as at an apse, where the curve is symmetrical about the normal.

The circle which has contact of the second order with a curve is therefore called the *circle of closest contact*, or the *osculating circle;* and it is in fact the *circle of curvature*, as we shall now show. For since the contact is of the second order, x, y, y_1, y_2 are the same for the curve and for the circle. But, for the curve,

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2};$$

hence the radius of curvature for the curve must be the same as that for the circle, which is obviously its own radius, showing that the latter is the circle of curvature.

EXAMPLES XLV.

Find the value of ρ in the following curves:—

1. (1) The parabola $p^2 = ar$. (2) The circle $ap = r^2 + b$.
   (3) The ellipse $p^2(a^2 + b^2 - r^2) = a^2b^2$.
   (4) The limaçon $r^4 = p^2(2ar - a^2 + b^2)$ at the point $p = r = a + b$.
   (5) The cardioide $r^3 = ap^2$. (6) The lemniscate $a^2p = r^3$.
   (7) The equiangular spiral $p = r\sin\alpha$.

2. (1) $xy = c^2$. (2) $b^2x^2 - a^2y^2 = a^2b^2$.
   (3) $4y = x^2 - 2\log x$. (4) $x^4 - 6a^2xy + 3a^4 = 0$.
   (5) The catenary $y = a\log\dfrac{x + \sqrt{x^2 - a^2}}{a}$.
   (6) $e^x = \sin y$. (7) $y = a\log\sin\dfrac{x}{a}$.
   (8) $\cos\dfrac{y}{a} = \cosh\dfrac{x}{a} + \sinh\dfrac{x}{a}$.

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three points, P, Q, R, on a curve, a circle can always be made to pass through them; but although it may cut the curve in other points as well, we cannot choose the positions of the latter on the curve.

3. (1) The hyperbola $x = a \sec \phi$; $y = b \tan \phi$.

(2) The parabola $x = a \cot^2 \psi$; $y = 2a \cot \psi$.

(3) $x = a\theta$; $y = a \log \sec \theta$. [Cf. Qu. 2; (6), (7), and (8).]

(4) The catenary $x = a \log (\sec \theta + \tan \theta)$; $y = a \sec \theta$.

(5) The tractrix $x = a \log \cot \dfrac{\theta}{2} - a \cos \theta$; $y = a \sin \theta$.

4. (1) The equiangular spiral $r = ae^{\theta \cot \alpha}$.

(2) The limaçon $r = a \cos \theta + b$, where $r = b/2$.

(3) The cardioide $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta$.

(4) The curve $r^3 = a^3 \cos 3\theta$.

(5) The curve $r^2 = a^2(2 + \cos 2\theta)$, at the point $(a, \pi/2)$.

5. Find the value of ρ at the origin for the following curves:—

(1) $2y = (x - 2y)^2$.
(2) $2 \cos \theta = r \cos 2\theta$.

(3) $x(a - y)^2 = y^2(a - x)$.
(4) $x(y^2 - a^2) = x^3 + y^3$.

(5) $x^3 + y^3 + 2x^2 - 4y + 3x = 0$.

(6) $(x\sqrt{3} - y)(1 - 4x) + y^2 + x^3 - 3y^3 = 0$.

(7) $a(x^2 - y^2) = 2x^3 + y^3$.

6. Use the (p, ψ) formula to find ρ for—

(1) The point $p = a \sin \psi$.
(2) The circle $p = a(1 + \sin \psi)$.

(3) The parabola $p = a \operatorname{cosec} \psi$.

(4) The rectangular hyperbola $p^2 = - a^2 \cos 2\psi$.

7. If $x = f(t)$; $y = \phi(t)$, prove that $\rho = (f'^2 + \phi'^2)^{\frac{3}{2}}/(f'\phi'' - f''\phi')$.

8. Prove that $\rho = (u^2 + u_1{}^2)^{\frac{3}{2}}/u^3(u + u_2)$ where $u = 1/r$.

Hence show that, for the parabola $2au = 1 + \cos \theta$, $\rho = 2a \sec^3 \frac{1}{2}\theta$.

9. Show that at a point of inflexion there is zero curvature.

10. Find the radius and coordinates of the centre of curvature at the point where $x = a$ of the curve $x^2y = a^2(x - y)$.

11. Find the equation to the circle of curvature of the parabola $y^2 = 4ax$, at one extremity of the latus rectum, and find where it again cuts the curve.

12. Find the value of ρ at the origin for the curve $y = x - \sin x$; also at the point at which $x = \pi/2$.

13. Show that the radius of curvature at the extremity of the major axis of an ellipse is equal to the semi-latus-rectum.

14. In the hyperbola show that the radius of curvature is proportional to the cube of the length intercepted on the normal by the curve and the x-axis.

15. In the cycloid show that $\rho =$ twice the normal. [Art. 308.]

16. O is a fixed point on a circle of diameter c; P is a moving point on the circle. If OP be produced to Q, so that $PQ =$ a constant b, find the value of ρ at the point on the locus of Q where it meets the diameter through O.

17. In the curve $r = a \sec 2\theta$, show that $\rho = -\dfrac{1}{3}\dfrac{r^4}{p^3}$.

18. Find for what point on the ellipse the radius of curvature is a mean proportional between its greatest and least values.

19. P is a point on the curve $r^2 = a^2 \cos 2\theta$; the normal at P meets at G the line drawn through O perpendicular to OP. Show that at P, $\rho = \frac{1}{3}PG$. [See Art. 233 (A).]

20. Show that for the curve

$$\frac{2y}{a} = \sin\frac{x}{a} - \log\tan\left(\frac{\pi}{4} + \frac{x}{2a}\right),\ \rho = -a\left(\cos^2\frac{x}{a} + 1\right)^2 \Big/ 4\sin\frac{x}{a}\cos\frac{x}{a}.$$

ANSWERS.

1. (1) $2p^3/a^2$. (2) $a/2$. (3) a^2b^2/p^3. (4) $(a+b)^2/(a+2b)$.
(5) $2ap/3r$. (6) $a^2/3r$. (7) $r \operatorname{cosec} \alpha$ or r^2/p.

2. (1) $r^3/2c^2$. (2) $-(a^4y^2 + b^4x^2)^{\frac{3}{2}}/a^4b^4 = -CD^3/ab$. (3) $(x^2+1)^2/4x$.
(4) $(x^4+a^4)^2/8x^3a^4$. (5) $-x^2/a$. (6) e^{-x}.
(7) $-a \operatorname{cosec} x/a$. (8) $-a \sec y/a$.

3. (1) $-(a^2\tan^2\phi + b^2\sec^2\phi)^{\frac{3}{2}}/ab = -CD^3/ab$. (2) $2a \operatorname{cosec}^3\psi$.
(3) $a\sec\theta = ae^{y/a}$. (4) $a\sec^2\theta = y^2/a$. (5) $-a\cot\theta$.

4. (1) $r \operatorname{cosec} \alpha$. (2) $2a^3/(4a^2 - b^2)$. (3) $\frac{2}{3}\sqrt{ar}$. (4) $a^3/4r^2$. (5) $-a$.

5. (1) 1. (2) -1. (3) $a/2$. (4) ∞. (5) $\frac{125}{64}$. (6) $\frac{4}{3}$.
(7) Differentiate three times; $-\dfrac{2\sqrt{2a}}{3}$, or $2\sqrt{2a}$.

6. (1) 0. (2) a. (3) $2a \operatorname{cosec}^3\psi$. (4) $-a^4/p^3$.

10. $-2a$; $(a, -\frac{3}{2}a)$.

11. $x^2 + y^2 - 10ax + 4ay = 3a^2$. Putting $y^2 = 4ax$, and rationalizing, we obtain an equation in x, which contains $(x - a)^3$ as a factor, the other factor being $x - 9a$. This should be so, since the circle of curvature cuts the parabola in three coincident points. The fourth point is $(9a, -6a)$.

12. ∞ ; $2\sqrt{2}$. **16.** $(b+c)^2/(b+2c)$. **18.** $x = a^{\frac{3}{2}}/\sqrt{a+b}$; $y = b^{\frac{3}{2}}/\sqrt{a+b}$.

299. Envelopes.—The equation to a plane curve usually consists of the variables x and y, together with other quantities termed *constants*.

Now, these "constants" may be made to have different values, in which case the curve will alter either in shape or position, or both, but the class of curve will not be altered. Thus, for example, the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a conic section ; and by altering the values of any of the constants we do not alter the class of curve, although this class includes the hyperbola, the ellipse, the parabola, together with the more particular forms, viz. the circle, two straight lines, and the point.

300. Family of Curves—Parameters.

Def.—The equation $f(x, y, a_1, a_2, \ldots) = 0$, where $a_1, a_2, \ldots$ are all supposed to vary gradually, is said to denote a *family of curves*, and the quantities $a_1, a_2, \ldots$ are called *variable parameters*, or *arbitrary constants*. It is not difficult to see that, if we *gradually* vary any of the parameters, the curve will *gradually* alter in shape or position, or both. This leads to the following definition :—

Def.—When the parameters of a given family of curves are made to vary according to given conditions, the locus of the ultimate intersections of consecutive members of the family is called the *envelope* of the family of curves.

The given conditions must be such that the curves pass systematically through a definite series of shapes and positions, and in a certain order, so that, given one member of the family. the preceding and succeeding members are quite definite.

301. *The envelope touches every member of the family;* for let OA, AB, BC, CD, DE be five consecutive members, then A, B, C, D are four consecutive points on the envelope. But each of the curves AB, BC, CD passes through two of them; hence, ultimately, the envelope will have two consecutive points in common with each of the curves, *i.e.* will touch each of them.

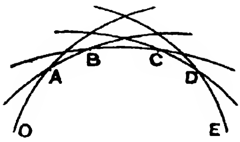

FIG 91.

302. Examples in Geometry.

Ex. 1. The envelope of a chord of a circle of constant length is a concentric circle, whose radius is equal to the distance of the chord from the centre.

Ex. 2. If S be a fixed point, and K a point on a fixed straight line, then the envelope of the perpendicular bisector of SK is a parabola, of which S is the focus, and the fixed line the directrix.

Ex. 3. It is known that if P be a point on an ellipse, foci S and S', and $S'P$ be joined and produced to W so that $PW = PS$, then $S'W = 2a$, the major axis; also the tangent at P bisects SW perpendicularly. Hence, conversely, if S be a fixed point and W a point on a fixed circle, centre S', the perpendicular bisector of SW will envelop an ellipse if $SS' < S'W$, or S is within the circle. Similarly, if S be without the circle, the envelope will be an hyperbola.

303. Analytical Method.

Let $f(x, y, a) = 0$ . . . . (1)

be a family of curves with a single variable parameter a; it is required to find the envelope.

A consecutive curve will be $f(x, y, a + \delta a) = 0$. . . . (2)

The envelope is the locus of the ultimate intersection of (1) and (2) when δa is infinitesimal; and since a is variable, the resulting equation, which represents a *single* locus, cannot contain a. Hence we must eliminate a between (1) and (2).

Now, expanding (2) by Taylor's Theorem, and neglecting small quantities beyond those of the first order, we have

$$f(x, y, a) + \frac{\partial f}{\partial a}.\delta a = 0\,;$$

or, from (1), $$\partial f/\partial a = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We shall therefore obtain the envelope by eliminating a between (1) and (3).

Ex. 1. *Find the envelope of the line* $a^2L - 2aM + N = 0 \ldots (1)$, *where* L, M, N *are linear expressions in* x *and* y.

Differentiating in a, we have

$$2aL - 2M = 0, \quad \text{or } a = M/L\,;$$

$$\therefore \text{ in (1) } \frac{M^2}{L} - \frac{2M^2}{L} + N = 0, \quad \text{or } M^2 = LN,$$

which is a conic.

It is important to notice that *this is the condition that the roots of* (1), *considered as a quadratic in* a, *are coincident.* Also $M^2 = LN$ is still the envelope if L, M, N, are *any* functions of x and y.

Ex. 2. *Find the envelope of the line* $y = mx + a\sqrt{1+m^2}$, m *being variable.*

We may differentiate in m, and eliminate as before; † or put $m = \tan\theta$. Adopting the latter method, we get

$$y = x\tan\theta + a\sec\theta,$$

$$\text{or} \quad x\sin\theta - y\cos\theta = -a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating in θ,

$$x\cos\theta + y\sin\theta = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Squaring (1) and (2) and adding, we get

$$x^2 + y^2 = a^2, \text{ a circle.}$$

Ex. 3. *Find the envelope of* $y + x\cot\frac{\theta}{2} = a\theta\cot\frac{\theta}{2}$, θ *being variable.*

---

† Or, we may rationalize, obtaining in m the quadratic $(y-mx)^2 = a^2(1+m^2)$. The condition for equal roots gives $x^2y^2 = (x^2-a^2)(y^2-a^2)$, etc.

Differentiating in θ,

$$-\frac{1}{2}x\operatorname{cosec}^2\frac{\theta}{2} = a\left(\cot\frac{\theta}{2} - \frac{1}{2}\theta\operatorname{cosec}^2\frac{\theta}{2}\right);$$

$$\therefore\ x = a\left(\theta - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right) = a(\theta - \sin\theta);$$

$$\therefore\ y = a\theta\cot\frac{\theta}{2} - x\cot\frac{\theta}{2} = a\theta\cot\frac{\theta}{2} - a(\theta - \sin\theta)\cot\frac{\theta}{2}$$

$$= 2a\cos^2\frac{\theta}{2} = a(1 + \cos\theta),$$

which is the cycloid.

304. Method of Undetermined Multipliers in a Special Case.

Consider the following example :—

Ex. 1. *Find the envelope of the line*

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{.... . . . .} \quad (1),$$

where a *and* b *are variable parameters connected by the equation*

$$a^2 + b^2 = c^2 \quad \text{. . . . .} \quad (2)$$

We may substitute $\sqrt{c^2 - a^2}$ for b in (1), and adopt the previous method; or, *noting that* $\frac{x}{a} + \frac{y}{b}$ *and* $a^2 + b^2$ *are homogeneous in* a *and* b, proceed thus :—

Let b stand for its equivalent $\sqrt{c^2 - a^2}$, so that b is a function of a; then differentiating in a, we have from (1)

$$-\frac{x}{a^2} - \frac{y}{b^2}\cdot\frac{db}{da} = 0.$$

And from (2)
$$2a + 2b\frac{db}{da} = 0.$$

Eliminating db/da by division, we have

$$-\frac{x}{a^2}\Big/2a = -\frac{y}{b^2}\Big/2b = \lambda, \text{ say.}$$

$$\therefore\ -\frac{x}{a^2} = 2a.\lambda;\quad -\frac{y}{b^2} = 2b.\lambda \quad \text{. . . . . .} \quad (3)$$

Taking advantage of Euler's Theorem of homogeneous functions, multiply the first equation by a, the second by b, and add:

$$\therefore\ -\left(\frac{x}{a}+\frac{y}{b}\right) = 2(a^2+b^2)\lambda,$$

$$\text{or}\quad -1 = 2c^2\lambda, \quad i.e.\ \lambda = -1/2c^2.$$

$$\therefore \text{ in (3) } x/a^2 = a/c^2; \quad y/b^2 = b/c^2;$$

$$\therefore\ a^3 = c^2x; \quad b^3 = c^2y.$$

Remembering that we have to eliminate a and b, we have in (1)

$$\frac{x}{c^{\frac{2}{3}}x^{\frac{1}{3}}} + \frac{y}{c^{\frac{2}{3}}y^{\frac{1}{3}}} = 1, \quad \text{or } x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}, \text{ the envelope.}$$

This curve, called the four-cusped hypocycloid, is the envelope of a straight line which moves so that a constant length c is intercepted between the axes of coordinates [see Ex. XXXIV., 4 (4)].

305. Generally (to show how Euler's Theorem is applied), let $F(x, y, a, b) = c_1$ ... (1) be the equation to a curve, in which a and b are variable parameters connected by the equation $f(a, b) = c_2$.

Also let F be homogeneous and of the mth degree, in a and b;
and let f „ „ „ „ nth „ „ .

Then reasoning as above, since b may be regarded as a function of a,

$$\frac{\partial F}{\partial a} + \frac{\partial F}{\partial b}\cdot\frac{db}{da} = 0; \quad \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\cdot\frac{db}{da} = 0;$$

$$\therefore\ \frac{\partial F}{\partial a}\Big/\frac{\partial f}{\partial a} = \frac{\partial F}{\partial b}\Big/\frac{\partial f}{\partial b} = \lambda \text{ say.}$$

$$\therefore\ \frac{\partial F}{\partial a} = \lambda.\frac{\partial f}{\partial a}; \quad \frac{\partial F}{\partial b} = \lambda.\frac{\partial f}{\partial b}. \quad \ldots\ldots \quad (a)$$

$$\therefore\ a\frac{\partial F}{\partial a} + b\frac{\partial F}{\partial b} = \lambda\left(a\frac{\partial f}{\partial a} + b\frac{\partial f}{\partial b}\right).$$

$$\therefore \text{ by Euler's Theorem, } mc_1 = \lambda.nc_2, \quad \therefore\ \lambda = \frac{m}{n}\frac{c_1}{c_2},$$

$$\therefore \text{ in } (a)\ \frac{\partial F}{\partial a} = \frac{m}{n}\frac{c_1}{c_2}.\frac{\partial f}{\partial a}; \quad \frac{\partial F}{\partial b} = \frac{m}{n}\frac{c_1}{c_2}.\frac{\partial f}{\partial b}.$$

From these and (1), a and b may be eliminated, the resulting equation being the envelope.

306. Evolutes.

Def.—The *evolute* of a given curve is the envelope of all the normals to the curve. It is also the locus of the centre of curvature, the latter being the intersection of two consecutive normals.

Ex. 1. *Find the evolute of the parabola* $y^2 = 4ax$.

The normal is $y = mx - 2am - am^3$ . . . . (1)

To find the envelope, differentiate in m,

$$\therefore 0 = x - 2a - 3am^2;$$

$$\therefore m^2 = (x - 2a)/3a \quad . . . . (2)$$

$\therefore$ from (1) and (2), $y^2 = m^2(x - 2a - am^2)^2$

$$= \frac{x - 2a}{3a}\left\{x - 2a - \frac{x - 2a}{3}\right\}^2 = \frac{x - 2a}{3a}\left\{\frac{2}{3}(x - 2a)\right\}^2;$$

or $ay^2 = \frac{4}{27}(x - 2a)^3$, which is the evolute.

Ex. 2. *Find the evolute of the equiangular spiral* $r = ae^{\theta \cot \alpha}$.

We know that $p = r \sin \alpha$ [Ex. XXXVI. 1, (1)].

$$\therefore \rho = r\frac{dr}{dp} = r \operatorname{cosec} \alpha.$$

Hence if EP be the normal, and $EP = \rho$; then since $\angle EPO = \frac{\pi}{2} - \alpha$, and $EP = OP \operatorname{cosec} \alpha = OP \sec EPO$, it follows that EO is $\perp$ to OP.

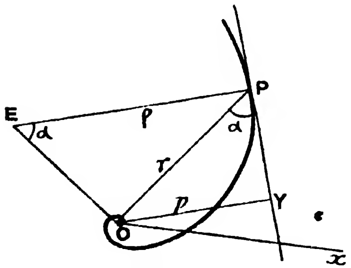

FIG. 92.

Now, E being the centre of curvature, the evolute is the locus of E.

Let (r', θ') be the coordinates of E.

Then

$r' = OE = r \cot \alpha$; or $r = r' \tan \alpha$;

$\theta' = EOx = \frac{\pi}{2} + \theta$; or $\theta = \theta' - \frac{\pi}{2}$.

But $r = ae^{\theta \cot \alpha}$;

$\therefore r' \tan \alpha = ae^{(\theta' - \frac{1}{2}\pi)\cot \alpha}$.

That is, the evolute is $r \tan \alpha = ae^{(\theta - \frac{1}{2}\pi)\cot \alpha}$, which is another equiangular spiral.

The latter can be shown to be a spiral equal to the given one, and turned through an angle, thus :—

The equation can be written $r = a \cot \alpha . e^{(\theta - \frac{1}{2}\pi)\cot\alpha}$

$$= ae^{\log\cot\alpha + (\theta - \frac{1}{2}\pi)\cot\alpha} = ae^{(\theta - \frac{1}{2}\pi + \log\cot\alpha/\cot\alpha)\cot\alpha}.$$

Hence, if the axis of x be turned through an angle $\frac{\pi}{2} - \frac{\log\cot\alpha}{\cot\alpha}$, the equation becomes $r = ae^{\theta\cot\alpha}$, as originally.

307. Caustic Curves.

Def.—Let A be a point of light in a plane, from which rays diverge, and suppose the latter to be reflected at, or refracted through, a surface. Then the envelope of the reflected or refracted rays is called the *caustic by reflection, or refraction,* accordingly.

We shall only take one example, namely the caustic by reflection of light issuing from a point A on the circumference of a circle APB, and reflected by the circle, regarded as a narrow cylinder.

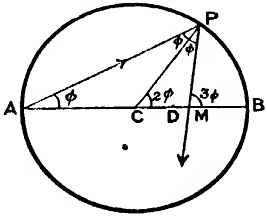

FIG. 93.

Let AP be a ray of light; then, if CP be joined, the reflected ray is PM, where CP bisects the angle APM. The caustic will be the envelope of PM.

If the $\angle PAB = \phi$, the coordinates of P referred to axes through C will be $x = a\cos 2\phi$, $y = a \sin 2\phi$.

The equation to PM is $y - a\sin 2\phi = \tan 3\phi(x - a\cos 2\phi)$;

or $$x \sin 3\phi - y\cos 3\phi = a \sin\phi. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating in ϕ, $3x\cos 3\phi + 3y\sin 3\phi = a\cos\phi$;

or $$x\cos 3\phi + y\sin 3\phi = \tfrac{1}{3}a\cos\phi. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solve for x and y; then

$$x = \frac{a}{3}\{3\sin\phi\sin 3\phi + \cos\phi\cos 3\phi\}$$

$$= \frac{a}{3}\{2\sin\phi\sin 3\phi + \cos 2\phi\} = \frac{a}{3}(2\cos 2\phi - \cos 4\phi)$$

$$= \frac{a}{3}(2\cos 2\phi - 2\cos^2 2\phi + 1) = \frac{2a}{3}\cos 2\phi(1 - \cos 2\phi) + \frac{a}{3}.$$

Similarly, $y = \frac{a}{3}(\cos\phi\sin 3\phi - 3\sin\phi\cos 3\phi) = \frac{a}{3}(2\sin 2\phi - \sin 4\phi)$

$$= \frac{2a}{3}\sin 2\phi(1 - \cos 2\phi).$$

Transfer the origin to D such that $CD = \frac{a}{3}$; we then have, using polar coordinates,

$$r\cos\theta = x' = x - \frac{a}{3} = \frac{2a}{3}\cos 2\phi(1 - \cos 2\phi);$$

$$r\sin\theta = y \qquad = \frac{2a}{3}\sin 2\phi(1 - \cos 2\phi).$$

$$\therefore \tan\theta = \tan 2\phi; \quad \therefore 2\phi = \theta.$$

Hence, by substitution, $r\sin\theta = \frac{2a}{3}\sin\theta(1 - \cos\theta)$;

or
$$r = \frac{2a}{3}(1 - \cos\theta),$$

which is a cardioide.

For a geometrical proof, see *Parkinson's Optics*; see, also, *Preston's Theory of Light.*

308. Note on the Cycloid.

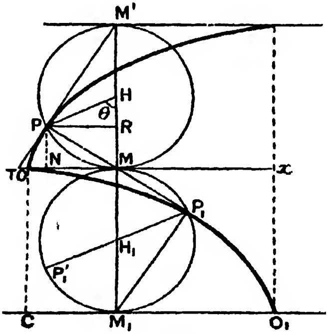

FIG. 94.

The cycloid is the locus of a point on the circumference of a circle which rolls on a straight line without slipping. To find the equation, let P be the point which traces out the cycloid, and take for origin its position (O) when in contact with the straight line Ox.

Let $\angle PHM = \theta$; radius of circle $= a$.

Then, since $OM = \text{arc } PM = a\theta$, we have

$$x = ON = OM - PR = a\theta - a\sin\theta$$
$$= a(\theta - \sin\theta);$$

$$y = PN = HM - HR = a - a\cos\theta = a(1 - \cos\theta).$$

Also
$$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = \frac{\sin\theta}{1 - \cos\theta} = \cot\frac{\theta}{2}.$$

Now draw the tangent PT, and join PM'; then

$$\tan PTM = \cot\frac{\theta}{2} = \cot PM'M;$$

whence TPM' is a straight line.

Hence PM' is the tangent, and PM the normal (the angle in a semicircle being a right angle).

309. The radius of curvature can be shown to be $4a\sin\frac{\theta}{2}$. [Cf. Art. 286, Ex. 2.]

But $$PM = MM'\sin\frac{\theta}{2} = 2a\sin\frac{\theta}{2}; \quad \therefore \rho = 2PM.$$

Produce PM to P_1 so that $MP_1 = MP$; then P_1 is the centre of curvature.

$$\therefore \text{ the locus of } P_1 \text{ is the evolute.}$$

Describe an equal circle touching Ox at M on the opposite side of the line, as in the figure; and produce $M'M$ to M_1. Draw the tangent CM_1O_1.

Now, the arc PM = arc MP_1 = arc M_1P_1';

$$\therefore \text{arc } M_1P_1' = \text{arc } MP = OM = CM_1,$$

if OC be parallel to MM_1. Hence, since C is a fixed point, being directly under O, and CM_1 = arc $P_1'M_1$, the circle is rolling on the line CM_1O_1, and P_1 is moving on a cycloid OP_1O_1, which is equal to the original one.

Hence, *the evolute of a cycloid is an equal cycloid.* [Cf. Ex. 21, below.]

Since PP_1 always touches the cycloid OP_1O_1, *and it is normal to the cycloid above*, it follows that if a thread be wrapped round the curve OP_1O_1, and a particle be attached to the point of the string at O, then on unwinding the string and keeping it stretched the particle will trace out the original cycloid, which is the *involute* of the other.

310. Application to Pendulums.

If we invert the figure so that O_1 is the highest point, we can regard O_1P_1P as the string of a pendulum, whose bob is at P. By this means we can make a pendulum swing with a cycloidal motion, and thus cause its oscillations to be isochronous for large or small vibrations.

See also Chapters XXVII. and XXVIII.

311. Epicycloid and Hypocycloid.

If a circle roll externally on a given fixed circle, and a definite point be taken on the former, then the locus of this carried point is called an *epicycloid*. If the circle roll internally, the locus is called a *hypocycloid*.

The equations of these curves are easily found in terms of an auxiliary variable.

(*a*) Let P be the position of the carried point, after the circle H has rolled from B to K; so that B is its initial position.

Hence arc KB = arc KP.

Let $\angle KAB = \phi$; $AK = a$; $KH = b$.

Then $\angle KHP = \dfrac{\text{arc } KP}{b} = \dfrac{\text{arc } KB}{b} = \dfrac{a\phi}{b}$.

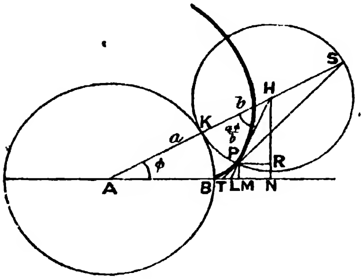

FIG. 95.

Also $\angle HPR = HLN = \phi + \dfrac{a\phi}{b} = \dfrac{a+b}{b}\phi$.

Hence, $x = AM = AN - PR = (a+b)\cos\phi - b\cos\dfrac{a+b}{b}\phi$;

$$y = PM = HN - HR = (a+b)\sin\phi - b\sin\frac{a+b}{b}\phi,$$

which are the required equations.

We may note that since K is momentarily at rest, and KPS is a right angle, therefore SP is the tangent at P; also

$$\psi = STN = \phi + PSK = \phi + \frac{a}{2b}\phi = \frac{a+2b}{2b}\phi.$$

If $a = b$, we have the *cardioide*, whose equations are

$$x = 2a\cos\phi - a\cos 2\phi; \quad y = 2a\sin\phi - a\sin 2\phi.$$

(*b*) By changing b into $-b$, or by a method similar to the above, we have, for the hypocycloid, the equations

$$x = (a-b)\cos\phi + b\cos\frac{a-b}{b}\phi;$$

$$y = (a-b)\sin\phi - b\sin\frac{a-b}{b}\phi.$$

If $b = \frac{1}{4}a$, the curve is the *four-cusped hypocycloid*, whose equations are

$$x = \frac{3}{4}a\cos\phi + \frac{a}{4}\cos 3\phi = a\cos^3\phi;$$

$$y = \frac{3}{4}a\sin\phi - \frac{a}{4}\sin 3\phi = a\sin^3\phi.$$

EXAMPLES XLVI.

1. Find the envelopes of the following:—

(1) $y = mx + a/m$; m variable.

(2) $x + y\sin\theta = a\cos\theta$; θ variable.

(3) $y = mx + c\sqrt{m}$; m variable.

(4) $m^3x = my + a$; m variable.

(5) $x\cos 2\theta + y\sin 2\theta = a$; θ variable.

(6) $x(1 - t^2) + 2yt = a(1 + t^2)$; t variable.

(7) $2ay = \left(m - \frac{1}{m}\right)bx + \left(m + \frac{1}{m}\right)ab$; m variable.

(8) $c^2(y - a)^2 + (cx - a^2)^2 = (a^2 + c^2)^2$; a variable.

(9) $x\cos\theta + y\sin\theta = a + b\sin\theta$; θ variable.

(10) $x^2\sec^2\theta + y^2\operatorname{cosec}^2\theta = a^2$.

2. Show that the ellipse whose equation is

$$2(y^2 - x^2)\cos\theta + 2(x^2 + y^2) = a^2\sin^2\theta,$$

where θ is variable, envelops the four straight lines $x \pm y \pm a = 0$.

3. Find the envelope of the curves $(a/x)^{\frac{1}{3}} + (b/y)^{\frac{1}{3}} = 1$, where the parameters a and b are connected by the equation $\sqrt{a} + \sqrt{b} = \sqrt{c}$.

4. Find the envelope of the parabolas $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where

$$a^{-\frac{1}{3}} + b^{-\frac{1}{3}} = c^{-\frac{1}{3}}.$$

5. Find the envelope of the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, in which $a + b = c$.

6. Find the envelope of $x\sin\theta + y\cos\theta = c\sin\theta\cos\theta$, θ being variable. Employ first the direct method; and then show that by putting $c\cos\theta = a$, $c\sin\theta = b$, the method of undetermined multipliers can be adopted.

7. Find the envelope of the family of curves $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$, where

$$a^m + b^m = c^m.$$

Show that if m and n are interchanged we get the same envelope.

8. Prove that if the corner of a rectangular piece of paper be folded down so that the sum of the edges left unfolded is constant, the crease will envelop a parabola.

9. Show that a system of concentric ellipses having their axes in the same direction envelop two equal rectangular hyperbolas.

10. Ellipses are described having their axes coincident in direction with those of a given ellipse, and the lengths of the axes are proportional to the coordinates of a variable point on the given ellipse. Prove that the ellipses all touch four straight lines.

11. The coordinates of a point on the parabola being given by

$$x = a\cot^2\psi;\ y = 2a\cot\psi,$$

show that the coordinates of the centre of curvature are given by

$$x = a(3\cot^2\psi + 2);\ y = -2a\cot^3\psi.$$

Hence show that the equation to the evolute is $27ay^2 = 4(x - 2a)^3$.

12. The equation to the normal to the ellipse at (x, y) being

$$\frac{a^2X}{x} - \frac{b^2Y}{y} = a^2 - b^2,$$

subject to the condition $b^2x^2 + a^2y^2 = a^2b^2$, show that the evolute to the ellipse is $(aX)^{\frac{2}{3}} + (bY)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

13. A system of circles is described passing through a fixed point (h, k) and having their centres on the circle $x^2 + y^2 = a^2$. Find their envelope.

14. Find the envelope of the circles passing through the centre of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, and having their centres on the circumference of the ellipse.

15. Show that the envelope of the lines given by

$$x\cos 3\theta + y\sin 3\theta = a\cos^{\frac{3}{2}} 2\theta,$$

is the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, or $r^2 = a^2\cos 2\theta$.

16. From a fixed point on the circumference of a circle, chords are drawn; and on these, as diameters, circles are described. Show that they envelop a cardioide.

17. Two perpendicular tangents are drawn to an ellipse. Show that

the envelope of the chord of contact is an ellipse whose eccentricity is $e\sqrt{2-e^2}$.

18. A parabola is described touching a given circle, and having a given point on the circumference as focus. Show that the envelope of the directrix is a cardioide.

19. A, B are two fixed points, and P is any point on the circle having AB as diameter; Q is the projection of P on AB, and between A and B on the line AB a point R is taken, such that $AR = BQ$. Find the envelope of the straight line PR.

20. Show that the polar equation between p and r of the envelope of the lines
$$x \cos 2\theta + y \sin 2\theta = 2a \cos \theta$$
is
$$p^2 = \tfrac{4}{3}(r^2 - a^2).$$

21. Show that the evolute of the cycloid
$$x = a(\theta + \sin\theta); \quad y = a(1 - \cos\theta)$$
is given by
$$x = a(\theta - \sin\theta); \quad y = a(3 + \cos\theta).$$
Show that this is an equal cycloid by transferring the origin to $(a\pi, 2a)$, and by putting $\theta - \pi = \phi$.

Answers.

1. **(1)** $y^2 = 4ax$. **(2)** $x^2 - y^2 = a^2$. **(3)** $4xy + c^2 = 0$. **(4)** $4y^3 = 27a^2x$.
(5) $x^2 + y^2 = a^2$. **(6)** $x^2 + y^2 = a^2$. **(7)** $b^2x^2 + a^2y^2 = a^2b^2$.
(8) $cy^2 + (c + 2x)(x^2 + y^2 - c^2) = 0$; *i.e.* $(x + c)\{2(x^2 + y^2) - cx - c^2\} = 0$.
(9) $x^2 + (y - b)^2 = a^2$. **(10)** $x \pm y \pm a = 0$.

3. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{c}$. **4.** $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{c}$. **5.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$. **6.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.

7. $x^{\frac{mn}{m+n}} + y^{\frac{mn}{m+n}} = c^{\frac{mn}{m+n}}$.

13. Let $(a \cos\theta, a \sin\theta)$ be the coordinates of the centre of one of the circles. The envelope is
$$4a^2\{(x - h)^2 + (y - k)^2\} = (x^2 + y^2 - h^2 - k^2)^2.$$

14. $(x^2 + y^2)^2 = 4(a^2x^2 + b^2y^2)$. **19.** $(2y)^{\frac{2}{3}} - x^{\frac{2}{3}} = a^{\frac{2}{3}}$.

INTEGRAL CALCULUS

CHAPTER XX.

IMMEDIATE INTEGRATION.

312. Indefinite Integrals.—Integration, though it is, strictly speaking, a summation, may be introduced to the beginner as the opposite process to that of differentiation; since, as we shall briefly show in the next article, the summation referred to depends on this opposite process. We therefore give the following definition :—

Def.—If $f'(x)dx$ is the *differential* of $f(x)$, then

$f(x)$ is the *integral* of $f'(x)dx$, and is written $\int f'(x)dx$.

Thus, $$f(x) = \int f'(x)dx.$$

Such an integral is called an *indefinite integral.*

Ex. 1. The differential of x^n is $nx^{n-1}dx$; $\therefore$ the integral of $nx^{n-1}dx$ is x^n. Or, since $d.x^n = nx^{n-1}dx$; $\therefore$ $\int nx^{n-1}dx = x^n$.

Ex. 2. $d.\log x = dx/x$; $\therefore$ $\int dx/x = \log x$.

Ex. 3. $d.x = dx$; $\therefore$ $\int dx = x$; which reads thus: "The differential of x is dx; $\therefore$ the integral of dx is x."

Similarly, if $y = f(x)$, then $\int dy = \int d.f(x) = \int f'(x)dx = y$.

For instance, $\int d.\sqrt{1-x^2} = \sqrt{1-x^2}$.

313. Definite Integrals — Reason for the Term Integral.

Let the curve in the figure denote the graph of the function $f(x)$.

Let CA, BD be two values of $f(x)$ corresponding to two given values of x, namely $OA = a$, and $OB = b$.

Then $\qquad CA = f(a)$, $DB = f(b)$.

Now, CA may be supposed to increase to BD by an infinite number of infinitely small increments, such as Pp, Qq, Rr; while x increases each time by dx. Each of these increments is the differential of $f(x)$, viz. $f'(x)dx$, a variable quantity whose value depends on the value of x. Now, by giving to x in turn all the values between a and b, and adding together the resulting values of $f'(x)dx$, we obtain the total increment, namely $DB - CA$, or $f(b) - f(a)$.

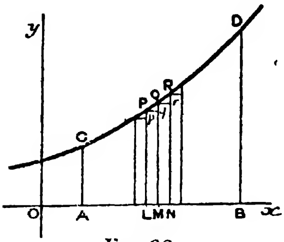

Fig. 96.

This may be written $\sum\limits_{x=a}^{x=b} f'(x)dx = f(b) - f(a)$.

Hence, to find the value of the left-hand side, we may first find what we have called the indefinite integral of $f'(x)dx$, namely $f(x)$, and then take the difference of its two extreme values, $f(b)$ and $f(a)$. And the usual notation is thus :—

$$\int_a^b f'(x)dx = f(b) - f(a), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a and b being termed the *limits* of integration; b being the *superior*, and a the *inferior*, limit.

Ex. 1. $\displaystyle\int_a^b nx^{n-1}dx = [x^n]_a^b$† $= b^n - a^n$. [See Ex. 1, above.]

Ex. 2. $\displaystyle\int_1^e \frac{dx}{x} = \Big[\log x\Big]_1^e = 1 - 0 = 1$. [See Ex. 2, above.]

A strict proof of the above statement (1) will be given in Chapter XXVI.

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† Generally, $[\phi(x)]_a^b$ denotes $\phi(b) - \phi(a)$.

314. Corrected Integrals.

Suppose b to be variable, though constant during the operation of integration; and replace it by z. Then we have

$$\int_a^z f'(x)dx = f(z) - f(a).$$

In practice, however, it is more convenient to write x itself instead of z; thus :—

$$\int_a^x f'(x)dx = f(x) - f(a).$$

The expression $\int_a^x f'(x)dx$ is termed a *corrected integral*; and is often written $\int_a f'(x)dx$.

315. We now see a reason for the terms "integral" and "integration," which are indeed synonymous with "sum" and "summation." The *sign of integration*, $\int$, is derived from the letter *s*, the initial of the word "sum." The expression $\int_b^a f'(x)dx$ really means nothing more nor less than the sum of an infinite number of infinitely small terms; but the actual method by which we obtain this sum, and which apparently has no connection with summation, is the inverse process to that of differentiation, exactly as subtraction, division, and finding the square root are inverse to addition, multiplication, and squaring.

It is evident that for the present we must put aside the idea of summation, and confine our attention to the finding of indefinite integrals, using the term "integration" in the sense of *passing from a given function to that of which it is the differential coefficient.*

Like most inverse processes, it is considerably more difficult than the direct one of differentiation. We must, therefore, consider in detail what expressions admit of being readily integrated, and the best methods of integrating them. Several chapters will be devoted to this, after which will follow the application of the Integral Calculus to the finding of areas, volumes, etc.

In the next chapter, however, we shall give a few simple applications to illustrate the subject.

316. Adding a Constant.

If $y = f(x) + C$, a constant, then we still have

$$dy = f'(x)dx; \text{ whence } \int f'(x)dx = f(x) + C.$$

Hence, in an indefinite integral, an *arbitrary constant* must always be supposed to be added.

Also, since $\int_a^b f'(x)dx = [f(x) + C]_a^b = \{f(b) + C\} - \{f(a) + C\}$
$$= f(b) - f(a),$$
the arbitrary constant disappears in the definite integral, as we should expect.

317. Integral of a Sum or Difference.

Suppose u, v, w, ... are functions of x, then the differential of
$$u \pm v \pm w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
is
$$du \pm dv \pm dw \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
But if every term of (2) is the differential of every term in (1) respectively, then every term in (1) is the integral of every term in (2).

Hence, *to integrate* du $\pm$ dv $\pm$ dw, *we need only integrate each of the terms and retain the signs between them.*

This may be expressed as follows :—

If P, Q, R ... be functions of x, then $\int(P \pm Q \pm R \ldots)dx$
$$= \int Pdx \pm \int Qdx \pm \int Rdx \ldots$$

Ex. If $y = nx^{n-1} - \dfrac{1}{x}$.
$$\int ydx = \int\left(nx^{n-1} - \frac{1}{x}\right)dx = \int nx^{n-1}dx - \int\frac{1}{x}dx = x^n - \log x.$$

318. Constant Factors or Coefficients.

Since $d\{c.f(x)\} = c.f'(x)dx$, c being a constant,
$$\therefore \int c.f'(x)dx = c.f(x) = c\int f'(x)dx.$$
Hence, *a constant factor may be placed outside the sign of integration.*

Ex. $\displaystyle\int\frac{dx}{2x} = \frac{1}{2}\int\frac{dx}{x} = \frac{1}{2}\log x.$

319. Fundamental Integrals. — The following table is almost immediately deducible from those in Arts. 58 and 80 (*q.v.*)

Thus, to find $\int x^n dx$, since $d.x^{n+1} = (n+1)x^n dx$,

$$\therefore \int (n+1)x^n dx = x^{n+1};$$

$$\text{i.e. } (n+1)\int x^n dx = x^{n+1} \text{ [by Art. 318]},$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1}.$$

In fact, if we differentiate $\dfrac{x^{n+1}}{n+1}$ we obtain x^n.

Similarly for $\int a^x dx$, and others; all the results of which may be verified by differentiation.

TABLE OF FUNDAMENTAL INTEGRALS.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

$$\int a^x\,dx = \frac{a^x}{\log_e a}.$$

$$\int e^x\,dx = e^x.$$

$$\int \frac{dx}{x} = \log x.$$

$$\int \cos x\,dx = \sin x.$$

$$\int \sin x\,dx = -\cos x.$$

$$\int \sec^2 x\,dx = \tan x.$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x.$$

$$\int \sec x \tan x\,dx = \int \frac{\sin x\,dx}{\cos^2 x} = \sec x$$

$$\int \operatorname{cosec} x \cot x\,dx = \int \frac{\cos x\,dx}{\sin^2 x} = -\operatorname{cosec} x.$$

$$\int \cosh x\,dx = \sinh x.$$

$$\int \sinh x\,dx = \cosh x.$$

$$*\int \operatorname{sech}^2 x\,dx = \tanh x.$$

$$*\int \operatorname{cosech}^2 x\,dx = -\coth x.$$

$$*\int \operatorname{sech} x \tanh x\,dx = \int \frac{\sinh x\,dx}{\cosh^2 x} = -\operatorname{sech} x.$$

$$*\int \operatorname{cosech} x \coth x\,dx = \int \frac{\cosh x\,dx}{\sinh^2 x} = -\operatorname{cosech} x.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} = -\frac{1}{a}\cot^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}.$$

$$\int \sec x\,dx = gd^{-1}x = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

$$*\int \operatorname{sech} x\,dx = gd\,x.$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} = \log(x + \sqrt{x^2 + a^2}) + \text{const.}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} = \log(x + \sqrt{x^2 - a^2}) + \text{const.}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} = \frac{1}{2a}\log\frac{a+x}{a-x} \quad [x < a].$$

$$*\int \frac{dx}{x^2 - a^2} = -\frac{1}{a}\coth^{-1}\frac{x}{a} = -\frac{1}{2a}\log\frac{x+a}{x-a} \quad [x > a].$$

$$*\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a} = -\frac{1}{a}\log\frac{a+\sqrt{a^2-x^2}}{x}.$$

$$*\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a}\operatorname{cosech}^{-1}\frac{x}{a} = -\frac{1}{a}\log\frac{a+\sqrt{a^2+x^2}}{x}.$$

Although all of the above integrals are of importance, yet those marked with an asterisk need not be learnt by the beginner at the first reading.

We shall now give several examples on the above forms, taking them in order.

320. Form (A)—$\int x^n dx = \dfrac{x^{n+1}}{n+1}$.

Ex. 1. *Find* $\int x^{\frac{1}{2}}dx$.

By the formula we obtain at once $x^{\frac{3}{2}} \div \frac{3}{2}$, *i.e.* $\frac{2}{3}x^{\frac{3}{2}}$.

The beginner may, however, notice the following rule :—

We know that, *as far as the* x *is concerned*, $x^{\frac{1}{2}}$ is the d.c. of $x^{\frac{3}{2}}$.

Write down $x^{\frac{3}{2}}$, leaving a space (or putting C) for the constant coefficient. Thus $\int x^{\frac{1}{2}}dx = Cx^{\frac{3}{2}}$.

Now C must be such as to neutralize the coefficient we should obtain by differentiating $x^{\frac{3}{2}}$, which is $\frac{3}{2}$ in this case. Since $\frac{2}{3}$ neutralizes $\frac{3}{2}$, the rule is :—*Invert whatever constant we should get in differentiating back the integral obtained.*

Ex. 2. $\int x^{-n}dx = Cx^{-n+1}$.

Since, by differentiating x^{-n+1} we get

$$(-n+1)x^{-n},\ C \text{ evidently} = \frac{1}{-n+1};$$

$$i.e.\ \int x^{-n}dx = \frac{x^{-n+1}}{-n+1};\ \text{or}\ \int\frac{dx}{x^n} = -\frac{1}{(n-1)x^{n-1}}.$$

Ex. 3. $\int \sin^n x \cos x\, dx$ ($= I$, say).

Here $\cos x dx = d.\sin x = dz$, if we put z for $\sin x$.

$$\therefore\ I = \int z^n\, dz = \frac{z^{n+1}}{n+1} = \frac{\sin^{n+1}x}{n+1}.$$

Ex. 4. $\int x(1+x^2)^n dx \; (=I$, say).

Here the differential of $1+x^2$ is $2x\,dx$, i.e. *some multiple of* x dx. Hence, putting $1+x^2=z$, $\therefore\; x\,dx=\frac{1}{2}dz$.

$$\therefore\; I=\int z^n.\frac{1}{2}dz=\frac{1}{2}\int z^n dz=\frac{1}{2}.\frac{z^{n+1}}{n+1}=\frac{(1+x^2)^{n+1}}{2(n+1)}.$$

Ex. 5. $\int \tan x\sec^2 x\sqrt{1-\tan^2 x}\,dx$.

The most involved part of the integral is $\sqrt{1-\tan^2 x}$, *and we therefore try the substitution* $1-\tan^2 x=z$; $\therefore\; -2\tan x\sec^2 x\,dx=dz$.

$$\therefore\; I=\int -\tfrac{1}{2}\sqrt{z}.dz=-\tfrac{1}{2}\int z^{\frac{1}{2}}dz=-\tfrac{1}{2}Cz^{\frac{1}{2}}=-\tfrac{1}{2}.\tfrac{2}{3}z^{\frac{3}{2}}$$

$$=-\tfrac{1}{3}z^{\frac{3}{2}}=-\tfrac{1}{3}(1-\tan^2 x)^{\frac{3}{2}}.$$

Ex. 6. $\displaystyle\int_0^1 \frac{(x+2)dx}{\sqrt{(x+1)(x+3)}}$.

Let the indefinite integral $\displaystyle =I=\int\frac{(x+2)dx}{\sqrt{x^2+4x+3}}$.

Put $\qquad x^2+4x+3=z$; $\therefore\; 2(x+2)dx=dz$.

$$\therefore\; I=\frac{1}{2}\int\frac{dz}{\sqrt{z}}=\sqrt{z}=\sqrt{x^2+4x+3};$$

$$\therefore\; [I]_0^1=[\sqrt{x^2+4x+3}]_0^1=2\sqrt{2}-\sqrt{3}.$$

NOTE.—When $n=-1$, the formula $\int x^n dx=\dfrac{x^{n+1}}{n+1}+C$ becomes

$$\int\frac{dx}{x}=\frac{x^0}{0}+C=\infty+C.$$

Hence, as n approaches -1, the variable term becomes infinitely large. If, however, we consider the definite integral

$$\int_a^b x^n\,dx=\frac{b^{n+1}-a^{n+1}}{n+1},$$

then putting $n=-1$, we have

$$\int_a^b\frac{dx}{x}=\mathop{lt}_{n+1=0}\frac{b^{n+1}-a^{n+1}}{n+1}=\mathop{lt}_{n+1=0}\frac{\{1+(n+1)\log b\ldots\}-\{1+(n+1)\log a\ldots\}}{n+1}$$

if we expand by the exponential theorem,

$$=\log b-\log a,$$

and this agrees with the formula $\displaystyle\int\frac{dx}{x}=\log x$.

We may, therefore, regard the first formula as universally true, even when $n + 1 = 0$.

321. Forms (B)—$\int a^x dx = \dfrac{a^x}{\log a}$; $\int e^x dx = e^x$; $\displaystyle\int \frac{dx}{x} = \log x$.

Ex. 7. $\displaystyle\int \frac{e^{\sqrt{x}}dx}{\sqrt{x}}$.

Put $\sqrt{x} = z$, $\therefore \dfrac{1}{2\sqrt{x}} \cdot dx = dz$.

$$\therefore I = 2\int e^z dz = 2e^z = 2e^{\sqrt{x}}.$$

Ex. 8. $\displaystyle\int \frac{(2x + 3)dx}{x^2 + 3x - 1} = \int \frac{dz}{z}$, if $z = x^2 + 3x - 1$,

$$= \log z = \log (x^2 + 3x - 1).$$

Or, at once without any substitution,

$$I = \int \frac{d(x^2 + 3x - 1)}{x^2 + 3x - 1} = \log (x^2 + 3x - 1).$$

Ex. 9. $\displaystyle\int \frac{f'(x)dx}{f(x)} = \log f(x)$.

Ex. 10. $\displaystyle\int_1^\infty \frac{dx}{(1+x^2)\tan^{-1} x}$.

Selecting $\tan^{-1} x$ as the most involved part of the integral, put

$$\tan^{-1} x = z; \quad \therefore \frac{1}{1 + x^2}dx = dz.$$

$$\therefore I = \int \frac{dz}{z} = \log z = \log \tan^{-1} x;$$

$$\therefore [I]_1^\infty = \log \tan^{-1} \infty - \log \tan^{-1} 1 = \log \frac{\pi}{2} - \log \frac{\pi}{4} = \log \left\{\frac{\pi}{2}\Big/\frac{\pi}{4}\right\} = \log_e 2.$$

322. Forms (C)—$\int \cos x\, dx = \sin x$, etc.; and $\int \cosh x\, dx = \sinh x$, etc.

Ex. 11. $\displaystyle\int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}\, dx}{\sqrt{x}}$.

Put $e^{\sqrt{x}} = z$, then $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}dx = dz$;

$$\therefore\ I = 2\int \cos z\,dz = 2\sin z = 2\sin e^{\sqrt{x}}.$$

Ex. 12. $\displaystyle\int\frac{dx}{x\cos^2(\log x)} = \int\frac{d(\log x)}{\cos^2(\log x)} = \int\frac{dz}{\cos^2 z} = \tan z = \tan(\log x).$

Ex. 13. $\int(\cos x - x\sin x)\cosh(x\cos x)dx.$

Put $x\cos x = z$; $\therefore\ (\cos x - x\sin x)dx = dz.$

$$\therefore\ I = \int\cosh z\,dz = \sinh z = \sinh(x\cos x).$$

323. Forms (D)—$\displaystyle\int\frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$, etc.; and

$$\int\frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} = \log(x + \sqrt{x^2 + a^2}) + C, \text{ etc.}$$

Ex. 14. $\displaystyle\int\frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b}\int\frac{d(bx)}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b}\sin^{-1}\frac{bx}{a}.$

Ex. 15. $\displaystyle\int\frac{\cos x\,dx}{a^2 + b^2\sin^2 x}.$

Observing that $\cos x\,dx = d(\sin x)$, this suggests putting $\sin x = z$;

$$\therefore\ I = \int\frac{dz}{a^2 + b^2z^2} = \frac{1}{b}\int\frac{d(bz)}{a^2 + b^2z^2} = \frac{1}{b}\int\frac{dv}{a^2 + v^2}\ [\text{if } v = bz = b\sin x]$$

$$= \frac{1}{b}\cdot\frac{1}{a}\tan^{-1}\frac{v}{a} = \frac{1}{ab}\tan^{-1}\frac{b\sin x}{a}.$$

Ex. 16. $\displaystyle\int\frac{e^x dx}{\sqrt{1 + e^{2x}}}.$

Put $e^x = z$; $\therefore\ e^x dx = dz.$

$$\therefore\ I = \int\frac{dz}{\sqrt{1 + z^2}} = \sinh^{-1} z = \sinh^{-1} e^x, \quad \text{or } \log(e^x + \sqrt{1 + e^{2x}}) + C.$$

Ex. 17. $\displaystyle\int\frac{\sec^2 x\,dx}{a^2 - b^2\tan^2 x}.$

Put $b\tan x = z$. $\therefore\ b\sec^2 x\,dx = dz.$

$$\therefore\ I = \frac{1}{b}\int\frac{dz}{a^2 - z^2} = \frac{1}{b}\cdot\frac{1}{a}\tanh^{-1}\frac{z}{a} = \frac{1}{ab}\tanh^{-1}\frac{b\tan x}{a},$$

$$\text{or } \frac{1}{2ab}\log\frac{a + b\tan x}{a - b\tan x}.$$

EXAMPLES XLVII.

1. Find what must be differentiated to give the following expressions, testing your answers by differentiating them back again [Form (A)]:—

(1) $x,\ x^2,\ x^3,\ x^{10},\ x^a$. (2) $ax,\ 3x^4$.

(3) $\sqrt{x},\ 2x\sqrt{x},\ 1/\sqrt{x}$. (4) $1/x^4,\ 1/x^{n+1},\ -2x^{-\frac{3}{4}}$.

(5) $(x-a)^3,\ (3-x)^{-\frac{1}{2}},\ \frac{2}{3}(2-3x)^{-2}$.

(6) $2x(1+x^2)^2,\ x(x^2-1)^{\frac{1}{3}},\ 3x\sqrt{x^2+1},\ -x/\sqrt{1-x^2}$.

(7) $(x+1)(x^2+2x+3)^2,\ \dfrac{2x-1}{\sqrt{x^2-x-1}}$.

2. Find the value of the following indefinite integrals [Form (A)]:—

(1) $\int \cos x \sin^3 x\, dx$. (2) $\int \cos^3 x \sin x\, dx$.

(3) $\int \cos x \sqrt{\sin x}\, dx$. (4) $\displaystyle\int \frac{\log x}{x} dx$.

(5) $\displaystyle\int \frac{dx}{x(\log x)^2}$. (6) $\displaystyle\int \frac{\sin^{-1} x\, dx}{\sqrt{1-x^2}}$.

(7) $\displaystyle\int \sqrt{\frac{\sin^{-1} x}{1-x^2}}\, dx$. (8) $\displaystyle\int \frac{dx}{\cos^2 x(1-\tan x)^2}$.

3. Find the value of the following [Forms (B)]:—

(1) $\int e^{2x+3}\, dx$. (2) $\int x e^{x^2}\, dx$.

(3) $\int \sin x\, e^{\cos x}\, dx$. (4) $\displaystyle\int \frac{dx}{2x+3}$

(5) $\displaystyle\int \frac{(2x+3)dx}{x^2+3x-1}$. (6) $\int \cot x\, dx$.

(7) $\int \sec^2 x\, e^{\tan x}\, dx$. (8) $\displaystyle\int \frac{e^{\sin^{-1} x}\, dx}{\sqrt{1-x^2}}$.

(9) $\displaystyle\int \frac{dx}{(1+x^2)\tan^{-1} x}$. (10) $\displaystyle\int \frac{a^{\sqrt{x}}\, dx}{\sqrt{x}}$.

(11) $\displaystyle\int \frac{x^3\, dx}{1+x^4}$. (12) $\displaystyle\int \frac{\sin x + x\cos x}{x \sin x} dx$.

(13) $\displaystyle\int \frac{\cos x - \sin x}{\cos x + \sin x} dx$. (14) $\displaystyle\int \frac{\sin 2x\, dx}{1+\sin^2 x}$.

(15) $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$.

(16) $\displaystyle\int \frac{dx}{2x+1+\sqrt{2x+1}}$.

(17) $\displaystyle\int \frac{dx}{x+x\log x}$.

(18) $\displaystyle\int \frac{\cos x \,.\, a^{\sqrt{1-\sin x}}}{\sqrt{1-\sin x}}\,dx$.

4. Find the value of the following [Forms (C)]:—

(1) $\int x \cos x^2\,dx$.

(2) $\displaystyle\int \frac{\sin\sqrt{x}\,dx}{\sqrt{x}}$.

(3) $\displaystyle\int \frac{\cos\log x\,dx}{x}$.

(4) $\displaystyle\int \frac{dx}{x^2\cos^2(1/x)}$.

(5) $\int \sin x \cos(\cos x)dx$.

(6) $\displaystyle\int \frac{x\sinh\sqrt{1-x^2}\,dx}{\sqrt{1-x^2}}$.

(7) $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}\cos^2\sqrt{1-x^2}}$.

(8) $\displaystyle\int \frac{\log x \sin\{1+(\log x)^2\}dx}{x}$.

(9) $\displaystyle\int \frac{e^x(1+x)dx}{\sin^2(xe^x)}$.

5. Find the value of the following [Forms (D)]:—

(1) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$.

(2) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$.

(3) $\displaystyle\int \frac{dx}{a^2+b^2x^2}$.

(4) $\displaystyle\int \frac{dx}{\sqrt{a^2x^2-1}}$.

(5) $\displaystyle\int \frac{x\,dx}{\sqrt{a^4+x^4}}$.

(6) $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-x^6}}$.

(7) $\displaystyle\int \frac{e^x\,dx}{1+e^{2x}}$.

(8) $\displaystyle\int \frac{\sec^2 x\,dx}{b^2+a^2\tan^2 x}$.

(9) $\displaystyle\int \frac{\sin x\,dx}{a-b\cos^2 x}$.

6. Find the value of the following definite integrals:—

(1) $\displaystyle\int_{-3}^{1} \frac{dx}{\sqrt{3-2x}}$.

(2) $\displaystyle\int_{0}^{2\sqrt{2}} \frac{x\,dx}{\sqrt{1+x^2}}$.

(3) $\displaystyle\int_{0}^{\frac{\pi}{4}} \cos x \sin x\,dx$.

(4) $\displaystyle\int_{0}^{\frac{\pi}{4}} \sec^2 x \tan x\,dx$.

(5) $\int_1^2 \frac{e^x\,dx}{e^x - 1}$. (6) $\int_0^{\frac{\pi}{6}} \cos x(1 - \sin x)^{-1}\,dx$.

(7) $\int_0^{\infty} \frac{1 + \tan^{-1} x}{1 + x^2}\,dx$.

ANSWERS.

1. (1) $\frac12 x^2$, $\frac13 x^3$, $\frac14 x^4$, $\frac{1}{11}x^{11}$, $\frac{x^{a+1}}{a+1}$. (2) $\frac{ax^2}{2}$, $\frac{3x^5}{5}$. (3) $\frac23 x^{\frac32}$, $\frac45 x^{\frac54}$, $2x^{\frac12}$.

   (4) $-\frac{1}{3x^3}$, $-\frac{1}{rx^r}$, $-6x^{\frac16}$. (5) $\frac14(x-a)^4$, $-2(3-x)^{\frac12}$, $\frac13(2-3x)^{-1}$.

   (6) $\frac13(1+x^2)^3$, $\frac38(x^2-1)^{\frac43}$, $(x^2+1)^{\frac32}$, $\sqrt{1-x^2}$.

   (7) $\frac16(x^2 + 2x + 3)^3$, $2\sqrt{x^2 - x - 1}$.

2. (1) $\frac14 \sin^4 x$. (2) $-\frac14 \cos^4 x$. (3) $\frac23(\sin x)^{\frac32}$. (4) $\frac12(\log x)^2$.

   (5) $-\frac{1}{\log x}$. (6) $\frac12(\sin^{-1} x)^2$. (7) $\frac23(\sin^{-1} x)^{\frac32}$. (8) $\frac{1}{1 - \tan x}$.

3. (1) $\frac12 e^{2x+3}$. (2) $\frac12 e^{x^2}$. (3) $-e^{\cos x}$. (4) $\frac12 \log(2x + 3)$.

   (5) $\log(x^2+3x-1)$. (6) $\log \sin x$. (7) $e^{\tan x}$. (8) $e^{\sin^{-1} x}$.

   (9) $\log \tan^{-1} x$. (10) $2a^{\sqrt{x}}/\log a$. (11) $\frac14 \log(1+x^4)$.

   (12) $\log(x \sin x) = \log x + \log \sin x$. (13) $\log(\cos x + \sin x)$.

   (14) $\log(1 + \sin^2 x)$. (15) $2\log(1 + \sqrt{x})$. (16) $\log(1 + \sqrt{2x + 1}$

   (17) $\log(1 + \log x)$. (18) $-2a^{\sqrt{1-\sin x}}/\log a$.

4. (1) $\frac12 \sin x^2$. (2) $-2\cos\sqrt{x}$. (3) $\sin \log x$. (4) $-\tan\frac{1}{x}$.

   (5) $-\sin(\cos x)$. (6) $-\cosh\sqrt{1 - x^2}$. (7) $-\tan\sqrt{1 - x^2}$.

   (8) $-\frac12 \cos\{1 + (\log x)^2\}$. (9) $-\cot(xe^x)$.

5. (1) $\sin^{-1}\frac{x}{2}$. (2) $\frac12 \sin^{-1}\frac{2x}{3}$. (3) $\frac{1}{ab}\tan^{-1}\frac{bx}{a}$.

   (4) $\frac{1}{a}\cosh^{-1} ax = \frac{1}{a}\log(ax + \sqrt{a^2x^2 - 1}) + C$.

   (5) $\frac12 \sinh^{-1}\frac{x^2}{a^2} = \frac12 \log(x^2 + \sqrt{x^4 + a^4}) + C$. (6) $\frac13 \sin^{-1} x^3$.

(7) $\tan^{-1} e^x$. (8) $\dfrac{1}{ab}\tan^{-1}\dfrac{a\tan x}{b}$.

(9) $-\dfrac{1}{\sqrt{ab}}\tanh^{-1}\left(\sqrt{\dfrac{b}{a}}\cos x\right) = -\dfrac{1}{2\sqrt{ab}}\log\dfrac{\sqrt{a}+\sqrt{b}\cos x}{\sqrt{a}-\sqrt{b}\cos x}$.

6. (1) 2. (2) 2. (3) $\frac{1}{4}$. (4) $\frac{1}{2}$. (5) $\log(e+1)$.

(6) $\log 2$. (7) $\frac{\pi}{8}(\pi+4)$.

CHAPTER XXI.

ELEMENTARY APPLICATIONS OF THE INTEGRAL CALCULUS.

324. Velocities and Accelerations.—We have shown [Art. 117] that ds/dt and d^2s/dt^2 (or dv/dt) represent the velocity and acceleration respectively of a moving particle at any instant. Suppose, now, that a particle moves under a given constant acceleration a; we shall show how to obtain the ordinary formulæ for the position and velocity at any time.

Let v = velocity at the time t.†

Then $dv/dt = a$, or $dv = adt$;

$$\therefore v = \int adt = at + C, \ a \text{ being constant.} \quad . \quad . \quad . \quad (1)$$

The value of C depends on the initial conditions.

Suppose that, when $t = 0$, the velocity is u, then in (1)

$$u = 0 + C, \text{ or } C = u.$$

$$\therefore v = u + at.$$

This can also be obtained by the use of corrected integrals, remembering that when $t = 0$, $v = u$; thus :—

$$\int_u^v dv = \int_0^t adt, \text{ or } v - u = at.$$

Here v and t are the variables, u and a being constants.

325. Again, since $v = ds/dt$, $\therefore ds/dt = u + at$;

or,
$$ds = (u + at)dt,$$

$$\therefore s = \int(u + at)dt = ut + \tfrac{1}{2}at^2 + C.$$

† To be precise, by "*the time* t" is meant a certain instant, namely, t seconds after some given instant from which the time is reckoned, and which is called "time 0."

If $s = 0$ when $t = 0$, then $C = 0$,

$$\therefore\ s = ut + \tfrac{1}{2}at^2\,;$$

or $$\int_0^s ds = \int_0^t (u + at)dt\,;\ \text{etc., as before.}$$

326. Since $a = \dfrac{dv}{dt} = \dfrac{dv}{ds}\cdot\dfrac{ds}{dt} = v\dfrac{dv}{ds}$,

$$\therefore\ vdv = ads\,;$$

$$\therefore\ \int_u^v vdv = a\int_0^s ds,\ \text{if } v = u \text{ when } s = 0.$$

$$\therefore\ \tfrac{1}{2}(v^2 - u^2) = as,\ \text{or } v^2 = u^2 + 2as.$$

327. *A particle moves along the axis of* x *so that its acceleration towards the origin is proportional to its distance from the same. To find its position at any time.*

The acceleration is vdv/ds; $\therefore\ vdv/ds = -ks$, since the acceleration, *as measured in the* $+^{ve}$ *direction*, is $-^{ve}$, k being a $+^{ve}$ constant.

$$\therefore\ vdv = -ksds.$$

Integrating both sides, $\tfrac{1}{2}v^2 = -\tfrac{1}{2}ks^2 + C$.

The value of C depends on an additional datum; suppose, then, that u is the velocity at O, or $v = u$ when $s = 0$,

$$\therefore\ \tfrac{1}{2}u^2 = 0 + C,\ \text{or } C = \tfrac{1}{2}u^2\,;$$

$$\therefore\ \tfrac{1}{2}v^2 = -\tfrac{1}{2}ks^2 + \tfrac{1}{2}u^2\,;$$

$$\therefore\ v^2 = u^2 - ks^2. \qquad (1)$$

Since $v = ds/dt$, $\therefore\ ds = \sqrt{u^2 - ks^2}\,dt$;

or, $$dt = \frac{ds}{\sqrt{u^2 - ks^2}},$$

$$\therefore\ t = \int \frac{ds}{\sqrt{u^2 - ks^2}} = \frac{1}{\sqrt{k}}\sin^{-1}\frac{\sqrt{k}}{u}s + C.$$

To find C, suppose the particle to be projected from the origin at time 0, *i.e.* let $s = 0$ when $t = 0$; then

$$0 = 0 + C,$$

and
$$\therefore t = \frac{1}{\sqrt{k}} \sin^{-1} \frac{\sqrt{k}}{u} s,$$

$$\therefore \frac{\sqrt{k}}{u} s = \sin(\sqrt{k}.t), \quad \therefore s = \frac{u}{\sqrt{k}} \sin(\sqrt{k}.t).$$

The particle therefore moves with simple harmonic motion.

If a be the amplitude, *i.e.* if $v = 0$ when $s = a$, then from (1)

$$0 = u^2 - ka^2, \quad \text{or } u = a\sqrt{k},$$

$$\therefore s = a \sin(\sqrt{k}.t).$$

328. Areas of Plane Curves.

Let PM, QN be two very near ordinates of the curve $y = f(x)$.

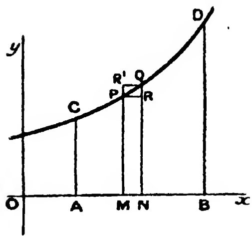

FIG. 97.

Draw PR, QR' parallel to Ox. Then if (x, y), $(x + \delta x, y + \delta y)$ be the coordinates of P and Q respectively, we have

Area of $PN = PM.MN = y\delta x$,
Area of $R'N = QN.MN = (y + \delta y)\delta x$.

The difference $= \delta y.\delta x$, which is of the second order of small quantities, and may be neglected in the summation, as we shall show.

The curvilinear area $PMNQ$, bounded by the arc PQ, is intermediate between PN and $R'N$; hence its area is $y\delta x$, the error being less than $\delta y.\delta x$.

Now let CA be a given ordinate, and let A denote the area $CAMP$. Then evidently $PMNQ = \delta A$, the increment of A as x increases to $x + \delta x$.

$$\therefore \delta A = y\delta x \text{ nearly.}$$

Using differentials we have, *exactly*, $dA = ydx$.

$$\therefore A = \int ydx.$$

If $OA = a$, $OB = b$, then area $CABD = \int_a^b y\,dx$.

To show that the accumulated error is negligible, we have

$$\Sigma_a^b \delta y . \delta x = \delta x\ \Sigma_a^b \delta y,$$

if δx be made the same for every term of the series (or OA be made to increase to OB by equal increments),

$$i.e. = (b - a)\delta x,$$

which is negligible compared with the whole area $CABD$.

Ex. In the parabola $y^2 = 4ax$.

$$\int y\,dx = \int 2\sqrt{ax}\,dx = 2\sqrt{a}\int\sqrt{x}\,dx = 2\sqrt{a}\,.\tfrac{2}{3}\,x^{\frac{3}{2}} + C$$
$$= \tfrac{4}{3}\sqrt{a}\,x^{\frac{3}{2}} + C.$$

To find the area of a segment cut off by a *double* ordinate for which $x = h$, $y = k$, we have

$$\text{area} = 2\int_0^h y\,dx = \tfrac{8}{3}\sqrt{a}\,h^{\frac{3}{2}} = \tfrac{8}{3}\cdot\sqrt{ah}\,.h = \tfrac{4}{3}kh, \text{ since } k^2 = 4ah.$$

For a further consideration of the subject, with examples on the same, see Chapter XXVII.

CHAPTER XXII.

ELEMENTARY TRANSFORMATIONS—INTEGRATION BY PARTS

329. In this chapter we shall give some general elementary methods of dealing with integrals which cannot be integrated at sight. The methods of partial fractions and successive reduction will be given further on.

330. Separation into Terms.

Ex. 1. $\int \frac{(2x-1)^3 dx}{x} = \int \frac{8x^3 - 12x^2 + 6x - 1}{x} dx$

$$= \int \left(8x^2 - 12x + 6 - \frac{1}{x}\right) dx = \tfrac{8}{3}x^3 - 6x^2 + 6x - \log x. \quad \text{[Art. 316.]}$$

Ex. 2. $\int \frac{ax+b}{\sqrt{1+x^2}} dx = a\int \frac{x\,dx}{\sqrt{1+x^2}} + b\int \frac{dx}{\sqrt{1+x^2}} = a\sqrt{1+x^2} + b\sinh^{-1}x$

$$= a\sqrt{1+x^2} + b\log(x + \sqrt{1+x^2}).$$

Ex. 3. $\int \frac{x^2 dx}{1-x^2} = \int \frac{1-(1-x^2)}{1-x^2} dx = \int \frac{dx}{1-x^2} - \int dx$

$$= \tanh^{-1}x - x = \frac{1}{2}\log\frac{1+x}{1-x} - x.$$

Ex. 4. $\int \frac{x^5 dx}{1+x} = \int \frac{(1+x^5)-1}{1+x} dx = \int \left(1 - x + x^2 - x^3 + x^4 - \frac{1}{1+x}\right) dx$

$$= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \log(1+x).$$

331. Substitution of a Single Letter for a Compound Expression.

Ex. $\int \frac{x^2 dx}{(2x-1)^5}.$

Put $2x - 1 = y$; $\therefore x = \frac{1}{2}(y+1)$, $dx = \frac{1}{2}dy$.

$$\therefore I = \int \frac{\frac{1}{4}(y+1)^2 . \frac{1}{2}dy}{y^5} = \frac{1}{8}\int \frac{y^2+2y+1}{y^5}\,dy$$

$$= \frac{1}{8}\int\left(\frac{1}{y^3} + \frac{2}{y^4} + \frac{1}{y^5}\right)dy = -\frac{1}{16y^2} - \frac{1}{12y^3} - \frac{1}{32y^4}$$

$$= -\frac{6y^2+8y+3}{96y^4} = -\frac{6(2x-1)^2+8(2x-1)+3}{96(2x-1)^4}$$

$$= -\frac{24x^2-8x+1}{96(2x-1)^4}.$$

332. Rationalization of the Numerator.

Ex. $\int\sqrt{\frac{1+x}{1-x}}\,dx = \int\frac{1+x}{\sqrt{1-x^2}}\,dx$, if we multiply above and below by $\sqrt{1+x}$,

$$= \int\frac{dx}{\sqrt{1-x^2}} + \int\frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \sqrt{1-x^2}.$$

333. Rationalization of the Denominator.

Ex. $\int\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}\,dx = \int\frac{(\sqrt{a^2+x^2}+x)^2\,dx}{a^2+x^2-x^2}$, if we multiply above and below by the numerator,

$$= \int\frac{a^2+2x^2+2x\sqrt{a^2+x^2}}{a^2}\,dx$$

$$= \int dx + \frac{2}{a^2}\int x^2dx + \frac{2}{a^2}\int x\sqrt{a^2+x^2}\,dx$$

$$= x + \frac{2x^3}{3a^2} + \frac{2}{3a^2}(a^2+x^2)^{\frac{3}{2}}.$$

EXAMPLES XLVIII.

1. Integrate :—

(1) $\frac{1-x}{x}\,dx.$ (2) $\frac{x\,dx}{1-x}.$ (3) $\frac{(ax+b)^3}{x^2}\,dx.$

(4) $\frac{x^3\,dx}{1+x}.$ (5) $\frac{3x-4}{2x+1}\,dx.$ (6) $\frac{(\sqrt{x}-1)^3}{x\sqrt{x}}\,dx.$

(7) $\dfrac{1-x^2}{1+x^2}\,dx.$ (8) $\dfrac{1-2x}{\sqrt{1-x^2}}\,dx.$

(9) $\dfrac{2a+x+\sqrt{a^2-x^2}}{a^2-x^2}\,dx.$ (10) $\dfrac{(x-\sqrt{1-x^2})^2}{1-x^2}\,dx.$

(11) $\dfrac{(x-\sqrt{1+x^2})^2}{1+x^2}\,dx.$ (12) $\dfrac{(2+x+\sqrt{1+x^2})^2}{1+x^2}\,dx.$

2. Integrate :—

(1) $\dfrac{x^3\,dx}{x-2}.$ (2) $\dfrac{x\,dx}{(x+1)^2}.$ (3) $\dfrac{(x+1)(x+2)}{x+3}\,dx.$

(4) $(x+1)(x-2)^4\,dx.$ (5) $x^2(x+a)^n\,dx.$ (6) $x^2\sqrt{x+a}\,dx.$

(7) $\dfrac{x\,dx}{\sqrt{x-1}}.$ (8) $x(x-1)^{\frac{3}{2}}\,dx.$ (9) $\dfrac{(1-x)^2\,dx}{\sqrt{1+x}}.$

(10) $\dfrac{x^3\,dx}{(2x-3)^{\frac{3}{2}}}.$

3. Integrate :—

(1) $\sqrt{\dfrac{2-x}{2+x}}\,dx.$ (2) $\sqrt{\dfrac{x-1}{x+1}}\,dx.$

(3) $\sqrt{\dfrac{ax+b}{ax-b}}\,dx.$ (4) $\dfrac{dx}{\sqrt{x+1}-\sqrt{x}}.$

(5) $\dfrac{\sqrt{1+x}+\sqrt{x}}{\sqrt{x}(\sqrt{1+x}-\sqrt{x})}\,dx.$ (6) $\dfrac{x\,dx}{\sqrt{2+9x^2}-3x}.$

(7) $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}\,x\,dx.$ (8) $\dfrac{\sqrt{x+1}}{x\sqrt{x-1}}\,dx.$

ANSWERS.

1. (1) $\log x - x.$ (2) $-\log(1-x)-x.$ (3) $\dfrac{1}{2}a^3x^2+3a^2bx+3ab^2\log x-\dfrac{b^3}{x}.$

(4) $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\log(1+x).$ (5) $\dfrac{3x}{2}-\dfrac{11}{4}\log(2x+1).$

(6) $x-6\sqrt{x}+3\log x+2x^{-\frac{1}{2}}.$ (7) $2\tan^{-1}x-x.$ (8) $\sin^{-1}x+2\sqrt{1-x^2}.$

(9) $2\tanh^{-1}\dfrac{x}{a}-\dfrac{1}{2}\log(a^2-x^2)+\sin^{-1}\dfrac{x}{a}=\dfrac{1}{2}\log\dfrac{a+x}{(a-x)^3}+\sin^{-1}\dfrac{x}{a}.$

(10) $\tanh^{-1}x+2\sqrt{1-x^2}.$ (11) $2x-\tan^{-1}x-2\sqrt{1+x^2}.$

(12) $2x+3\tan^{-1}x+2\log(1+x^2)+4\sinh^{-1}x+2\sqrt{1+x^2}.$

2. (1) $\frac{1}{3}x^3 + x^2 + 4x + C + 8\log(x-2)$. (2) $\log(1+x) + \frac{1}{1+x}$.

(3) $\frac{1}{2}x^2 + 2\log(x+3)$. (4) $\frac{1}{30}(x-2)^5(5x+8)$.

(5) $(x+a)^{n+1}\left\{\frac{(x+a)^2}{n+3} - \frac{2a(x+a)}{n+2} + \frac{a^2}{n+1}\right\}$

(6) $\frac{2}{105}(15x^2 - 12ax + 8a^2)(x+a)^{\frac{3}{2}}$. (7) $\frac{2}{3}(x+2)\sqrt{x-1}$.

(8) $\frac{2}{35}(5x+2)(x-1)^{\frac{3}{2}}$. (9) $\frac{2}{15}(3x^2 - 14x + 43)\sqrt{x+1}$.

(10) $\frac{1}{5}(x^3 + 3x^2 + 18x - 54)(2x-3)^{-\frac{1}{2}}$.

3. (1) $2\sin^{-1}\frac{x}{2} + \sqrt{4 - x^2}$. (2) $\sqrt{x^2 - 1} - \cosh^{-1}x$.

(3) $\frac{1}{a}\sqrt{a^2x^2 - b^2} + \frac{b}{a}\cosh^{-1}\frac{ax}{b}$. (4) $\frac{2}{3}\{(x+1)^{\frac{3}{2}} + x^{\frac{3}{2}}\}$.

(5) $2\sqrt{x} + \frac{4}{3}x^{\frac{3}{2}} + \frac{4}{5}(1+x)^{\frac{5}{2}}$. (6) $\frac{1}{54}(2 + 9x^2)^{\frac{3}{2}} + \frac{1}{2}x^3$.

(7) $\frac{1}{3a}\{x^3 + (x^2 - a^2)^{\frac{3}{2}}\}$. (8) $\cosh^{-1}x + \sec^{-1}x$.

334. Integration by Parts.

Let u and v be functions of x.

Then
$$d(uv) = udv + vdu,$$
$$\therefore uv = \int udv + \int vdu$$
$$\therefore \int udv = uv - \int vdu. \qquad \text{(i)}$$

This formula, which should be committed to memory, is of great use when vdu is easier to integrate than udv. It moreover plays an important part in the chapter on Successive Reduction.

Ex. 1. $\int x\log x\,dx$. Let dv stand for $x\,dx$, and u for $\log x$.

Then $v = \frac{x^2}{2}$, and $du = \frac{dx}{x}$; whence $v\,du = \frac{x}{2}dx$;

$$\therefore I = \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}.$$

It should be noticed that (1) to get uv, the part dv is integrated;
(2) there is a negative sign;
(3) to get $\int vdu$, the part v that was

integrated is now left unaltered, while the part u that was left unaltered is now *differentiated.*

Ex. 2. $\int \sin^{-1} x\,dx = x\sin^{-1}x - \int x\,d(\sin^{-1}x)$

$$= x\sin^{-1}x - \int\frac{x\,dx}{\sqrt{1-x^2}} = x\sin^{-1}x + \sqrt{1-x^2}.$$

Ex. 3. $\displaystyle\int\frac{x\,dx}{\sqrt{x+1}} = x.2\sqrt{x+1} - 2\int\sqrt{x+1}\,dx = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{\frac{3}{2}}$

$$= \frac{2}{3}\sqrt{x+1}\,(x-2).$$

Ex. 4. $\int e^x\cos x\,dx = e^x\sin x - \int e^x \sin x\,dx$, if we make $dv = \cos x\,dx$.
Again $\int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx$, if we make $dv = e^x\,dx$.
Adding and halving,

$$\int e^x\cos x\,dx = \frac{e^x(\sin x + \cos x)}{2}.$$

Ex. 5. $\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int x\,.\,d\sqrt{a^2-x^2}$

$$= x\sqrt{a^2-x^2} + \int\frac{x^2dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} - \int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}};$$

$$\therefore\ 2\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}.$$

$$\dagger\ \therefore\ \int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Ex. 6. † Similarly $\displaystyle\int\sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right) + C.$$

Ex. 7. † Similarly $\displaystyle\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right) + C.$$

---

† The three last integrals are very important, and the student should endeavour to commit them to memory.

EXAMPLES XLIX.

Integrate by parts:—

1. $x^2 \log x\, dx$.
2. $x \sin x\, dx$.
3. $\log x\, dx$.
4. $x e^x\, dx$.
5. $\log(x+\sqrt{1+x^2})\, dx$.
6. $x^n \log x\, dx$.
7. $\tan^{-1} x\, dx$.
8. $\dfrac{x\, dx}{\sqrt{2x+1}}$.
9. $x(x+a)^3\, dx$.
10. $(x+a)(x+b)^n\, dx$.
11. $x \sin^{-1} x\, dx$.
12. $e^x \sin x\, dx$.
13. $\dfrac{x \sin^{-1} x\, dx}{\sqrt{1-x^2}}$.
14. $x \tan^{-1} x\, dx$.
15. $(\log x)^2\, dx$.
16. $\sqrt{a^2+x^2}\, dx$.
17. $\sqrt{x^2-a^2}\, dx$.
18. $e^x \cos 2x\, dx$.
19. $e^x \sin x \cos x\, dx$.
20. $e^x \cos nx\, dx$.
21. $e^x \sin x \sin 3x\, dx$.
22. $x^3 \cos x\, dx$.

Answers.

1. $\frac{1}{9}x^3(3\log x - 1)$.
2. $\sin x - x\cos x$.
3. $x(\log x - 1)$.
4. $e^x(x-1)$.
5. $x\log(x+\sqrt{1+x^2}) - \sqrt{1+x^2}$.
6. $\dfrac{x^{n+1}}{(n+1)^2}\{(n+1)\log x - 1\}$.
7. $x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$.
8. $\frac{1}{3}(x-1)\sqrt{2x+1}$.
9. $\frac{1}{20}(4x-a)(x+a)^4$.
10. $\dfrac{(x+b)^{n+1}}{(n+1)(n+2)}\{(n+1)x + (n+2)a - b\}$.
11. $\frac{1}{4}(2x^2-1)\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$.
12. $\frac{1}{2}e^x(\sin x - \cos x)$.
13. $x - \sqrt{1-x^2}\sin^{-1}x$.
14. $\frac{1}{2}(1+x^2)\tan^{-1}x - \frac{1}{2}x$.
15. $x(\log x)^2 - 2x\log x + 2x$.

16 and 17. See text.

18. $\frac{1}{5}e^x(\cos 2x + 2\sin 2x)$.
19. $\frac{1}{10}e^x(\sin 2x - 2\cos 2x)$.
20. $\dfrac{e^x}{1+n^2}(\cos nx + n\sin nx)$.
21. $\frac{1}{2}e^x\{\frac{1}{5}(\cos 2x + 2\sin 2x) - \frac{1}{17}(\cos 4x + 4\sin 4x)\}$.
22. $x(x^2-6)\sin x + 3(x^2-2)\cos x$.

335. Rationalization by Substitution.—This is useful when a surd of the form $\sqrt[n]{x}$, $\sqrt[n]{a+bx}$, or under certain circumstances $\sqrt[n]{a+bx^p}$, occurs in the expression to be integrated.

Ex. 1. $\int \frac{dx}{\sqrt{a} + \sqrt{x}}$.

Put $\sqrt{x} = y$, or $x = y^2$; $\therefore dx = 2y dy$.

$$\therefore I = 2\int \frac{y\,dy}{\sqrt{a} + y} = 2\int \frac{(\sqrt{a} + y) - \sqrt{a}}{\sqrt{a} + y} dy = 2y - 2\sqrt{a}\log(\sqrt{a} + y)$$

$$= 2\sqrt{x} - 2\sqrt{a}\log(\sqrt{a} + \sqrt{x}).$$

Ex. 2. $\int \frac{x+1}{x\sqrt{x-2}} dx$.

Put $x - 2 = y^2$; $\therefore dx = 2y\,dy$.

$$I = 2\int \frac{(y^2+3)y\,dy}{(y^2+2)y} = 2\int \left(1 + \frac{1}{y^2+2}\right) dy = 2y + \sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}}$$

$$= 2\sqrt{x-2} + \sqrt{2}\tan^{-1}\sqrt{\frac{x-2}{2}}.$$

Ex. 3. $\int \frac{\sqrt{x} + 2}{x - \sqrt[3]{x^2}} dx$.

Put $x = y^6$; $\therefore dx = 6y^5\,dy$.

$$\therefore I = 6\int \frac{(y^3+2)y^5\,dy}{y^6 - y^4} = 6\int \frac{y^4 + 2y}{y^2 - 1} dy = 6\int \frac{(y^4-1) + (2y+1)}{y^2-1} dy$$

$$= 6\int (y^2+1)dy + 6\int \frac{2y\,dy}{y^2-1} + 6\int \frac{dy}{y^2-1}$$

$$= 2y^3 + 6y + 6\log(y^2-1) + 3\log\frac{y-1}{y+1}. \quad \text{[See Table, Art. 319.]}$$

$$= 2y^3 + 6y + 6\log(y+1) + 6\log(y-1) + 3\log(y-1) - 3\log(y+1)$$

$$= 2x^{\frac{1}{2}} + 6x^{\frac{1}{6}} + 9\log(x^{\frac{1}{6}} - 1) + 3\log(x^{\frac{1}{6}} + 1).$$

Ex. 4. $\int \frac{x(1 - \sqrt{1-x^2})\,dx}{2 - \sqrt{1-x^2}}$.

Put $1 - x^2 = y^2$; $\therefore -x\,dx = y\,dy$.

$$\therefore I = -\int \frac{(1-y)y\,dy}{2-y}. \quad \text{Put } 2 - y, \text{ or } 2 - \sqrt{1-x^2}, = z;$$

$$\therefore I = \int \frac{(z-1)(2-z)dz}{z} = \int \left(-\frac{2}{z} + 3 - z\right) dz$$

$$= -2\log z + 3z - \tfrac{1}{2}z^2 = -2\log(2-y) + 6 - 3y - \tfrac{1}{2}(4 - 4y + y^2)$$

$$= -2\log(2-y) + \tfrac{1}{2}(8 - 2y - y^2)$$

$$= -2\log(2 - \sqrt{1-x^2}) + \tfrac{1}{2}\{8 - 2\sqrt{1-x^2} - 1 + x^2\}$$

$$= -2\log(2 - \sqrt{1-x^2}) + \tfrac{1}{2}\{x^2 - 2\sqrt{1-x^2}\} + C.$$

EXAMPLES L.

Integrate :—

1. $\dfrac{\sqrt{x}\,dx}{1+\sqrt{x}}$.
2. $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\,dx$.
3. $\dfrac{x\sqrt{x}\,dx}{1+\sqrt{x}}$.
4. $\dfrac{x\,dx}{\sqrt{2x+1}}$.
5. $\dfrac{dx}{\sqrt{x}\sqrt{x+1}}$.
6. $\dfrac{dx}{\sqrt{x-x^2}}$.
7. $\dfrac{x^3\,dx}{\sqrt{x^2+1}}$.
8. $\dfrac{2-3x}{x\sqrt{1-x}}\,dx$.
9. $\dfrac{x^2-1}{x\sqrt{x^2-2}}\,dx$.
10. $\dfrac{dx}{x^{\frac{4}{5}}+x^{\frac{3}{5}}}$.
11. $\dfrac{dx}{x^{\frac{2}{3}}\sqrt{x^{\frac{1}{6}}+1}}$.
12. $\dfrac{x-6}{2+\sqrt{x-2}}\,dx$.
13. $\dfrac{x^3+2x}{1+\sqrt{1+x^2}}\,dx$.

ANSWERS.

1. $x-2\sqrt{x}+2\log(1+\sqrt{x})$.
2. $x-4\sqrt{x}+4\log(1+\sqrt{x})$.
3. $\frac{1}{2}x(x+2)-\frac{2}{3}\sqrt{x}(x+3)+2\log(1+\sqrt{x})$.
4. $\frac{1}{3}\sqrt{2x+1}(x-1)$.
5. $2\cosh^{-1}\sqrt{x+1}=\cosh^{-1}(2x+1)$.
6. $2\sin^{-1}\sqrt{x}=\sin^{-1}(2\sqrt{x-x^2})$. Put $x=y^2$; see Ex. (5).
7. $\frac{1}{3}\sqrt{x^2+1}(x^2-2)$.
8. $6\sqrt{1-x}-2\log\dfrac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$.
9. $\sqrt{x^2-2}-\dfrac{1}{\sqrt{2}}\tan^{-1}\sqrt{\dfrac{x^2-2}{2}}$.
10. $5\{x^{\frac{1}{5}}-\log(x^{\frac{1}{5}}+1)\}$.
11. $4\sqrt{x^{\frac{1}{6}}+1}(x^{\frac{1}{6}}-2)$.
12. $\frac{2}{3}(x-2)^{\frac{3}{2}}-2x$.
13. $\frac{1}{3}y^3-\frac{1}{2}y^2+2y-2\log(y+1)$, where $y=\sqrt{x^2+1}$.

336. Use of Trigonometrical Formulæ.—By making use of the various formulæ in trigonometry, many integrals can be transformed into others immediately integrable.

Ex. 1. $$\int\frac{\sin^3\theta\,d\theta}{\cos^2\theta}=\int\frac{\sin\theta\,(1-\cos^2\theta)d\theta}{\cos^2\theta}=\int\frac{\sin\theta\,d\theta}{\cos^2\theta}-\int\sin\theta\,d\theta$$
$$=\sec\theta+\cos\theta.$$

Ex. 2. $\int \sin^3\theta\, d\theta = \frac{1}{4}\int(3\sin\theta - \sin 3\theta)d\theta = -\frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta.$

Or:—$I = \int(1-\cos^2\theta)\sin\theta\, d\theta = -\cos\theta + \frac{1}{3}\cos^3\theta$, which agrees with the preceding result.

Ex. 3. $$\int\frac{d\theta}{\sin\theta\cos\theta} = \int\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}d\theta = \int\frac{\sin\theta\, d\theta}{\cos\theta} + \int\frac{\cos\theta\, d\theta}{\sin\theta}$$
$$= -\log\cos\theta + \log\sin\theta = \log\tan\theta.$$

Ex. 4. $\int\sin m\theta\cos n\theta\, d\theta = \frac{1}{2}\int\{\sin(m+n)\theta + \sin(m-n)\theta\}d\theta$
$$= -\frac{1}{2(m+n)}\cos(m+n)\theta - \frac{1}{2(m-n)}\cos(m-n)\theta.$$

Ex. 5. $\int\cos^4\theta\, d\theta = \frac{1}{4}\int(2\cos^2\theta)^2 d\theta = \frac{1}{4}\int(1+\cos 2\theta)^2 d\theta$
$$= \frac{1}{4}\int(1 + 2\cos 2\theta + \cos^2 2\theta)d\theta$$
$$= \frac{1}{4}\int d\theta + \frac{1}{2}\int\cos 2\theta\, d\theta + \frac{1}{8}\int(1+\cos 4\theta)d\theta$$
$$= \frac{\theta}{4} + \frac{\sin 2\theta}{4} + \frac{\theta}{8} + \frac{\sin 4\theta}{32} = \frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}.$$

Ex. 6. $$\int\frac{1+\cos^4\theta}{\sin^4\theta}d\theta = \int\frac{(1-\cos^2\theta)^2 + 2\cos^2\theta}{\sin^4\theta}d\theta = \int\frac{\sin^4\theta + 2\cos^2\theta}{\sin^4\theta}d\theta$$
$$= \int d\theta + 2\int\cot^2\theta\operatorname{cosec}^2\theta\, d\theta = \theta - \frac{2}{3}\cot^3\theta.$$

Ex. 7. $$\int\sec\theta\, d\theta = \int\frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int\frac{dx}{\sqrt{1+x^2}} \text{ (if } x = \tan\theta)$$
$$= \log(x + \sqrt{1+x^2}) = \log(\tan\theta + \sec\theta) = \mathrm{gd}^{-1}\theta.$$

Ex. 8. $\int\tan^2\theta\, d\theta = \int(\sec^2\theta - 1)d\theta = \tan\theta - \theta.$

EXAMPLES LI.

Integrate:—

1. $\dfrac{\cos^3\theta}{\sin^2\theta}d\theta.$
2. $\cos^3\theta\, d\theta.$
3. $\dfrac{1+\sin^3\theta}{\cos^2\theta}d\theta.$
4. $\cos^2\theta\, d\theta.$
5. $\sin^2\theta\, d\theta.$
6. $\sin\theta\sin 2\theta\, d\theta.$
7. $\cos 3\theta\sin 2\theta\, d\theta.$
8. $\sin^4\theta\, d\theta.$
9. $\sin^2\theta\cos^2\theta\, d\theta.$
10. $\dfrac{1+\sin^4\theta}{\cos^4\theta}d\theta.$
11. $\cot^2\theta\, d\theta.$
12. $\tan^3\theta\, d\theta.$
13. $\dfrac{\sin^4 x + \cos^4 x}{\sin^3 x\cos^3 x}dx.$
14. $\dfrac{dx}{\sin^2 x\cos^2 x}.$

ANSWERS.

1. $-(\operatorname{cosec}\theta+\sin\theta)$. 2. $\sin\theta-\frac{1}{3}\sin^3\theta$. 3. $\tan\theta+\sec\theta+\cos\theta$.
4. $\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta$. 5. $\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta$. 6. $\frac{1}{2}\sin\theta-\frac{1}{6}\sin 3\theta$.
7. $\frac{1}{2}\cos\theta-\frac{1}{10}\cos 5\theta$. 8. $\frac{3}{8}\theta-\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta$.
9. $\frac{1}{8}\theta-\frac{1}{32}\sin 4\theta$. 10. $\theta+\frac{2}{3}\tan^3\theta$. 11. $-(\cot\theta+\theta)$.
12. $\frac{1}{2}\tan^2\theta+\log\cos\theta$. 13. $\frac{1}{2}(\sec^2 x-\operatorname{cosec}^2 x)$. 14. $\tan x-\cot x$.

337. Trigonometrical Substitution.—This is often useful when the surds $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$ occur. The substitutions are given in the following table :—

Surd	Put for x	Value of surd
(a) $\sqrt{a^2-x^2}$	(1) $a\sin\theta$ or (2) $a\tanh u$	(1) $a\cos\theta$ (2) $a\operatorname{sech} u$
(b) $\sqrt{a^2+x^2}$	(1) $a\tan\theta$ or (2) $a\sinh u$	(1) $a\sec\theta$ (2) $a\cosh u$
(c) $\sqrt{x^2-a^2}$	(1) $a\sec\theta$ or (2) $a\cosh u$	(1) $a\tan\theta$ (2) $a\sinh u$

NOTE.—If $a=1$, these surds become respectively $\sqrt{1-x^2}$, $\sqrt{1+x^2}$, and $\sqrt{x^2-1}$.

There are other substitutions which might have been made; thus in (a) we might have had in the second column $a\cos\theta$ or $a\operatorname{sech} u$; but there is no advantage to be derived from using them instead of those given.

These substitutions may also be made in other cases. [See Ex. 5, below.]

Ex. 1. $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}$.

Put $x=a\sin\theta$; $\therefore dx=a\cos\theta\,d\theta$.

$$\therefore I=\int\frac{a\cos\theta d\theta}{a\cos\theta}=\int d\theta=\theta=\sin^{-1}\frac{x}{a}.$$

Ex. 2. $\int\sqrt{a^2 - x^2}\,dx$.

Put $x = a\sin\theta$; $dx = a\cos\theta\,d\theta$.

$$\therefore\ I = a^2\int\cos^2\theta\,d\theta = \frac{a^2}{2}\int(1+\cos 2\theta)d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{4}\sin 2\theta = \frac{a^2}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{2} \text{ as before.}$$

Ex. 3. $\int\sqrt{a^2 + x^2}\,dx$.

(1) Put $x = a\sinh u$; $dx = a\cosh u\,du$.

$$\therefore I = a^2\int\cosh^2 u\,du = \frac{a^2}{2}\int(1+\cosh 2u)du$$

$$= \frac{a^2}{2}u + \frac{a^2}{4}\sinh 2u = \frac{a^2}{2}u + \frac{a^2}{2}\sinh u\cosh u$$

$$= \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x\sqrt{a^2+x^2}}{2}.$$

(2) Otherwise, put $x = a\tan\theta$; $\therefore\ dx = a\sec^2\theta\,d\theta$.

$\therefore\ I = a^2\int\sec^3\theta\,d\theta = a^2\int\sec\theta\,.\,d(\tan\theta) = a^2\sec\theta\tan\theta - a^2\int\sec\theta\tan^2\theta\,d\theta$ (integrating by parts)

$$= a^2\sec\theta\tan\theta - a^2\int(\sec^3\theta - \sec\theta)d\theta = a^2\sec\theta\tan\theta - I + a^2\int\sec\theta\,d\theta.$$

$$= a^2\sec\theta\tan\theta - I + a^2\log(\sec\theta + \tan\theta) \text{ [see Ex. 7, Art. 336];}$$

$$\therefore\ I = \frac{a^2}{2}\sec\theta\tan\theta + \frac{a^2}{2}\log(\sec\theta + \tan\theta)$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log(x + \sqrt{a^2+x^2}) + C, \text{ where } C = -\frac{a^2}{2}\log a.$$

Ex. 4. $\int\frac{x^2(x+1)}{(x^2+1)^{\frac{3}{2}}}dx$.

Put $x = \tan\theta$; $dx = \sec^2\theta\,d\theta$.

$$\therefore\ I = \int\frac{\tan^2\theta(\tan\theta + 1)\sec^2\theta\,d\theta}{\sec^3\theta} = \int\frac{\sin^2\theta(\sin\theta+\cos\theta)}{\cos^2\theta}d\theta$$

$$= \int\frac{\sin^3\theta}{\cos^2\theta}d\theta + \int\frac{\sin^2\theta}{\cos\theta}d\theta = \int\frac{\sin\theta(1-\cos^2\theta)}{\cos^2\theta}d\theta + \int\frac{1-\cos^2\theta}{\cos\theta}d\theta$$

$$= \int\frac{\sin\theta}{\cos^2\theta}d\theta - \int\sin\theta\,d\theta + \int\sec\theta\,d\theta - \int\cos\theta\,d\theta$$

$$= \sec\theta + \cos\theta + \log(\tan\theta + \sec\theta) - \sin\theta$$

$$= \sqrt{1+x^2} + \frac{1}{\sqrt{1+x^2}} + \log(x + \sqrt{1+x^2}) - \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{2 - x + x^2}{\sqrt{1+x^2}} + \log(x + \sqrt{1+x^2}).$$

Ex. 5. $\int x\sqrt{\frac{a-x}{a+x}}dx.$

Put $x = a\cos\theta$; $\therefore\ dx = -a\sin\theta\, d\theta$.

$$\therefore\ I = -a^2\int\cos\theta\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta\, d\theta = -a^2\int\cos\theta(\operatorname{cosec}\theta - \cot\theta)\sin\theta\, d\theta$$

[see *Misc. Theorems.*]

$$= -a^2\int\cos\theta\, d\theta + a^2\int\cos^2\theta\, d\theta = -a^2\sin\theta + \tfrac{1}{2}a^2\int(1+\cos 2\theta)d\theta$$

$$= -a^2\sin\theta + \frac{a^2}{2}(\theta + \sin\theta\cos\theta)$$

$$= -a\sqrt{a^2-x^2} + \frac{a^2}{2}\cos^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}.$$

Otherwise, $I = \int\frac{(ax-x^2)dx}{\sqrt{a^2-x^2}} = \int\frac{(a^2-x^2)+ax-a^2}{\sqrt{a^2-x^2}}dx$

$$= \int\sqrt{a^2-x^2}\,dx + a\int\frac{x\,dx}{\sqrt{a^2-x^2}} - a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$= \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - a\sqrt{a^2-x^2} - a^2\sin^{-1}\frac{x}{a}$$

$$= \frac{x\sqrt{a^2-x^2}}{2} - a\sqrt{a^2-x^2} - \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

which differs by the constant $\frac{a^2}{2}\cdot\frac{\pi}{2}$ from the preceding result.

Ex. 6. $\int\frac{dx}{(a^2-b^2x^2)^{\frac{3}{2}}}.$

Put $bx = a\sin\theta$; $dx = \frac{a}{b}\cos\theta\, d\theta$

$$I = \frac{a}{b}\cdot\frac{1}{a^3}\int\frac{\cos\theta\, d\theta}{\cos^3\theta} = \frac{1}{a^2b}\int\sec^2\theta\, d\theta = \frac{1}{a^2b}\tan\theta$$

$$= \frac{1}{a^2b}\,\frac{bx}{\sqrt{a^2-b^2x^2}} = \frac{x}{a^2\sqrt{a^2-b^2x^2}}.$$

EXAMPLES LII.

Integrate :—

1. $\dfrac{dx}{\sqrt{a^2+x^2}}$. 2. $\dfrac{dx}{\sqrt{x^2-a^2}}$. 3. $\sqrt{x^2-a^2}\,dx$.

4. $\dfrac{dx}{(a^2+x^2)^{\frac{3}{2}}}$. 5. $\dfrac{\sqrt{x^2-a^2}\,dx}{x}$. 6. $\dfrac{dx}{(a^2+b^2x^2)^{\frac{3}{2}}}$.

7. $\dfrac{x-2}{\sqrt{x^2-1}}\,dx$. 8. $\dfrac{x^3\,dx}{\sqrt{1-x^2}}$. 9. $\dfrac{a+x}{(a^2+x^2)^{\frac{3}{2}}}\,dx$.

10. $\sqrt{\dfrac{a-x}{a+x}}\,dx$. 11. $\dfrac{\sqrt{2-x}}{\sqrt{2x^2+x^3}}\,dx$. 12. $\dfrac{x+1}{x(x^2+1)}\,dx$.

ANSWERS.

1, 2, 3. See above. 4. $\dfrac{x}{a^2\sqrt{a^2+x^2}}$. 5. $\sqrt{x^2-a^2}-a\cos^{-1}\dfrac{a}{x}$.

6. $\dfrac{x}{a^2\sqrt{a^2+b^2x^2}}$. 7. $\sqrt{x^2-1}-2\log(x+\sqrt{x^2-1})$.

8. $-\frac{1}{3}(x^2+2)\sqrt{1-x^2}$. 9. $\dfrac{1}{a}\dfrac{x-a}{\sqrt{x^2+a^2}}$. 10. $a\sin^{-1}\dfrac{x}{a}+\sqrt{a^2-x^2}$.

11. $\cos^{-1}\dfrac{x}{2}-\log\dfrac{2+\sqrt{4-x^2}}{x}=\cos^{-1}\dfrac{x}{2}-\cosh^{-1}\dfrac{2}{x}$ [put $x=2\cos\theta$].

12. $\log\dfrac{x}{\sqrt{1+x^2}}+\tan^{-1}x$.

338. Miscellaneous Substitutions.—Innumerable substitutions may be made which can hardly be classified. Some are given in the following example :—

Ex. $\int\sqrt{2+\tan^2\theta}\,d\theta$.

Put $\tan\theta=x$, or $\theta=\tan^{-1}x$; $\therefore\ d\theta=\dfrac{dx}{1+x^2}$.

$$\therefore\ I=\int\frac{\sqrt{2+x^2}\,dx}{1+x^2}.\quad \text{Put } x^2=y;\quad \therefore\ 2x\,dx=dy.$$

$$\therefore\ I=\tfrac{1}{2}\int\frac{\sqrt{2+y}\,dy}{(1+y)\sqrt{y}}.\quad \text{Put } y=\frac{1}{z};\quad \therefore\ dy=-\frac{1}{z^2}\,dz.$$

$$\therefore I = -\tfrac{1}{2}\int \frac{\sqrt{2+\frac{1}{z}}\,dz}{z^2\sqrt{\frac{1}{z}\left(1+\frac{1}{z}\right)}} = -\tfrac{1}{2}\int\frac{\sqrt{2z+1}\,dz}{z(z+1)}.$$

Put $2z+1=v^2$; $\therefore dz = v\,dv$.

$$\therefore I = -\tfrac{1}{2}\int\frac{4v^2\,dv}{(v^2-1)(v^2+1)} = -\int\left(\frac{1}{v^2-1}+\frac{1}{v^2+1}\right)dv$$

$$= \coth^{-1} v + \cot^{-1} v, \text{ since } v > 1.\dagger \quad \text{[See Arts. 80, 319.]}$$

But $v = \sqrt{2z+1} = \sqrt{\frac{2+y}{y}} = \sqrt{\frac{2+x^2}{x^2}} = \sqrt{2\cot^2\theta + 1}$;

$$\therefore I = \coth^{-1}\sqrt{2\cot^2\theta+1} + \cot^{-1}\sqrt{2\cot^2\theta+1}.$$

***339.** Since $v^2 - 1 = 2\cot^2\theta$, or $\tan\theta = \frac{\sqrt{2}}{\sqrt{v^2-1}}$, we might make this substitution at once.

Thus, differentiating,

$$2v\,dv = -4\cot\theta\,\mathrm{cosec}^2\theta\,d\theta$$

$$= -4\frac{\sqrt{v^2-1}}{\sqrt{2}}.\left(\frac{v^2-1}{2}+1\right)d\theta = -\sqrt{2}\sqrt{v^2-1}.(v^2+1)\,d\theta,$$

$$\therefore d\theta = -\sqrt{2}.\frac{v\,dv}{\sqrt{v^2-1}.(v^2+1)}.$$

Also
$$\sqrt{2+\tan^2\theta} = v\tan\theta = \frac{\sqrt{2}v}{\sqrt{v^2-1}},$$

$$\therefore I = -2\int\frac{v^2\,dv}{(v^2-1)(v^2+1)} = \text{etc., as before.}$$

Otherwise:—$\int\sqrt{2+\tan^2\theta}\,d\theta = \int\sqrt{1+\sec^2\theta}\,d\theta$.

Put $\sec^2\theta = x$; $\therefore \theta = \sec^{-1}\sqrt{x}$; $d\theta = \frac{dx}{2x\sqrt{x-1}}$;

$$\therefore I = \frac{1}{2}\int\frac{\sqrt{1+x}\,dx}{x\sqrt{x-1}} = \frac{1}{2}\int\frac{(x+1)dx}{x\sqrt{x^2-1}} = \tfrac{1}{2}\cosh^{-1}x + \tfrac{1}{2}\sec^{-1}x$$

$$= \tfrac{1}{2}\cosh^{-1}\sec^2\theta + \tfrac{1}{2}\sec^{-1}\sec^2\theta.$$

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† For $v^2 = 1 + \frac{2}{x^2} > 1$.

***340.** To show that these two results agree—

Let $\frac{1}{2}\sec^{-1}\sec^2\theta = \frac{1}{2}\phi$;

$$\therefore \sec\phi = \sec^2\theta,$$

$$\cos\phi = \cos^2\theta,$$

$$\cot^2\tfrac{1}{2}\phi = \frac{1+\cos\phi}{1-\cos\phi} = \frac{1+\cos^2\theta}{\sin^2\theta} = \operatorname{cosec}^2\theta + \cot^2\theta = 2\cot^2\theta + 1.$$

$$\therefore \tfrac{1}{2}\phi = \cot^{-1}\sqrt{2\cot^2\theta + 1}.$$

The other part may be established trigonometrically, or by logs, thus:—

$$\tfrac{1}{2}\cosh^{-1}\sec^2\theta = \tfrac{1}{2}\log\{\sec^2\theta + \sqrt{\sec^4\theta - 1}\}$$

$$= \tfrac{1}{2}\log\{\sec^2\theta + \tan\theta\sqrt{\sec^2\theta + 1}\}. \quad . \quad . \quad . \quad (1)$$

Also $\coth^{-1}\sqrt{2\cot^2\theta + 1} = \coth^{-1}\dfrac{\sqrt{\sec^2\theta + 1}}{\tan\theta} = \dfrac{1}{2}\log\dfrac{\sqrt{\sec^2\theta + 1} + \tan\theta}{\sqrt{\sec^2\theta + 1} - \tan\theta}$

$$= \tfrac{1}{2}\log\frac{\sec^2\theta + 1 + \tan^2\theta + 2\tan\theta\sqrt{\sec^2\theta + 1}}{\sec^2\theta + 1 - \tan^2\theta}$$

$$= \tfrac{1}{2}\log(\sec^2\theta + \tan\theta\sqrt{\sec^2\theta + 1}) = \tfrac{1}{2}\cosh^{-1}\sec^2\theta, \text{ by (1).}$$

****341.** The following proof may be noticed:—

If $\cosh u = \sec\phi$, we know that $\coth\frac{1}{2}u = \cot\frac{1}{2}\phi$. [Exs. XIV., 24.]

$$\therefore u = \cosh^{-1}\sec\phi, \text{ and } \tfrac{1}{2}u = \coth^{-1}\cot\tfrac{1}{2}\phi;$$

$$\therefore \tfrac{1}{2}\cosh^{-1}\sec\phi = \coth^{-1}\cot\tfrac{1}{2}\phi. \quad . \quad . \quad . \quad . \quad (2)$$

Now, from Art. 340,

$$\sec\phi = \sec^2\theta, \text{ and } \cot\tfrac{1}{2}\phi = \sqrt{2\cot^2\theta + 1}.$$

$$\therefore \text{ by (2) } \tfrac{1}{2}\cosh^{-1}\sec^2\theta = \coth^{-1}\sqrt{2\cot^2\theta + 1}.$$

EXAMPLES LIII.

Integrate, by means of the substitutions given:—

1. $\dfrac{dx}{\sqrt{x}\sqrt{a-x}}$; put $x = a\sin^2\theta$.

2. $\dfrac{dx}{\sqrt{2ax - x^2}}$; $x = a(1-\cos\theta) = a\operatorname{vers}\theta$.

3. $\dfrac{(x+a)dx}{\sqrt{x}\sqrt{a-x}}$; $x = a\sin^2\theta$.

4. $\dfrac{(x-a)dx}{(x+a)\sqrt{x}}$; $x = a\tan^2\theta$; also $x = y^2$.

5. $\dfrac{dx}{\sqrt{x(x-a)^{\frac{3}{2}}}}$; $x = \dfrac{ay^2}{y^2-a^2}$.

6. $\dfrac{2a+x}{a+x}\sqrt{\dfrac{a-x}{a+x}}\,dx$; $x = a\cos 2\theta$; also $x = a\sin\theta$

7. $\sqrt{a^2+x^2}\,dx$; $x = \dfrac{a}{2}\left(y - \dfrac{1}{y}\right)$.

8. $\sqrt{x^2-a^2}\,dx$; $x = \dfrac{a}{2}(e^y + e^{-y})$.

9. $\dfrac{(x^2-a^2)dx}{x\sqrt{x^4+3x^2a^2+a^4}}$; $\dfrac{x}{a} + \dfrac{a}{x} = z$.

EXAMPLES LIV.—MISCELLANEOUS.

Integrate by inspection, or by any of the preceding methods :—

1. $\dfrac{x^2\,dx}{x-1}$.

2. $\dfrac{x\,dx}{\sqrt{x+a}}$.

3. $\left(\dfrac{x-1}{x+1}\right)^2 dx$.

4. $\dfrac{e^{\tan x}\,dx}{\cos^2 x}$.

5. $x\log(1+x)dx$.

6. $x^2\tan^{-1}x\,dx$.

7. $\dfrac{dx}{\sqrt{x^2+1}+x}$.

8. $\dfrac{\log x\,dx}{x}$.

9. $\dfrac{\log x\,dx}{x^2}$.

10. $\dfrac{x\,dx}{(1+x^2)^2}$.

11. $\dfrac{\sin 2x\,dx}{1+\sin^2 x}$.

12. $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$.

13. $\dfrac{x\sqrt{1-x^2}}{\sqrt{1+x^2}}\,dx$.

14. $\dfrac{\sin x\,dx}{\sqrt{1+\cos^2 x}}$.

15. $\dfrac{\sin 2x\,dx}{\sqrt{1+\cos^2 x}}$.

16. $\dfrac{x\,dx}{\sqrt{a^4-x^4}}$.

17. $\dfrac{\sin^{-1}x\,dx}{\sqrt{1-x^2}}$.

18. $\sec^4\theta\,d\theta$.

19. $\tan\theta\sec^3\theta\,d\theta$.

20. $\dfrac{d\theta}{\sec\theta+\tan\theta}$.

21. $\dfrac{\sqrt{x}\,dx}{1+x}$.

22. $\tan^{-1}\sqrt{x}\,dx$.

23. $\sqrt{x}\tan^{-1}\sqrt{x}\,dx$.

24. $x(1+x^2)\tan^{-1}x\,dx$.

25. $\dfrac{x\tan^{-1}x}{\sqrt{1+x^2}}\,dx$.

26. $x\sqrt{x^2-1}\cosh^{-1}x\,dx$.

27. $\dfrac{x^2\,dx}{\sqrt{x^2-a^2}}$. **28.** $\dfrac{x^2\,dx}{\sqrt{x^2+a^2}}$. **29.** $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}dx$.

30. $\dfrac{x\sqrt{x-1}}{\sqrt{x+1}}dx$. **31.** $\dfrac{(x-2)\sqrt{x-3}}{\sqrt{x+3}}dx$.

ANSWERS (LIII.).

1. $2\sin^{-1}\sqrt{\dfrac{x}{a}}$. **2.** $\operatorname{vers}^{-1}\dfrac{x}{a}=\cos^{-1}\dfrac{a-x}{a}$. **3.** $3a\sin^{-1}\sqrt{\dfrac{x}{a}}-\sqrt{x(a-x)}$.

4. $2\sqrt{x}-4\sqrt{a}\tan^{-1}\sqrt{\dfrac{x}{a}}$. **5.** $-\dfrac{2}{a}\sqrt{\dfrac{x}{x-a}}$. **6.** $-\dfrac{(a-x)^{\frac{3}{2}}}{(a+x)^{\frac{1}{2}}}$.

9. $\sinh^{-1}\left(\dfrac{x}{a}+\dfrac{a}{x}\right)=\log(x^2+a^2+\sqrt{x^4+3x^2a^2+a^4})-\log(ax)$.

ANSWERS (LIV.).

1. $\frac{1}{2}x^2+x+\log(x-1)$. **2.** $\frac{2}{3}(x-2a)\sqrt{x+a}$. **3.** $\dfrac{x^2+x-4}{x+1}-4\log(x+1)$.

4. $e^{\tan x}$. **5.** $\dfrac{x^2-1}{2}\log(x+1)-\frac{1}{4}(x-1)^2$. **6.** $\frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log(1+x^2)$.

7. $\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\sinh^{-1}x-\frac{1}{2}x^2$. **8.** $\frac{1}{2}(\log x)^2$. **9.** $-\dfrac{1}{x}(1+\log x)$.

10. $-\dfrac{1}{2(1+x^2)}$. **11.** $\log(1+\sin^2 x)$. **12.** $\log(e^x+e^{-x})$.

13. $\frac{1}{2}\sin^{-1}x^2+\frac{1}{2}\sqrt{1-x^4}$. **14.** $-\sinh^{-1}\cos x$. **15.** $-2\sqrt{1+\cos^2 x}$.

16. $\frac{1}{2}\sin^{-1}\dfrac{x^2}{a^2}$. **17.** $\frac{1}{2}(\sin^{-1}x)^2$. **18.** $\tan\theta+\frac{1}{3}\tan^3\theta$. **19.** $\frac{1}{3}\sec^3\theta$.

20. $\log(1+\sin\theta)$. **21.** $2(\sqrt{x}-\tan^{-1}\sqrt{x})$. **22.** $(1+x)\tan^{-1}\sqrt{x}-\sqrt{x}$.

23. $\frac{2}{3}x\sqrt{x}\tan^{-1}\sqrt{x}+\frac{1}{3}\log(1+x)-\frac{1}{3}x$. **24.** $\frac{1}{4}(1+x^2)^2\tan^{-1}x-\frac{1}{4}(x+\frac{1}{3}x^3)$.

25. $\sqrt{1+x^2}\tan^{-1}x-\sinh^{-1}x$. **26.** $\frac{1}{3}(x^2-1)^{\frac{3}{2}}\cosh^{-1}x-\frac{1}{3}(\frac{1}{3}x^3-x)$.

27. $\dfrac{x\sqrt{x^2-a^2}}{2}+\dfrac{a^2}{2}\cosh^{-1}\dfrac{x}{a}$. **28.** $\dfrac{x\sqrt{x^2+a^2}}{2}-\dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a}$.

29. $\dfrac{1}{2a}\left(x^2+x\sqrt{x^2-a^2}-a^2\cosh^{-1}\dfrac{x}{a}\right)$.

30. $\dfrac{x\sqrt{x^2-1}}{2}+\frac{1}{2}\cosh^{-1}x-\sqrt{x^2-1}$.

31. $\frac{1}{2}(x-10)\sqrt{x^2-9}+\dfrac{21}{2}\cosh^{-1}\dfrac{x}{3}$.

CHAPTER XXIII.

STANDARD FORMS.

342. Integration of $\dfrac{dx}{\sqrt{ax^2+2hx+b}}$.

We shall show that (except when $ax^2+2hx+b$ is a perfect square) this integral is reducible to one or other of the three fundamental forms:—

$$(a)\quad \int\frac{dz}{\sqrt{a^2-z^2}}=\sin^{-1}\frac{z}{a}\text{ or }-\cos^{-1}\frac{z}{a}.$$

$$(b)\quad \int\frac{dz}{\sqrt{a^2+z^2}}=\sinh^{-1}\frac{z}{a}=\log\left(z+\sqrt{a^2+z^2}\right)-\log a.$$

$$(c)\quad \int\frac{dz}{\sqrt{z^2-a^2}}=\cosh^{-1}\frac{z}{a}=\log\left(z+\sqrt{z^2-a^2}\right)-\log a.$$

That this is most probably the case will appear if we begin with any one of the latter integrals and put $z=x+c$.

Thus $$\int\frac{dz}{\sqrt{a^2+z^2}}=\int\frac{dx}{\sqrt{a^2+(x+c)^2}}=\int\frac{dx}{\sqrt{x^2+2cx+c^2+a^2}},$$

which is of the form we are considering.

We shall first give some numerical examples and then take the general form.

343. Numerical Examples.

Ex. 1. $\displaystyle\int\frac{dx}{\sqrt{(2-x)(3x-1)}}.$

We have $(2 - x)(3x - 1) = -2 + 7x - 3x^2 = -3(x^2 - \frac{7}{3}x + \frac{2}{3})$

$$= -3\{(x - \tfrac{7}{6})^2 - (\tfrac{5}{6})^2\} = 3\{(\tfrac{5}{6})^2 - (x - \tfrac{7}{6})^2\}$$

$$\therefore\ I = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(\frac{5}{6})^2 - (x - \frac{7}{6})^2}} = \frac{1}{\sqrt{3}}\int\frac{dz}{\sqrt{(\frac{5}{6})^2 - z^2}},\ \text{if } z = x - \tfrac{7}{6},$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{6z}{5} = \frac{1}{\sqrt{3}}\sin^{-1}\frac{6x - 7}{5}.$$

Ex. 2. $\displaystyle\int\frac{dx}{\sqrt{3x^2 - 10x + 3}}$.

We have $3x^2 - 10x + 3 = 3(x^2 - \frac{10}{3}x + 1) = 3\{(x - \frac{5}{3})^2 - (\frac{4}{3})^2\}$,

$$\therefore\ I = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(x - \frac{5}{3})^2 - (\frac{4}{3})^2}} = \frac{1}{\sqrt{3}}\int\frac{dz}{\sqrt{z^2 - (\frac{4}{3})^2}},\ \text{if } z = x - \tfrac{5}{3},$$

$$= \frac{1}{\sqrt{3}}\cosh^{-1}\frac{3z}{4},\ \text{or}\ \frac{1}{\sqrt{3}}\log(3z + \sqrt{9z^2 - 16}) + C$$

(C being the constant $-\dfrac{1}{\sqrt{3}}\log 4$, which we ignore) †

$$= \frac{1}{\sqrt{3}}\cosh^{-1}\frac{3x - 5}{4},\ \text{or}\ \frac{1}{\sqrt{3}}\log\{3x - 5 + \sqrt{3(3x^2 - 10x + 3)}\} + C.$$

Ex. 3. $$\int\frac{dx}{\sqrt{3x^2 - 10x + 9}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(x - \frac{5}{3})^2 + \frac{2}{9}}} = \frac{1}{\sqrt{3}}\int\frac{dz}{\sqrt{z^2 + (\sqrt{2}/3)^2}}$$

$$= \frac{1}{\sqrt{3}}\sinh^{-1}\frac{3z}{\sqrt{2}},\ \text{or}\ \frac{1}{\sqrt{3}}\log(3z + \sqrt{9z^2 + 2}) + C$$

$$= \frac{1}{\sqrt{3}}\sinh^{-1}\frac{3x - 5}{\sqrt{2}},\ \text{or}\ \frac{1}{\sqrt{3}}\log\{3x - 5 + \sqrt{3(3x^2 - 10x + 9)}\} + C.$$

Ex. 4. $$\int\frac{dx}{\sqrt{3x^2 + 6x + 3}} = \frac{1}{\sqrt{3}}\int\frac{dx}{x + 1} = \frac{1}{\sqrt{3}}\log(x + 1).$$

In these examples we have given all the different cases which might occur (except, perhaps, that in which the expression under the radical sign is the sum of two negative squares, which gives an imaginary result).

We now take the general form.

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† We shall denote by C some constant, the actual value of which is not material. Moreover, *if C be used twice in the same example, the two do not necessarily denote the same constant.*

344. General Form.

We have $ax^2 + 2hx + b = a\left\{\left(x+\dfrac{h}{a}\right)^2 + \dfrac{ab-h^2}{a^2}\right\}$.

Case I.—Let a be $+^{ve}$, and $h^2 > ab$, so that $ax^2 + 2hx + b$ breaks up into two real linear factors, or can be expressed as the difference of two squares.

Then $I = \dfrac{1}{\sqrt{a}}\displaystyle\int \dfrac{dx}{\sqrt{\left(x+\dfrac{h}{a}\right)^2 - \dfrac{h^2-ab}{a^2}}} = \dfrac{1}{\sqrt{a}}\int\dfrac{dz}{\sqrt{z^2-m^2}}$,

if $z = x + \dfrac{h}{a}$; and $m^2 = \dfrac{h^2-ab}{a^2}$, *the latter being* $+^{ve}$,

$$= \frac{1}{\sqrt{a}}\cosh^{-1}\frac{z}{m} = \frac{1}{\sqrt{a}}\cosh^{-1}\frac{ax+h}{\sqrt{h^2-ab}},$$

or,
$$= \frac{1}{\sqrt{a}}\log(z+\sqrt{z^2-m^2}) - \frac{1}{\sqrt{a}}\log m,$$

$$= \frac{1}{\sqrt{a}}\log\left(\frac{ax+h}{a} + \sqrt{\frac{(ax+h)^2-(h^2-ab)}{a^2}}\right) + C$$

$$= \frac{1}{\sqrt{a}}\log\{ax+h+\sqrt{a(ax^2+2hx+b)}\} + C.$$

[See footnote, Art. 343.]

Case II.—Let a be $+^{ve}$, and $ab > h^2$, so that $ax^2 + 2hx + b$ does not break up into real linear factors, *i.e.* can only be expressed as the sum of two squares.

Then $I = \dfrac{1}{\sqrt{a}}\displaystyle\int \dfrac{dx}{\sqrt{\left(x+\dfrac{h}{a}\right)^2 + \dfrac{ab-h^2}{a^2}}} = \dfrac{1}{\sqrt{a}}\int\dfrac{dz}{\sqrt{z^2+m_1^2}}$,

if $z = x + \dfrac{h}{a}$; and $m_1^2 = \dfrac{ab-h^2}{a^2}$, *the latter being* $+^{ve}$;

and by reasoning similar to the above this becomes

$$\frac{1}{\sqrt{a}}\sinh^{-1}\frac{ax+h}{\sqrt{ab-h^2}}, \text{ or } \frac{1}{\sqrt{a}}\log\{ax+h+\sqrt{a(ax^2+2hx+b)}\} + C.$$

NOTE.—So long as a is $+^{ve}$, the logarithm form of answer is the same whether $h^2 >$ or $< ab$.

Case III.—Let a be $+^{ve}$, and $ab = h^2$, so that $ax^2 + 2hx + b$ is a times a perfect square.

$$\text{Then } I = \frac{1}{\sqrt{a}}\int \frac{dx}{x + \frac{h}{a}} = \frac{1}{\sqrt{a}} \log \frac{ax + h}{a} = \frac{1}{\sqrt{a}} \log (ax + h) + C.$$

This can be deduced from either of the preceding cases.

Thus, in case I, $m = 0$;

$$\therefore I = \frac{1}{\sqrt{a}} \log (z + \sqrt{z^2 - m^2}), \text{ omitting the constant,}$$

$$= \frac{1}{\sqrt{a}} \log 2z = \frac{1}{\sqrt{a}} \log z + C = \frac{1}{\sqrt{a}} \log (ax + h) + C.$$

It is true that the constant $-\frac{1}{\sqrt{a}} \log m$ given in Case I. becomes infinite, but this counts for nothing as there is already omitted another constant which may have any value from $+\infty$ to $-\infty$.

Case IV.—Let a be $-^{ve}$, and $h^2 > ab$.

Then taking the $+^{ve}$ factor $-a$ outside, we have

$$ax^2 + 2hx + b = -a\left\{-x^2 - \frac{2hx}{a} - \frac{b}{a}\right\}$$

$$= -a\left\{-\left(x + \frac{h}{a}\right)^2 + \frac{h^2 - ab}{a^2}\right\} = -a(m^2 - z^2)$$

if $z = x + \frac{h}{a}$; and $m^2 = \frac{h^2 - ab}{a^2}$, *the latter being* $+^{ve}$.

$$\therefore I = \frac{1}{\sqrt{-a}}\int \frac{dz}{\sqrt{m^2 - z^2}} = \frac{1}{\sqrt{-a}} \sin^{-1}\frac{z}{m} = \frac{1}{\sqrt{-a}} \sin^{-1} \frac{ax + h}{\sqrt{h^2 - ab}}.$$

Case V.—If a is $-^{ve}$, and $ab > h^2$, then $ax^2 + 2hx + b$ is only expressible as the sum of two $-^{ve}$ squares. Hence I is imaginary, its form being the same as in Case II.

345. Included Forms—$\displaystyle\int \frac{dx}{\sqrt{(a-x)(x-b)}}$.

This, of course, is included in the form we are considering; but it will be an instructive exercise to discuss it separately.

Now $(a-x)(x-b)$ is bound to be expressible as the difference of two *perfect* squares; we shall therefore get a rational answer. As the coefficient of x^2 is $-^{ve}$ the resulting form will be

$$\int \frac{dz}{\sqrt{a^2-z^2}} \text{ or } \sin^{-1}\frac{z}{a}.$$

Thus $(a-x)(x-b)$

$$= -x^2 + x(a+b) - ab$$

$$= -\left\{\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2\right\}$$

$$= \left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2 = m^2 - z^2 \text{ say, so that } dx = dz;$$

$$\therefore I = \int \frac{dz}{\sqrt{m^2-z^2}} = \sin^{-1}\frac{z}{m} = \sin^{-1}\frac{2x-a-b}{a-b}.$$

***346.** The following method is worthy of notice:—

First, suppose $a > b$, so that $a-b$ is $+^{ve}$. Then

$$I = \int \frac{dx}{\sqrt{(a-b+b-x)(x-b)}} = \int \frac{dx}{\sqrt{x-b}\sqrt{(a-b)-(x-b)}}.$$

Now let $\sqrt{x-b} = z$; $\quad \therefore \dfrac{dx}{2\sqrt{x-b}} = dz.$

$$\therefore I = 2\int \frac{dz}{\sqrt{(a-b)-z^2}} = 2\sin^{-1}\frac{z}{\sqrt{a-b}} = 2\sin^{-1}\sqrt{\frac{x-b}{a-b}};$$

and noting that $2\sin^{-1}a = \sin^{-1}2a\sqrt{1-a^2}$, this becomes

$$\sin^{-1}\left\{2\sqrt{\frac{x-b}{a-b}}\sqrt{1-\frac{x-b}{a-b}}\right\} = \sin^{-1}\frac{2\sqrt{(a-x)(x-b)}}{a-b}.$$

Next suppose $b > a$, and we shall get

$$I = \int \frac{dx}{\sqrt{x-a}\sqrt{(b-a)-(x-a)}},$$

which will become

$$2\sin^{-1}\sqrt{\frac{x-a}{b-a}} = \sin^{-1}\frac{2\sqrt{(a-x)(x-b)}}{b-a}.$$

***347.** These last two results apparently disagree with the first one.

Now, it must be remembered that (besides the consideration of the omitted constant) the inverse sine is a many-valued function.

Let $\sin^{-1} v$ stand for the *principal value* in the first result, *i.e.* the smallest $+^{ve}$ angle whose sine is v, where $v = \dfrac{2x - a - b}{a - b}$. If $\text{Sin}^{-1} v$ denote the *general* value of $\sin^{-1} v$, we have

$$\text{Sin}^{-1} v = n\pi + (-1)^n \sin^{-1} v,$$

which may be written four different ways:—

$$\begin{aligned}
(1) &= C + \sin^{-1} v,\\
(2) &= C - \sin^{-1} v,\\
(3) &= C + \frac{\pi}{2} - \sin^{-1} v = C + \cos^{-1} v,\\
(4) &= C - \frac{\pi}{2} + \sin^{-1} v = C - \cos^{-1} v.
\end{aligned}$$

Hence, in reality, neglecting the constant, our first result should be written $\pm \sin^{-1} v$, or $\pm \cos^{-1} v$.

$$\begin{aligned}
\text{Also, } \cos^{-1} v = \cos^{-1}\frac{2x - a - b}{a - b} &= \sin^{-1}\sqrt{1 - \left(\frac{2x - a - b}{a - b}\right)^2}\\
&= \sin^{-1}\sqrt{\left(1 - \frac{2x - a - b}{a - b}\right)\left(1 + \frac{2x - a - b}{a - b}\right)}\\
&= \sin^{-1}\frac{2\sqrt{(a - x)(x - b)}}{a \sim b},
\end{aligned}$$

which agrees with the two latter results.

348. A third method is to put $x = a\cos^2\theta + b\sin^2\theta$.

Then $(a - x)(x - b)$

$$= (a - b)\sin^2\theta . (a - b)\cos^2\theta = (a - b)^2 \sin^2\theta\cos^2\theta;$$

$$dx = (-2a\cos\theta\sin\theta + 2b\sin\theta\cos\theta)d\theta = 2(b - a)\sin\theta\cos\theta\, d\theta,$$

$$\therefore I = -2\int d\theta = -2\theta.$$

Now $2x = a(1 + \cos 2\theta) + b(1 - \cos 2\theta) = a + b + (a - b)\cos 2\theta$,

$$\therefore \cos 2\theta = \frac{2x - a - b}{a - b};$$

$$\therefore I = -2\theta = -\cos^{-1}\frac{2x - a - b}{a - b},$$

which, by Art. 347, agrees with the previous results.

349. $\displaystyle\int\frac{dx}{\sqrt{(x-a)(x-b)}}.$

Since $(x-a)(x-b)$ is expressible as the difference of two squares, and the coefficient of x^2 is $+^{\text{ve}}$, this must lead to the inverse hyperbolic cosine.

Thus $I=\displaystyle\int dx\Big/\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}=\cosh^{-1}\frac{2x-a-b}{a-b},$

or $$\log\{2x-a-b+2\sqrt{(x-a)(x-b)}\}+C.$$

The latter may be written

$$\log\{(x-a)+(x-b)+2\sqrt{(x-a)(x-b)}\}=2\log(\sqrt{x-a}+\sqrt{x-b}).$$

Otherwise, $I=\displaystyle\int\frac{dx}{\sqrt{x-a}\sqrt{(x-a)+(a-b)}}$, supposing $a>b$,

$$=2\int\frac{dz}{\sqrt{z^2+a-b}},\text{ if }\sqrt{x-a}=z,\ \therefore\ \frac{dx}{2\sqrt{x-a}}=dz,$$

$$=2\sinh^{-1}\frac{z}{\sqrt{a-b}}=2\sinh^{-1}\sqrt{\frac{x-a}{a-b}},$$

or $2\log\left\{\sqrt{\dfrac{x-a}{a-b}}+\sqrt{\dfrac{x-a}{a-b}+1}\right\}$, *i.e.* $2\log(\sqrt{x-a}+\sqrt{x-b})+C.$

Also, since $\cosh 2u=2\sinh^2u+1$,

$$\therefore\ 2u=\cosh^{-1}(2\sinh^2u+1);$$

$$\therefore\ 2\sinh^{-1}\sqrt{\frac{x-a}{a-b}}=\cosh^{-1}\left(2\frac{x-a}{a-b}+1\right)=\cosh^{-1}\frac{2x-a-b}{a-b},$$

as before.

The integral may also be obtained by putting

$$x=a\sec^2\theta-b\tan^2\theta,\ \text{ or }x=a\cosh^2u-b\sinh^2u.$$

EXAMPLES LV.

1. Integrate :—

(1) $\dfrac{dx}{\sqrt{x^2-4x+3}}.$ (2) $\dfrac{dx}{\sqrt{3+4x-4x^2}}.$

(3) $\dfrac{dx}{\sqrt{x^2-4x+5}}$.
(4) $\dfrac{dx}{\sqrt{(x-3)(5-x)}}$.

(5) $\dfrac{dx}{\sqrt{(x-3)(x-5)}}$.
(6) $\dfrac{dx}{\sqrt{x^2+2x}}$.

(7) $\dfrac{dx}{\sqrt{(2-x)x}}$.
(8) $\dfrac{dx}{\sqrt{5x^2-12x+4}}$.

(9) $\dfrac{dx}{\sqrt{4+8x-5x^2}}$.
(10) $\dfrac{dx}{\sqrt{2x^2-5x-1}}$.

(11) $\dfrac{dx}{\sqrt{4x^2+12x+9}}$.
(12) $\dfrac{dx}{\sqrt{(2x-3a)(3x-2a)}}$.

(13) $\dfrac{dx}{\sqrt{3x^2-2ax+a^2}}$.
(14) $\dfrac{dx}{\sqrt{x^2+2px+q}}$.

(15) $\dfrac{dx}{\sqrt{(x-a)(x+b)}}$.
(16) $\dfrac{dx}{\sqrt{(ax+b)(bx+a)}}$.

2. Evaluate $\displaystyle\int\frac{x\,dx}{\sqrt{(x^2-1)(x^2-2)}}$ by putting

(1) $x^2=z$; (2) $\sqrt{x^2-1}=z$;

and show that the two results agree.

3. Evaluate $\displaystyle\int\frac{dx}{\sqrt{(x+a)(x+b)}}$, $(a>b)$, by putting

(1) $x+a=z^2$; (2) $x=a\tan^2\theta-b\sec^2\theta$;

(3) $x=a\sinh^2 u-b\cosh^2 u$;

and show that the results agree.

4. If α, β be the roots of $ax^2+2hx+b=0$, prove that

$$\int\frac{dx}{\sqrt{ax^2+2hx+b}}=\frac{2}{\sqrt{a}}\log(\sqrt{x-\alpha}+\sqrt{x-\beta}),$$

or $\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{2x-\alpha-\beta}{\alpha-\beta}$, according as a is $+^{ve}$ or $-^{ve}$.

Show that these results agree with those of Art. 344.

5. Integrate (1) $\displaystyle\sqrt{\frac{x-a}{x-b}}\,dx$; (2) $\displaystyle\sqrt{\frac{x-a}{b-x}}\,dx$.

ANSWERS.

1. (1) $\cosh^{-1}(x-2)$. (2) $\frac{1}{2}\sin^{-1}\dfrac{2x-1}{2}$. (3) $\sinh^{-1}(x$

(4) $\sin^{-1}(x-4)$. (5) $\cosh^{-1}(x-4)$. (6) $\cosh^{-1}(x+1)$.

(7) $\sin^{-1}(x-1)$. (8) $\frac{1}{\sqrt{5}}\cosh^{-1}\frac{5x-6}{4}$. (9) $\frac{1}{\sqrt{5}}\sin^{-1}\frac{5x-4}{6}$.

(10) $\frac{1}{\sqrt{2}}\cosh^{-1}\frac{4x-5}{\sqrt{33}}$. (11) $\frac{1}{2}\log(2x+3)$.

(12) $\frac{1}{\sqrt{6}}\cosh^{-1}\frac{12x-13a}{5a}$. (13) $\frac{1}{\sqrt{3}}\sinh^{-1}\frac{3x-a}{a\sqrt{2}}$.

(14) $\log(x+p+\sqrt{x^2+2px+q})$.

(15) $2\log(\sqrt{x-a}+\sqrt{x+b})=2\sinh^{-1}\sqrt{\frac{x-a}{a+b}}=\cosh^{-1}\frac{2x-a+b}{a+b}$.

$$(16)\ \frac{1}{\sqrt{ab}}\cosh^{-1}\frac{2abx+a^2+b^2}{a^2-b^2}.$$

2. $\frac{1}{2}\cosh^{-1}(2x^2-3)=\cosh^{-1}\sqrt{x^2-1}$; for each $=\log(\sqrt{x^2-1}+\sqrt{x^2-2})$.

5. (1) $\sqrt{(x-a)(x-b)}-(a-b)\log(\sqrt{x-a}+\sqrt{x-b})$.

(2) $-\sqrt{(x-a)(b-x)}-\frac{1}{2}(a-b)\sin^{-1}\frac{2x-a-b}{a-b}$.

350. Integration of $\sqrt{ax^2+2hx+b}\,dx$.

We can show that this integral is reducible to one or other of the three forms: —

$$\dagger\ (a)\ \int\sqrt{a^2-z^2}\,dz=\frac{z\sqrt{a^2-z^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{z}{a},$$

$$(b)\ \int\sqrt{a^2+z^2}\,dz=\frac{z\sqrt{a^2+z^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{z}{a},$$

$$(c)\ \int\sqrt{z^2-a^2}\,dz=\frac{z\sqrt{z^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\frac{z}{a}.$$

The discussion, however, is exactly the same as before, and we shall therefore only work examples.

Ex. 1. $\int\sqrt{(x-1)(x-3)}\,dx$

$=\int\sqrt{(x-2)^2-1}\,dx=\int\sqrt{z^2-1}\,dz$, if $z=x-2$,

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† See Art. 334, Exs. 5, 6, and 7.

$$= \frac{z\sqrt{z^2-1}}{2} - \tfrac{1}{2}\cosh^{-1} z$$

$$= \frac{(x-2)\sqrt{(x-1)(x-3)}}{2} - \tfrac{1}{2}\cosh^{-1}(x-2)$$

$$= \frac{(x-2)\sqrt{(x-1)(x-3)}}{2} - \tfrac{1}{2}\log\{x-2+\sqrt{(x-1)(x-3)}\}.$$

Ex. 2. $\int\sqrt{2-3x-4x^2}\,dx$

$$= \int\sqrt{\tfrac{41}{16}-(2x+\tfrac{3}{4})^2}\,dx = \tfrac{1}{2}\int\sqrt{\tfrac{41}{16}-z^2}\,dz, \text{ if } z = 2x+\tfrac{3}{4},$$

$$= \tfrac{1}{2}\left\{\tfrac{1}{2}z\sqrt{\tfrac{41}{16}-z^2} + \tfrac{1}{2}\cdot\tfrac{41}{16}\sin^{-1}\left(z\Big/\frac{\sqrt{41}}{4}\right)\right\}$$

$$= \frac{z}{4}\sqrt{\frac{41}{16}-z^2} + \frac{41}{32}\sin^{-1}\frac{4z}{\sqrt{41}}$$

$$= \frac{8x+3}{16}\sqrt{2-3x-4x^2} + \frac{41}{32}\sin^{-1}\frac{8x+3}{\sqrt{41}}.$$

351. Integration of

$$\frac{dx}{x\sqrt{ax^2+2hx+b}}, \text{ and } \frac{dx}{(px+q)\sqrt{ax^2+2hx+b}}.$$

In the first integral put $x = \frac{1}{z}$; $\therefore dx = -\frac{1}{z^2}dz$.

$$\therefore \int\frac{dx}{x\sqrt{ax^2+2hx+b}} = \int\frac{dx}{x^2\sqrt{a+\frac{2h}{x}+\frac{b}{x^2}}} = -\int\frac{dz}{\sqrt{a+2hz+bz^2}}$$

which is of the form considered above. [Art. 342.]

In the second integral, first put $px+q=y$; then $ax^2+2hx+b$ reduces to *some* quadratic expression in y, say $ly^2+2my+n$.

$$\therefore \int\frac{dx}{(px+q)\sqrt{ax^2+2hx+b}} = \frac{1}{p}\int\frac{dy}{y\sqrt{ly^2+2my+n}},$$

which is reduced to the previous form; and putting $y = 1/z$ we reduce it again to the form $\int\frac{dz}{\sqrt{az^2+2hz+b}}$. This is equivalent to putting at once $px+q=1/z$.

Ex. 1. $\int \frac{dx}{x\sqrt{(2x-1)(3-x)}}.$

Put $x = \frac{1}{z}$; $\therefore dx = -\frac{1}{z^2}dz.$

$$\therefore I = -\int \frac{dz}{\sqrt{(2-z)(3z-1)}} = -\frac{1}{\sqrt{3}}\sin^{-1}\frac{6z-7}{5} \text{ [by Ex. 1, Art. 343]}$$

$$= -\frac{1}{\sqrt{3}}\sin^{-1}\frac{6-7x}{5x}.$$

Ex. 2. $\int \frac{dx}{(2x-3)\sqrt{2x^2-3x+4}}.$

Put $2x-3 = \frac{1}{z}$; $\therefore dx = -\frac{1}{2z^2}dz.$

Also $2x^2-3x+4 = \frac{1}{2}\cdot 2x(2x-3)+4 = \frac{1}{2}\left(3+\frac{1}{z}\right)\frac{1}{z}+4$

$$= \frac{3z+1}{2z^2}+4 = \frac{8z^2+3z+1}{2z^2}.$$

$$\therefore I = -\tfrac{1}{2}\int \frac{dz}{z^2\cdot\frac{1}{z}\sqrt{\frac{8z^2+3z+1}{2z^2}}} = -\frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{8z^2+3z+1}}$$

$$= -\frac{1}{\sqrt{2}}\int\frac{dz}{2\sqrt{2}\sqrt{z^2+\frac{3}{8}z+\frac{1}{8}}} = -\frac{1}{4}\int\frac{dz}{\sqrt{(z+\frac{3}{16})^2+\frac{23}{256}}}$$

$$= -\frac{1}{4}\sinh^{-1}\frac{16z+3}{\sqrt{23}} = -\frac{1}{4}\sinh^{-1}\frac{16+3(2x-3)}{\sqrt{23}(2x-3)}$$

$$= -\frac{1}{4}\sinh^{-1}\frac{6x+7}{\sqrt{23}(2x-3)}.$$

352. Integration of $\frac{px+q}{\sqrt{ax^2+2hx+b}}dx.$

Since $d(ax^2+2hx+b) = 2(ax+h)\,dx$, we shall write

$$\frac{px+q}{\sqrt{ax^2+2hx+b}} = k.\frac{2(ax+h)}{\sqrt{ax^2+2hx+b}} + l.\frac{1}{\sqrt{ax^2+2hx+b}},$$

k being chosen so that the numerator of the first fraction contains the term px, in consequence of which that of the second fraction is without x, and is therefore of the form considered in Art. 342.

To find k and l, compare coefficients in the numerators of the two sides.

We have $px + q = 2k(ax + h) + l$.

$$\therefore p = 2ka, \text{ or } k = p/2a,$$

$$q = 2kh + l, \text{ or } l = q - 2kh = q - \frac{hp}{a} = \frac{aq - hp}{a}.$$

Or, at once, $px + q = \dfrac{p}{2a}(2ax) + q = \dfrac{p}{2a}2(ax + h) - \dfrac{2hp}{2a} + q$

$$= \frac{p}{2a}\cdot 2(ax + h) + \frac{aq - hp}{a}.$$

$$\therefore I = k\int \frac{2(ax + h)dx}{\sqrt{ax^2 + 2hx + b}} + l\int \frac{dx}{\sqrt{ax^2 + 2hx + b}}.$$

The first integral $= k\displaystyle\int \frac{dz}{\sqrt{z}}$, if $z = ax^2 + 2hx + b$,

$$= 2k\sqrt{z} = \frac{p}{a}\sqrt{ax^2 + 2hx + b}.$$

The second is found as above.

In practice k and l can often be guessed at once.

Ex. 1. $\displaystyle\int \frac{4x - 3}{\sqrt{x^2 - 2x + 5}}\,dx = 2\int \frac{(2x - 2)dx}{\sqrt{x^2 - 2x + 5}} + \int \frac{dx}{\sqrt{x^2 - 2x + 5}}$

$$= 2\int \frac{dz}{\sqrt{z}} + \int \frac{dx}{\sqrt{(x - 1)^2 + 2^2}}, \text{ if } z = x^2 - 2x + 5,$$

$$= 4\sqrt{x^2 - 2x + 5} + \sinh^{-1}\frac{x - 1}{2}.$$

Ex. 2. $\displaystyle\int \frac{8x^2 - 14x + 1}{\sqrt{x(1 - x)}}\,dx = 2\int \frac{8x^2 - 14x + 1}{\sqrt{1 - (2x - 1)^2}}\,dx.$

Put $2x - 1 = y$; $\therefore dx = \frac{1}{2}dy$, and $x = \frac{1}{2}(y + 1)$.

$$\therefore I = 2\int \frac{2(y + 1)^2 - 7(y + 1) + 1}{\sqrt{1 - y^2}}\cdot\frac{dy}{2} = \int \frac{2y^2 - 3y - 4}{\sqrt{1 - y^2}}\,dy$$

$$= \int \frac{2(y^2 - 1) - 3y - 2}{\sqrt{1 - y^2}}\,dy$$

$$= -2\int \sqrt{1 - y^2}\,dy - 3\int \frac{y\,dy}{\sqrt{1 - y^2}} - 2\int \frac{dy}{\sqrt{1 - y^2}}$$

$$= -y\sqrt{1-y^2} - \sin^{-1} y + 3\sqrt{1-y^2} - 2\sin^{-1} y$$
$$= (3-y)\sqrt{1-y^2} - 3\sin^{-1} y$$
$$= (4-2x)\sqrt{4x(1-x)} - 3\sin^{-1}(2x-1)$$
$$= 4(2-x)\sqrt{x(1-x)} - 3\sin^{-1}(2x-1).$$

Ex. 3. $\displaystyle\int \frac{2x^2 - x - 2}{(x-2)\sqrt{x^2-2x+5}}\,dx.$

Let $x^2 - 2x + 5 = R$, for brevity.

Then, by division, $\dfrac{2x^2 - x - 2}{x-2} = 2x + 3 + \dfrac{4}{x-2}.$

$$\therefore I = \int \frac{2x+3}{\sqrt{R}}\,dx + 4\int \frac{dx}{(x-2)\sqrt{R}}$$
$$= A + B \text{ say.}$$

Now, since $dR = (2x-2)dx$,

$$A = \int \frac{(2x-2)+5}{\sqrt{R}}\,dx = \int \frac{dR}{\sqrt{R}} + 5\int \frac{dx}{\sqrt{(x-1)^2+2^2}}$$
$$= 2\sqrt{R} + 5\sinh^{-1}\frac{x-1}{2}.$$

In B put $x - 2 = y$, $\therefore x^2 - 2x + 5 = xy + 5 = y^2 + 2y + 5.$

$$\therefore B = 4\int \frac{dy}{y\sqrt{y^2+2y+5}} = -4\int \frac{d(1/y)}{\sqrt{1 + \dfrac{2}{y} + \dfrac{5}{y^2}}} = -4\int \frac{dz}{\sqrt{5z^2+2z+1}}$$
$$= -\frac{4}{\sqrt{5}}\int \frac{dz}{\sqrt{(z+\frac{1}{5})^2 + \frac{4}{25}}} = -\frac{4}{\sqrt{5}}\sinh^{-1}\frac{5z+1}{2}, \text{ where } z = \frac{1}{x-2},$$
$$= -\frac{4}{\sqrt{5}}\sinh^{-1}\frac{x+3}{2(x-2)}.$$
$$\therefore I = 2\sqrt{x^2-2x+5} + 5\sinh^{-1}\frac{x-1}{2} - \frac{4}{\sqrt{5}}\sinh^{-1}\frac{x+3}{2(x-2)}$$
$$= 2\sqrt{R} + 5\log(x - 1 + \sqrt{R}) - \frac{4}{\sqrt{5}}\log\frac{x+3+\sqrt{5R}}{x-2} + C.$$

EXAMPLES LVI.

Integrate the following :—

1\. (1) $\sqrt{x^2 - 6x + 5}\,dx.$ (2) $\sqrt{2ax - x^2}\,dx.$

(3) $\sqrt{(x-a)^2 + (x-b)^2}\,dx.$

2. (1) $\dfrac{dx}{x\sqrt{x^2-4}}$. (2) $\dfrac{dx}{x\sqrt{5x^2-4x+1}}$.

(3) $\dfrac{dx}{x\sqrt{-(2x+1)(x+1)}}$. (4) $\dfrac{dx}{(2x+3)\sqrt{(x+1)(x+2)}}$.

(5) $\dfrac{dx}{(x-2)\sqrt{x-x^2}}$. (6) $\dfrac{dx}{x^{\frac{1}{2}}(x-1)^{\frac{3}{2}}}$.

3. (1) $\dfrac{2x-3}{\sqrt{x^2-2x+5}}\,dx$. (2) $\dfrac{x\,dx}{\sqrt{(x-2)(3-x)}}$.

(3) $\dfrac{x+2}{x\sqrt{x^2-1}}\,dx$. (4) $\dfrac{x^2-1}{x\sqrt{x^2+1}}\,dx$.

4. (1) $\dfrac{(x+1)^2\,dx}{x\sqrt{x^2-x}}$. (2) $\left(\dfrac{x+1}{x}\right)^{\frac{1}{2}}dx$.

(3) $\dfrac{(x-2)(x-3)\,dx}{\sqrt{(x+2)(x+3)}}$. (4) $\dfrac{x^2+x+1}{(x+2)\sqrt{x^2-4x+3}}\,dx$.

5. (1) $(x+1)\sqrt{x^2+1}\,dx$. (2) $(x+2)\sqrt{x^2-2x-3}\,dx$.

(3) $(px+q)\sqrt{ax^2+2hx+b}\,dx$.

ANSWERS.

1. (1) $\frac{1}{2}(x-3)\sqrt{x^2-6x+5}-2\cosh^{-1}\dfrac{x-3}{2}$.

(2) $\frac{1}{2}(x-a)\sqrt{2ax-x^2}+\frac{1}{2}a^2\sin^{-1}\dfrac{x-a}{a}$.

(3) $\dfrac{2x-a-b}{4}\sqrt{(x-a)^2+(x-b)^2}+\dfrac{(a-b)^2}{4\sqrt{2}}\sinh^{-1}\dfrac{2x-a-b}{a-b}$.

2. (1) $\dfrac{1}{2}\cos^{-1}\dfrac{2}{x}$. (2) $\sinh^{-1}\dfrac{2x-1}{x}$. (3) $\cos^{-1}\dfrac{2+3x}{x}$.

(4) $\sec^{-1}(2x+3)$. (5) $\dfrac{1}{\sqrt{2}}\cos^{-1}\dfrac{3x-2}{x-2}$. (6) $-2\sqrt{\dfrac{x}{x-1}}$.

3. (1) $2\sqrt{x^2-2x+5}-\sinh^{-1}\dfrac{x-1}{2}$.

(2) $-\sqrt{(x-2)(3-x)}+\frac{5}{2}\sin^{-1}(2x-5)$.

(3) $\cosh^{-1}x+2\sec^{-1}x$. (4) $\sqrt{x^2+1}+\operatorname{cosech}^{-1}x$.

4. (1) $\sqrt{x^2 - x} + 2\sqrt{\dfrac{x-1}{x}} + \dfrac{5}{2}\cosh^{-1}(2x - 1).$

(2) $\sqrt{x^2 + x} - 2\sqrt{\dfrac{x+1}{x}} + \dfrac{3}{2}\cosh^{-1}(2x + 1).$

(3) $\dfrac{2x - 35}{4}\sqrt{(x+2)(x+3)} + \dfrac{199}{8}\cosh^{-1}(2x + 5).$

(4) $\sqrt{x^2 - 4x + 3} + \cosh^{-1}(x - 2) - \sqrt{\dfrac{3}{5}}\cosh^{-1}\dfrac{7 - 4x}{x + 2}.$

5. (1) $\frac{1}{6}(2x^2 + 3x + 2)\sqrt{x^2 + 1} + \frac{1}{2}\sinh^{-1} x.$

(2) $\frac{1}{6}(2x^2 + 5x - 15)\sqrt{x^2 - 2x - 3} - 6\cosh^{-1}\dfrac{x-1}{2}.$

(3) $\dfrac{p}{3a}(ax^2 + 2hx + b)^{\frac{3}{2}} + \dfrac{aq - hp}{a}\int\sqrt{ax^2 + 2hx + b}\,dx.$

353. Integration of $\dfrac{dx}{ax^2 + 2hx + b}$.

We shall show that this integral is reducible to one or other of he three forms :—

$$(a)\quad \int\frac{dz}{a^2 + z^2} = \frac{1}{a}\tan^{-1}\frac{z}{a},$$

$$(b)\quad \int\frac{dz}{a^2 - z^2} = \frac{1}{a}\tanh^{-1}\frac{z}{a} = \frac{1}{2a}\log\frac{a + z}{a - z}\ [z < a],$$

$$(c)\quad \int\frac{dz}{z^2 - a^2} = -\frac{1}{a}\coth^{-1}\frac{z}{a} = \frac{1}{2a}\log\frac{z - a}{z + a}\ [z > a];$$

ıe case in which $ax^2 + 2hx + b$ is a perfect square being ‹cepted.

354. Method of Partial Fractions.

—Before taking the ›neral form we shall adopt this method to prove the above ›rmulæ, although they belong to the fundamental forms. The ethod itself will be more fully discussed below (Chapter XXIV.)

Taking (*b*) first, we have, using x instead of z,

$$\frac{1}{a^2 - x^2} = \frac{1}{(a - x)(a + x)} = \frac{1}{2a}\left\{\frac{1}{a - x} + \frac{1}{a + x}\right\},$$

in which the expression is broken up into two partial fractions, the numerators being found by rough trial.

$$\therefore \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int \frac{dx}{a - x} + \frac{1}{2a}\int \frac{dx}{a + x}$$

$$= -\frac{1}{2a}\log(a - x) + \frac{1}{2a}\log(a + x)$$

$$= \frac{1}{2a}\log\frac{a + x}{a - x}\dagger \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{or } = \frac{1}{a}\tanh^{-1}\frac{x}{a}.$$

Similarly in (c),

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \frac{dx}{x - a} - \frac{1}{2a}\int \frac{dx}{x + a} = \frac{1}{2a}\log\frac{x - a}{x + a}\dagger \quad . \quad . \quad (2)$$

$$\text{or } = -\frac{1}{a}\coth^{-1}\frac{x}{a}.$$

***355.** This method may also be adopted in the case of (a).

Thus $\dfrac{1}{a^2 + x^2} = \dfrac{1}{(a + xi)(a - xi)} = \dfrac{1}{2a}\left(\dfrac{1}{a + xi} + \dfrac{1}{a - xi}\right)$;

$$\therefore \int \frac{dx}{a^2 + x^2} = \frac{1}{2ai}\{\log(a + xi) - \log(a - xi)\}$$

$$= \frac{1}{2ai}\log\frac{a + xi}{a - xi} = I, \text{ say.}$$

$$\therefore \frac{a + xi}{a - xi} = e^{2aIi} = \frac{e^{aIi}}{e^{-aIi}}.$$

Hence, componendo and dividendo,

$$\frac{xi}{a} = \frac{e^{aIi} - e^{-aIi}}{e^{aIi} + e^{-aIi}} = i\tan aI.$$

$$\therefore I = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

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† To aid the memory, it should be noticed in (1) that "*on both sides, a* comes before x (alphabetical order); and *addition* in the numerator comes before *subtraction* in the denominator (as in learning Arithmetic)." In (2) *both conditions are reversed*, i.e. a comes *after* x, and addition comes *after* subtraction. The coefficient $1/2a$ must not be forgotten.

We shall get the same result if we call $a^2 + x^2 = (x + ai)(x - ai)$, and remember that $\tan^{-1}\dfrac{x}{a}$ is many-valued.

Otherwise :—

$$\frac{1}{2ai}\log\frac{a+xi}{a-xi} = \frac{1}{ai}\,\frac{1}{2}\log\frac{1+\dfrac{xi}{a}}{1-\dfrac{xi}{a}} = \frac{1}{ai}\tanh^{-1}\frac{xi}{a} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

[See Exs. XIV., 3.]

356. General Form.—We have

$$\int\frac{dx}{ax^2+2hx+b} = \frac{1}{a}\int\frac{dx}{\left(x+\dfrac{h}{a}\right)^2+\dfrac{ab-h^2}{a^2}} = \frac{1}{a}\int\frac{dz}{z^2+\dfrac{ab-h^2}{a^2}}.$$

Case I.—If $ab > h^2$, or $ax^2 + 2hx + b = 0$ has imaginary roots, then

$$I = \frac{1}{a}\cdot\frac{a}{\sqrt{ab-h^2}}\tan^{-1}\frac{az}{\sqrt{ab-h^2}} = \frac{1}{\sqrt{ab-h^2}}\tan^{-1}\frac{ax+h}{\sqrt{ab-h^2}}.$$

Case II.—If $ab < h^2$, or $ax^2 + 2hx + b = 0$ has real roots (α and β say), then

$$I = \frac{1}{a}\int\frac{dz}{z^2-\dfrac{h^2-ab}{a^2}}.$$

First, suppose $z^2 > \dfrac{h^2-ab}{a^2}$; that is, $z^2 - \dfrac{h^2-ab}{a^2}$ is $+^{\text{ve}}$.

But this

$$= \left(z - \frac{\sqrt{h^2-ab}}{a}\right)\left(z + \frac{\sqrt{h^2-ab}}{a}\right)$$

$$= \left(x + \frac{h-\sqrt{h^2-ab}}{a}\right)\left(x + \frac{h+\sqrt{h^2-ab}}{a}\right) = (x-\alpha)(x-\beta).$$

Hence $(x-\alpha)(x-\beta)$ is $+^{\text{ve}}$, *i.e.* x *does not lie between* α *and* β. Then

$$I = \frac{1}{a}\,\frac{-a}{\sqrt{h^2-ab}}\coth^{-1}\frac{az}{\sqrt{h^2-ab}} = \frac{-1}{\sqrt{h^2-ab}}\coth^{-1}\frac{ax+h}{\sqrt{h^2-ab}};$$

$$\text{or, } = \frac{1}{2\sqrt{h^2-ab}}\log\frac{ax+h-\sqrt{h^2-ab}}{ax+h+\sqrt{h^2-ab}}.$$

Secondly, suppose $z^2 < \frac{h^2 - ab}{a^2}$, so that $(x - \alpha)(x - \beta)$ is $-^{ve}$;

i.e. x *does lie between* α *and* β.

$$\text{Then } I = -\frac{1}{a}\int \frac{dz}{\frac{h^2 - ab}{a^2} - z^2} = -\frac{1}{a}\frac{a}{\sqrt{h^2 - ab}}\tanh^{-1}\frac{az}{\sqrt{h^2 - ab}}$$

$$= -\frac{1}{\sqrt{h^2 - ab}}\tanh^{-1}\frac{ax + h}{\sqrt{h^2 - ab}},$$

$$\text{or } = -\frac{1}{2\sqrt{h^2 - ab}}\log\frac{\sqrt{h^2 - ab} + (ax + h)}{\sqrt{h^2 - ab} - (ax + h)}$$

$$= \frac{1}{2\sqrt{h^2 - ab}}\log\frac{\sqrt{h^2 - ab} - (ax + h)}{\sqrt{h^2 - ab} + (ax + h)}.$$

NOTES.—In either case the $+^{ve}$ value of the surd is to be taken.

We have assumed a to be $+^{ve}$. If a is $-^{ve}$, we may take the $-^{ve}$ sign outside, and the discussion will be the same as before. It may be remarked that, in considering the above different cases, the object is to obtain an answer free from imaginaries. But, if we agree to admit imaginary values, then there is nothing to choose between the three forms of answer. Moreover, it is far simpler to be able to state that the integral $\int \frac{dx}{ax^2 + 2hx + b^2}$ gives rise to (say) the inverse tangent (real or imaginary), this being understood to include the other two inverse forms.

In Case II., when we are not concerned with the limits of the integral, it is convenient to use either form of answer indiscriminately.

Case III.—If $h^2 = ab$, then

$$I = \frac{1}{a}\int \frac{dx}{\left(x + \frac{h}{a}\right)^2} = -\frac{1}{a}\cdot\frac{1}{x + \frac{h}{a}} = -\frac{1}{ax + h}.$$

***357.** The last case may be deduced from the preceding cases, though not at first sight.

Thus, in Case I., if we put $h^2 = ab$, $I = \infty \times \tan^{-1}\infty = \infty \times \frac{\pi}{2} = \infty$.

But, noting that $\tan^{-1} m = \dfrac{\pi}{2} - \cot^{-1} m = \dfrac{\pi}{2} - \tan^{-1}\dfrac{1}{m}$, we have

$$I = \frac{1}{\sqrt{ab-h^2}}\left(\frac{\pi}{2} - \tan^{-1}\frac{\sqrt{ab-h^2}}{ax+h}\right) = C - \frac{1}{\sqrt{ab-h^2}}.\tan^{-1}\frac{\sqrt{ab-h^2}}{ax+h}$$

$$= C - \frac{1}{ax+h} \times \tan^{-1}\frac{\sqrt{ab-h^2}}{ax+h} \Big/ \frac{\sqrt{ab-h^2}}{ax+h},$$

the limit of which, when $ab - h^2 = 0$, is $C - \dfrac{1}{ax+h}$, as above.

Otherwise, if we put $\sqrt{ab-h^2} = r$, $a\alpha + h = A$, $a\beta + h = B$

$$\int_\alpha^\beta \frac{dx}{ax^2+2hx+b} = \frac{1}{r}\left\{\tan^{-1}\frac{B}{r} - \tan^{-1}\frac{A}{r}\right\}, \quad \text{[see Case I.]},$$

$$= \frac{1}{r}\tan^{-1}\frac{B/r - A/r}{1+\dfrac{AB}{r^2}} = \frac{1}{r}\tan^{-1}\frac{(B-A)r}{r^2+AB},$$

the limit of which, when $r = 0$, is

$$\frac{B-A}{AB} = \frac{1}{A} - \frac{1}{B} = \frac{1}{a\alpha+h} - \frac{1}{a\beta+h} = \left[-\frac{1}{ax+h}\right]_\alpha^\beta.$$

In Case II., first part, putting $ax + h = y$, $\sqrt{h^2 - ab} = r$,

then $\dfrac{1}{2r}\log\dfrac{y-r}{y+r} = \dfrac{1}{2r}\left\{\log\left(1-\dfrac{r}{y}\right) - \log\left(1+\dfrac{r}{y}\right)\right\}$

$$= -\frac{1}{2r}.2\left\{\frac{r}{y} + \frac{r^3}{3y^3}\dots\right\} = -\frac{1}{y} + \text{terms containing } r.$$

Hence when $r = 0$, this becomes $-\dfrac{1}{y}$ or $-\dfrac{1}{ax+h}$.

The second part cannot be considered, since (r being zero) z^2 would have to be —ve to satisfy the condition given. In the first part z may have *any* value.

358. The method of partial fractions may be adopted in the general case. Thus, if α and β be the roots of $ax^2 + 2hx + b = 0$, whether real or imaginary; and $\alpha > \beta$ so that

$$\alpha = \frac{-h+\sqrt{h^2-ab}}{a}, \quad \beta = \frac{-h-\sqrt{h^2-ab}}{a},$$

a being supposed $+^{ve}$; then

$$\frac{1}{ax^2+2hx+b}=\frac{1}{a(x-\alpha)(x-\beta)}=\frac{1}{a}\cdot\frac{1}{\alpha-\beta}\left\{\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right\},$$

supposing $x>\alpha>\beta$;

$$\therefore \int\frac{dx}{ax^2+2hx+b}=\frac{1}{a(\alpha-\beta)}\log\frac{x-\alpha}{x-\beta}=\frac{1}{2\sqrt{h^2-ab}}\log\frac{x+\dfrac{h-\sqrt{h^2-ab}}{a}}{x+\dfrac{h+\sqrt{h^2-ab}}{a}}$$

$$=\frac{1}{2\sqrt{h^2-ab}}\log\frac{ax+h-\sqrt{h^2-ab}}{ax+h+\sqrt{h^2-ab}}.$$

If $x<\beta<\alpha$,

$$I=\frac{1}{a(\alpha-\beta)}\left\{\int\frac{dx}{\beta-x)}-\int\frac{dx}{\alpha-x}\right\}=\frac{1}{a(\alpha-\beta)}\log\frac{\alpha-x}{\beta-x},$$

leading to the same result as above.

If $\alpha>x>\beta$,

$$I=\frac{1}{a(\alpha-\beta)}\left\{-\int\frac{dx}{\alpha-x}-\int\frac{dx}{x-\beta}\right\}=\frac{1}{a(\alpha-\beta)}\log\frac{\alpha-x}{x-\beta},$$

which reduces to

$$\frac{1}{2\sqrt{h^2-ab}}\log\frac{\sqrt{h^2-ab}-(ax+h)}{\sqrt{h^2-ab}+(ax+h)},$$

as in Case II., 2nd part.

359. Examples.

Ex. 1. $$\int\frac{dx}{3x^2-10x+9}=\tfrac{1}{3}\int\frac{dx}{(x-\frac{5}{3})^2+(\sqrt{2}/3)^2}$$

$$=\tfrac{1}{3}\cdot\frac{1}{\sqrt{2}/3}\tan^{-1}\frac{x-\frac{5}{3}}{\sqrt{2}/3}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{3x-5}{\sqrt{2}}.$$

Ex. 2. $$\int\frac{dx}{(2-x)(3x-1)}=\tfrac{1}{3}\int\frac{dx}{(\frac{5}{6})^2-(x-\frac{7}{6})^2}=\tfrac{1}{3}\cdot\tfrac{6}{5}\tanh^{-1}\frac{6x-7}{5}$$

$$=\tfrac{2}{5}\tanh^{-1}\frac{6x-7}{5}.$$

Or, using the logarithmic form, $$=\frac{1}{3}\cdot\frac{1}{2.\frac{5}{6}}\log\frac{\frac{5}{6}+(x-\frac{7}{6})}{\frac{5}{6}-(x-\frac{7}{6})}=\tfrac{1}{5}\log\frac{6x-2}{12-6x}$$

$$=\tfrac{1}{5}\log\frac{3x-1}{2-x}+C.$$

Ex. 3. $$\int\frac{dx}{3x^2-10x+3}=\tfrac{1}{3}\int\frac{dx}{(x-\frac{5}{3})^2-(\frac{4}{3})^2}=-\tfrac{1}{3}\cdot\tfrac{3}{4}\coth^{-1}\frac{3x-5}{4}$$
$$=-\tfrac{1}{4}\coth^{-1}\frac{3x-5}{4}.$$

Or, in logarithms,
$$=\tfrac{1}{3}\cdot\tfrac{1}{2}\cdot\tfrac{3}{4}\log\frac{(x-\frac{5}{3})-\frac{4}{3}}{(x-\frac{5}{3})+\frac{4}{3}}=\tfrac{1}{8}\log\frac{x-3}{x-\frac{1}{3}}=\tfrac{1}{8}\log\frac{x-3}{3x-1}+C.$$

360. When the denominator factorizes, the method of partial fractions may be used. Thus in Ex. 2 above, let us write
$$\frac{1}{(2-x)(3x-1)}=\frac{\text{some quantity}}{2-x}+\frac{\text{some quantity}}{3x-1}.$$
To find the numerators, they must be such that, on reducing to a common denominator, the x term in the numerator vanishes. Hence they are respectively $+1$ and $+3$; and since
$$\frac{1}{2-x}+\frac{3}{3x-1}=\frac{5}{(2-x)(3x-1)},$$
we must evidently divide by 5.
$$\therefore\ \frac{1}{(2-x)(3x-1)}=\frac{1}{5}\left\{\frac{1}{2-x}+\frac{3}{3x-1}\right\}.$$
$$\therefore\ \int\frac{dx}{(2-x)(3x-1)}=-\tfrac{1}{5}\log(2-x)+\tfrac{1}{5}\log(3x-1)=\tfrac{1}{5}\log\frac{3x-1}{2-x}.$$

361. Integration of $\dfrac{px+q}{ax^2+2hx+b}dx$.

Exactly as in Art. 352, we have
$$px+q=\frac{p}{2a}.2(ax+h)+\frac{aq-hp}{a}=k.2(ax+h)+l,\text{ say.}$$
$$\therefore\ \int\frac{px+q}{ax^2+2hx+b}dx=k\int\frac{2(ax+h)}{ax^2+2hx+b}dx+l\int\frac{dx}{ax^2+2hx+b}.$$

The first integral is $k\log(ax^2+2hx+b)$, and the second is of the form considered above.

If the denominator factorizes, however, it is better to adopt the method of partial fractions. (See Chapter XXIV.)

Ex. $$\int \frac{x^3\,dx}{x^2 - 2x + 2} = \int \left\{x + 2 + \frac{2x - 4}{x^2 - 2x + 2}\right\} dx$$
$$= \tfrac{1}{2}x^2 + 2x + \int \frac{2x - 2}{x^2 - 2x + 2}\,dx - 2\int \frac{dx}{x^2 - 2x + 2}$$
$$= \tfrac{1}{2}x^2 + 2x + \log(x^2 - 2x + 2) - 2\tan^{-1}(x - 1).$$

EXAMPLES LVII.

1. Integrate by the method of partial fractions :—

(1) $\dfrac{dx}{4 - x^2}$. (2) $\dfrac{dx}{4x^2 - 9}$.

(3) $\dfrac{dx}{2 - 3x^2}$. (4) $\dfrac{dx}{(x - 1)(x - 3)}$.

(5) $\dfrac{dx}{x(x - 3)}$. (6) $\dfrac{dx}{(2x - 3)(3x - 2)}$.

(7) $\dfrac{dx}{5 + 13x - 6x^2}$. (8) $\dfrac{dx}{(ax + b)(bx + a)}$.

2. Integrate by reducing to the fundamental forms :—

(1) $\dfrac{dx}{x^2 - 4x + 5}$. (2) $\dfrac{dx}{4x(1 - x)}$.

(3) $\dfrac{dx}{6x^2 - 11x + 3}$. (4) $\dfrac{dx}{3x^2 + 4x + 5}$.

(5) $\dfrac{dx}{(x - a)(b - x)}$. (6) $\dfrac{dx}{(ax + b)(cx + d)}$.

3. Integrate :—

(1) $\dfrac{2x + 5}{x^2 + 4x + 5}\,dx$. (2) $\dfrac{x - 2}{x^2 - 2x - 1}\,dx$.

(3) $\dfrac{x^3 + 1}{x^2 + 1}\,dx$. (4) $\dfrac{(x - 1)^3}{x(x - 2)}\,dx$.

(5) $\dfrac{x + 1}{2x^2 - 5x + 2}\,dx$. (6) $\dfrac{x^2 - ax - a^2}{x^2 + ax + a^2}\,dx$.

(7) $\dfrac{3x - 7}{2x^2 - 3x + 4}\,dx$.

ANSWERS.

1. (1) $\frac{1}{4}\log\frac{2+x}{2-x}$. (2) $\frac{1}{12}\log\frac{2x-3}{2x+3}$. (3) $\frac{1}{2\sqrt{6}}\log\frac{\sqrt{2}+\sqrt{3}x}{\sqrt{2}-\sqrt{3}x}$.

(4) $\frac{1}{2}\log\frac{x-3}{x-1}$. (5) $\frac{1}{3}\log\frac{x-3}{x}$. (6) $\frac{1}{5}\log\frac{2x-3}{3x-2}$.

(7) $\frac{1}{17}\log\frac{3x+1}{5-2x}$. (8) $\frac{1}{a^2-b^2}\log\frac{ax+b}{bx+a}$.

2. (1) $\tan^{-1}(x-2)$. (2) $\frac{1}{2}\tanh^{-1}(2x-1)$. (3) $-\frac{2}{7}\coth^{-1}\frac{12x-11}{7}$.

(4) $\frac{1}{\sqrt{11}}\tan^{-1}\frac{3x+2}{\sqrt{11}}$. (5) $\frac{2}{a-b}\tanh^{-1}\frac{2x-a-b}{a-b}$.

(6) $-\frac{2}{ad-bc}\coth^{-1}\frac{2acx+ad+bc}{ad-bc}$.

3. (1) $\log(x^2+4x+5)+\tan^{-1}(x+2)$.

(2) $\frac{1}{2}\log(x^2-2x-1)+\frac{1}{\sqrt{2}}\coth^{-1}\frac{x-1}{\sqrt{2}}$.

(3) $\frac{x^2}{2}-\frac{1}{2}\log(x^2+1)+\tan^{-1}x$. (4) $\frac{1}{2}x^2-x+\frac{1}{2}\log(x^2-2x)$.

(5) $\frac{1}{4}\log(2x^2-5x+2)-\frac{3}{2}\coth^{-1}\frac{4x-5}{3}=\log(x-2)-\frac{1}{2}\log(2x-1)$.

(6) $x-a\log(x^2+ax+a^2)-\frac{2a}{\sqrt{3}}\tan^{-1}\frac{2x+a}{\sqrt{3}a}$.

(7) $\frac{3}{4}\log(2x^2-3x+4)-\frac{19}{2\sqrt{23}}\tan^{-1}\frac{4x-3}{\sqrt{23}}$.

362. Integration of $\frac{d\theta}{a\sin^2\theta + 2h\cos\theta\sin\theta + b\cos^2\theta}$.

It should be noticed that the denominator of the above is *homogeneous in sines and cosines, and of the second degree.*

The integral

$$= \int \frac{\sec^2\theta\, d\theta}{a\tan^2\theta + 2h\tan\theta + b} = \int \frac{dx}{ax^2+2hx+b}, \text{ if } x=\tan\theta;$$

and is thus reduced to the preceding form.

This includes a number of important special cases which we give below.

363. $\dfrac{d\theta}{a + b\cos\theta}$.

We can transform the denominator by using the half angle. First, let $a > b$.

$$\text{Then } I = \int \frac{d\theta}{a\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) + b\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)}$$

$$= \int \frac{d\theta}{(a+b)\cos^2\frac{\theta}{2} + (a-b)\sin^2\frac{\theta}{2}}$$

$$= \int \frac{\sec^2\frac{\theta}{2}\,d\theta}{(a+b) + (a-b)\tan^2\frac{1}{2}\theta}$$

$$= 2\int \frac{dx}{(a+b) + (a-b)x^2}, \text{ if } x = \tan\frac{\theta}{2},$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\,x\right\}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right\}.$$

This result can be put into a different form.

For let $\sqrt{a+b}\cos\frac{\theta}{2} = m$, $\sqrt{a-b}\sin\frac{\theta}{2} = n$, so that

$$I = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{n}{m}.$$

$$\text{Now put } 2\tan^{-1}\frac{n}{m} = \phi, \quad \therefore \frac{n}{m} = \tan\frac{\phi}{2};$$

$$\therefore \cos\phi = \frac{1-\tan^2\frac{1}{2}\phi}{1+\tan^2\frac{1}{2}\phi} = \frac{m^2-n^2}{m^2+n^2}$$

$$= \frac{(a+b)\cos^2\frac{\theta}{2} - (a-b)\sin^2\frac{\theta}{2}}{(a+b)\cos^2\frac{\theta}{2} + (a-b)\sin^2\frac{\theta}{2}} = \frac{a\cos\theta + b}{a + b\cos\theta};$$

$$\therefore I = \frac{\phi}{\sqrt{a^2-b^2}} = \frac{1}{\sqrt{a^2-b^2}} \cos^{-1}\frac{a\cos\theta+b}{a+b\cos\theta}.$$

If $a < b$ we shall get

$$I = \frac{2}{\sqrt{b^2-a^2}} \tanh^{-1}\left\{\sqrt{\frac{b-a}{b+a}}\tan\frac{\theta}{2}\right\},$$

$$\text{or } = \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b+a}\cos\frac{\theta}{2} + \sqrt{b-a}\sin\frac{\theta}{2}}{\sqrt{b+a}\cos\frac{\theta}{2} - \sqrt{b-a}\sin\frac{\theta}{2}}.$$

We can show that this $= \dfrac{1}{\sqrt{b^2-a^2}} \cosh^{-1}\dfrac{a\cos\theta+b}{a+b\cos\theta}$.

364. $\int\frac{d\theta}{\sin\theta}$ and $\int\frac{d\theta}{\cos\theta}$.

We have

$$\int\frac{d\theta}{\sin\theta} = \frac{1}{2}\int\frac{d\theta}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1}{2}\int\frac{\sec^2\frac{\theta}{2}\,d\theta}{\tan\frac{\theta}{2}} = \int\frac{dx}{x}, \text{ if } x = \tan\frac{\theta}{2}$$

$$= \log x = \log\tan\frac{\theta}{2}.$$

Also
$$\int\frac{d\theta}{\cos\theta} = \int\frac{d\theta}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} = \int\frac{\sec^2\frac{\theta}{2}\,d\theta}{1-\tan^2\frac{\theta}{2}} = 2\int\frac{dx}{1-x^2}$$

$$= 2\tanh^{-1}x = 2\tanh^{-1}\left(\tan\frac{\theta}{2}\right),$$

or
$$= \log\frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta} = \log(\sec\theta + \tan\theta)$$

$$= \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \text{gd}^{-1}\theta.$$ [See *Misc. Theorems.*]

This might have been deduced from the preceding integral $\int\frac{d\theta}{\sin\theta}$;

thus
$$\int\frac{d\theta}{\cos\theta} = \int\frac{d\theta}{\sin\left(\frac{\pi}{2}+\theta\right)} = \int\frac{d\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right)} = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Again, as in Ex. 7, Art. 336,

$$\int\frac{d\theta}{\cos\theta}=\int\frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}}=\int\frac{dx}{\sqrt{1+x^2}},\text{ if } x=\tan\theta,$$

$$=\log(x+\sqrt{1+x^2})=\log(\tan\theta+\sec\theta)\text{ as before,}$$

or $$=\sinh^{-1}\tan\theta.$$

Also $$I=\int\frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}}=\int\frac{dx}{\sqrt{x^2-1}},\text{ if } x=\sec\theta,$$

$$=\cosh^{-1}x=\cosh^{-1}\sec\theta,$$

the logarithmic form being as before.

From the above results, we have

$$\int\frac{d\theta}{\cos\theta}=\mathrm{gd}^{-1}\theta=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\log(\sec\theta+\tan\theta)$$

$$=2\tanh^{-1}\tan\tfrac{1}{2}\theta=\sinh^{-1}\tan\theta=\cosh^{-1}\sec\theta.$$

365. $\displaystyle\int\frac{d\theta}{a+h\sin\theta+b\cos\theta}.$

$$\text{This}=\int\frac{d\theta}{a\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)+2h\cos\frac{\theta}{2}\sin\frac{\theta}{2}+b\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)}$$

$$=\int\frac{d\theta}{(a+b)\cos^2\frac{\theta}{2}+2h\cos\frac{\theta}{2}\sin\frac{\theta}{2}+(a-b)\sin^2\frac{\theta}{2}},$$

which is reduced to the form we are considering.

Ex. $\displaystyle\int\frac{d\theta}{2+3\cos\theta+\sin\theta}=\int\frac{d\theta}{2(c^2+s^2)+3(c^2-s^2)+2cs},$

if $c=\cos\frac{1}{2}\theta$, $s=\sin\frac{1}{2}\theta$,

$$=\int\frac{d\theta}{5c^2+2cs-s^2}=\int\frac{\sec^2\frac{\theta}{2}\,d\theta}{5+2\tan\frac{\theta}{2}-\tan^2\frac{\theta}{2}}$$

$$=2\int\frac{dx}{5+2x-x^2},\text{ if } x=\tan\frac{\theta}{2},$$

$$= 2\int\frac{dx}{6-(x-1)^2} = \frac{2}{\sqrt{6}}\tanh^{-1}\frac{x-1}{\sqrt{6}} \equiv \frac{1}{\sqrt{6}}\log\frac{\sqrt{6}-1+x}{\sqrt{6}+1-x}$$

$$= \frac{2}{\sqrt{6}}\tanh^{-1}\frac{\tan\frac{1}{2}\theta-1}{\sqrt{6}} \equiv \frac{1}{\sqrt{6}}\log\frac{\sqrt{6}-1+\tan\frac{1}{2}\theta}{\sqrt{6}+1-\tan\frac{1}{2}\theta}.$$

366. Integration of $\frac{a'+b'\cos\theta+c'\sin\theta}{a+b\cos\theta+c\sin\theta}d\theta.$

The numerator can be expressed as the sum of three quantities, viz. :—

(1) A multiple of the denominator, $a+b\cos\theta+c\sin\theta$,
(2) A multiple of its differential coefficient, $-b\sin\theta+c\cos\theta$,
(3) A constant.

For, let $a'+b'\cos\theta+c'\sin\theta$

$$=l(a+b\cos\theta+c\sin\theta)+m(-b\sin\theta+c\cos\theta)+n\,;$$

then, comparing coefficients of 1, $\cos\theta$, $\sin\theta$, we have the following equations for finding l, m, and n :—

$$\begin{aligned} la+n &= a'\,; \\ lb+mc &= b'\,; \\ lc-mb &= c'. \end{aligned}$$

$$\therefore\ l=\frac{bb'+cc'}{b^2+c^2}\,;\quad m=\frac{b'c-bc'}{b^2+c^2}\,;\quad n=\frac{b(a'b-ab')-c(c'a-ca')}{b^2+c^2}.$$

We therefore have

$$I=l\int\frac{a+b\cos\theta+c\sin\theta}{a+b\cos\theta+c\sin\theta}d\theta+m\int\frac{-b\sin\theta+c\cos\theta}{a+b\cos\theta+c\sin\theta}d\theta$$
$$+n\int\frac{d\theta}{a+b\cos\theta+c\sin\theta}.$$

The first integral is $l\theta$.

The second is $m\log(a+b\cos\theta+c\sin\theta)$.

The third may be obtained as before.

Ex. 1. $\int\frac{2+3\sin\theta-\cos\theta}{1+\cos\theta+\sin\theta}d\theta.$

Let $2+3s-c=l(1+c+s)+m(-s+c)+n.$

Then $l + n = 2$; $l + m = -1$; $l - m = 3$;
whence $l = 1$, $m = -2$, $n = 1$.

$$\therefore\ I = \int d\theta - 2\int \frac{-\sin\theta + \cos\theta}{1 + \cos\theta + \sin\theta}\,d\theta + \int \frac{d\theta}{1 + \cos\theta + \sin\theta}$$

$$= \theta - 2\log(1 + \cos\theta + \sin\theta) + I' \text{ (say)}.$$

Then $$I' = \frac{1}{2}\int \frac{d\theta}{\cos^2\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2}} = \int \frac{\sec^2\dfrac{\theta}{2}\,d\left(\dfrac{\theta}{2}\right)}{1 + \tan\dfrac{\theta}{2}} = \log\left(1 + \tan\frac{\theta}{2}\right)$$

$$= \log\left(1 + \frac{\sin\theta}{1 + \cos\theta}\right) = \log\frac{1 + \cos\theta + \sin\theta}{1 + \cos\theta}.$$

$$\therefore\ I = \theta - \log(1 + \cos\theta + \sin\theta) - \log(1 + \cos\theta).$$

Ex. 2. $\displaystyle\int \frac{1 - \sin\theta}{\cos\theta + 2}\,d\theta.$

In this example, l, m, and n can be guessed at once.

Thus $$I = \int \frac{d\theta}{\cos\theta + 2} + \int \frac{-\sin\theta\,d\theta}{\cos\theta + 2}$$

$$= \int \frac{d\theta}{3\cos^2\dfrac{\theta}{2} + \sin^2\dfrac{\theta}{2}} + \log(\cos\theta + 2).$$

The first integral $$= 2\int \frac{\sec^2\dfrac{\theta}{2}.\,d\left(\dfrac{\theta}{2}\right)}{3 + \tan^2\dfrac{\theta}{2}} = 2\int \frac{dz}{3 + z^2} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right).$$

$$\therefore\ I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right) + \log(\cos\theta + 2).$$

EXAMPLES LVIII.

1. Integrate :—

(1) $\dfrac{d\theta}{\cos^2\theta - 2\sin^2\theta}$. (2) $\dfrac{d\theta}{\sin\theta\cos\theta}$.

(3) $\dfrac{d\theta}{1+\cos^2\theta}$. (4) $\dfrac{d\theta}{\cos\theta(\sin\theta+\cos\theta)}$.

(5) $\dfrac{d\theta}{(\cos\theta-\sin\theta)^2}$. (6) $\dfrac{d\theta}{(\sin\theta-2\cos\theta)(2\sin\theta+\cos\theta)}$.

(7) $\dfrac{d\theta}{5\cos^2\theta-4\cos\theta\sin\theta-2\sin^2\theta}$.

2. Integrate:—

(1) $\dfrac{d\theta}{1+\cos\theta}$. (2) $\dfrac{d\theta}{1+\sin\theta}$.

(3) $\dfrac{d\theta}{2+3\cos\theta}$. (4) $\dfrac{d\theta}{3+2\cos\theta}$.

(5) $\dfrac{d\theta}{4+5\sin\theta}$. (6) $\dfrac{d\theta}{\cos\theta+\sin\theta}$.

(7) $\dfrac{d\theta}{4\cos\theta-3\sin\theta}$. (8) $\dfrac{d\theta}{5-4\cos\theta+3\sin\theta}$.

3. Integrate:—

(1) $\dfrac{d\theta}{\sin 2\theta}$. (2) $\dfrac{d\theta}{2-3\cos 2\theta}$.

(3) $\dfrac{d\theta}{\sin 3\theta}$. (4) $\dfrac{d\theta}{2\cos n\theta-\sin n\theta}$.

4. Integrate:—

(1) $\dfrac{\sin\theta\,d\theta}{\sin\theta+\cos\theta}$. (2) $\dfrac{2-\sin\theta}{2\cos\theta+3}\,d\theta$.

(3) $\dfrac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}\,d\theta$. (4) $\dfrac{3\sin\theta-4\cos\theta-5}{2+\cos\theta-2\sin\theta}\,d\theta$.

***5.** Integrate:—

(1) $\dfrac{d\theta}{\cos\theta+\cos^2\theta}$. (2) $\tan\theta\sec 2\theta\,d\theta$.

(3) $\dfrac{d\theta}{\sin\theta+\cos 2\theta}$. (4) $\dfrac{1+\tan\theta}{2+\sin 2\theta}\,d\theta$.

(5) $\dfrac{\tan\theta\,d\theta}{\sqrt{\cos^4\theta+\sin^4\theta}}$.

ANSWERS.

1. (1) $\dfrac{1}{\sqrt{2}}\tanh^{-1}(\sqrt{2}\tan\theta)$. (2) $\log\tan\theta$. (3) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\tan\theta\right)$.

(4) $\log(1+\tan\theta)$. (5) $\dfrac{1}{1-\tan\theta}$. (6) $\frac{1}{5}\log\dfrac{\sin\theta-2\cos\theta}{2\sin\theta+\cos\theta}$.

(7) $\dfrac{1}{\sqrt{14}}\tanh^{-1}\left\{\sqrt{\dfrac{2}{7}}\left(\tan\theta+1\right)\right\}$.

2. (1) $\tan\dfrac{\theta}{2}$. (2) $\tan\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)=C-\dfrac{2\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta+\cos\frac{1}{2}\theta}$.

(3) $\dfrac{2}{\sqrt{5}}\tanh^{-1}\left(\dfrac{1}{\sqrt{5}}\tan\dfrac{\theta}{2}\right)$. (4) $\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{1}{\sqrt{5}}\tan\dfrac{\theta}{2}\right)$.

(5) $\frac{1}{3}\log\dfrac{2\tan\dfrac{\theta}{2}+1}{\tan\dfrac{\theta}{2}+2}$. (6) $\sqrt{2}\tanh^{-1}\dfrac{1}{\sqrt{2}}\left(\tan\dfrac{\theta}{2}-1\right)$.

(7) $\frac{1}{5}\log\dfrac{2+\tan\frac{1}{2}\theta}{1-2\tan\frac{1}{2}\theta}$. (8) $-\dfrac{2}{3}\dfrac{1}{1+3\tan\frac{1}{2}\theta}$.

3. (1) $\frac{1}{2}\log\tan\theta$. (2) $-\dfrac{1}{\sqrt{5}}\coth^{-1}(\sqrt{5}\tan\theta)$. (3) $\frac{1}{3}\log\tan\dfrac{3\theta}{2}$.

(4) $\dfrac{2}{n\sqrt{5}}\tanh^{-1}\dfrac{2\tan\dfrac{n\theta}{2}+1}{\sqrt{5}}$.

4. (1) $\dfrac{\theta}{2}-\dfrac{1}{2}\log(\sin\theta+\cos\theta)$.

(2) $\frac{1}{2}\log(2\cos\theta+3)+\dfrac{4}{\sqrt{5}}\tan^{-1}\left(\dfrac{1}{\sqrt{5}}\tan\dfrac{\theta}{2}\right)$.

(3) $\frac{6}{5}\theta+\frac{3}{5}\log(\sin\theta+2\cos\theta+3)-\frac{8}{5}\tan^{-1}\{\frac{1}{2}(1+\tan\frac{1}{2}\theta)\}$.

(4) $2\left\{\log\left(\sin\dfrac{\theta}{2}-\cos\dfrac{\theta}{2}\right)-\theta\right\}$.

5. (1) $\mathrm{gd}^{-1}\theta-\tan\frac{1}{2}\theta$. (2) $-\frac{1}{2}\log(1-\tan^2\theta)$.

(3) $-\dfrac{4}{3\sqrt{3}}\coth^{-1}\dfrac{\tan\dfrac{\theta}{2}+2}{\sqrt{3}}-\dfrac{2}{3\left(\tan\dfrac{\theta}{2}-1\right)}$.

(4) $\frac{1}{4}\log(1+\tan\theta+\tan^2\theta)+\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{2\tan\theta+1}{\sqrt{3}}$.

(5) $\frac{1}{2}\sinh^{-1}\tan^2\theta$.

CHAPTER XXIV.

METHOD OF PARTIAL FRACTIONS.

367. We shall now consider the integration of $\frac{f(x)dx}{\phi(x)}$, where $f(x)$ and $\phi(x)$ are rational integral algebraical functions of x [Art. 16, footnote], by resolving $f(x)/\phi(x)$ into partial fractions—that is, into a series of fractions whose denominators are linear or quadratic factors of $\phi(x)$. The resulting expression can then be integrated by the preceding methods.

If $f(x)$ be of higher degree than $\phi(x)$, we can always divide down until the remainder is at least one degree lower than the divisor; the quotient, too, is immediately integrable. For example—

$$\int \frac{x^5 - 3x^4 - 2x^2 - 1}{x^3 + x - 2} dx = \int \left\{ x^2 - 4x - 1 + \frac{4x^2 - 7x - 3}{x^3 + x - 2} \right\} dx$$
$$= \frac{x^3}{3} - 2x^2 - x + \int \frac{4x^2 - 7x - 3}{x^3 + x - 2} dx;$$

and in the latter integral the numerator is one degree lower than the denominator.

We shall therefore assume, unless stated to the contrary, that $f(x)$ is one degree lower than $\phi(x)$, which is of the nth degree, say; we shall also assume that the roots, $a_1, a_2, \ldots a_n$, of the equation $\phi(x) = 0$, are known, whether they be real or imaginary.

There are four cases to be considered :—

(A) Real and different roots.

(B) Multiple real roots; *i.e.* real roots, two or more of which are equal.

(C) Imaginary and different roots.

(D) Multiple imaginary roots.

In any case we may write $\phi(x) = a(x - a_1)(x - a_2)\ldots(x - a_n)$ a being the coefficient of x^n.

368. (A) Real and Different Roots.

Let $a_1, a_2, \ldots$ be real, and let no two of them be equal.

Assume
$$\frac{f(x)}{\phi(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \ldots + \frac{A_n}{x - a_n} \quad . \quad . \quad (1)$$

in which $A_1, A_2, \ldots$ do not involve x. Required to find A_1, A_2, etc

Clearing of fractions, since $\phi(x) = a(x - a_1)(x - a_2)\ldots(x - a_n)$ we have

$$f(x) = A_1 a(x - a_2)(x - a_3)\ldots(x - a_n)$$
$$+ \text{ terms containing } x - a_1 \text{ as a factor} \quad . \quad (2)$$

Now, suppose A_1, A_2, etc., were known; then (2) would be an identity, and therefore the coefficients of like powers of x would be identically the same on both sides of the equation.

Conversely, if we assume that (2) is an identity, and equate coefficients of like powers of x, we shall have equations for finding A_1, A_2, etc. Moreover, $f(x)$ is of the $(n - 1)$th degree, as also is the right-hand side; hence there are n coefficients to equate (including that of x^0), and these n equations are just sufficient to determine the n numerators $A_1, A_2, \ldots A_n$.

Again, if we assume (2) to be an identity, then it will be true for all values of x. We may therefore find A_1, A_2, etc., by giving special values to x.

Thus, let $x = a_1$. Then, since $x - a_1 = 0$, we have from (2),

$$f(a_1) = A_1 a(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n);$$

$$\therefore A_1 = \frac{f(a_1)}{a(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n)}.$$

Hence
$$\frac{A_1}{x - a_1} = \frac{f(a_1)}{a(x - a_1)(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n)},$$

and it will be seen that this can be written down, if in $f(x)/\phi(x)$

we substitute a_1 for x *in all parts except the factor* $x - a_1$, which we write down unaltered.

This may be remembered by noting that, unless we made $x - a_1$ the exception, the fraction would become infinite.

Similarly, $\dfrac{A_2}{x-a_2} = \dfrac{f(a_2)}{a(a_2-a_1)(x-a_2)(a_2-a_3)\ldots(a_2-a_n)}$; etc.

NOTE.—The above method may be adopted even where a_1, a_2, etc., are imaginary, but in practice the method of Art. 374 is preferable.

369. Suppose $f(x)$ to be of higher degree than $\phi(x)$, and let $\psi(x)$ be the quotient when $f(x)$ is divided by $\phi(x)$, the remainder being one degree lower than $\phi(x)$.

Assume $\dfrac{f(x)}{\phi(x)} = \psi(x) + \dfrac{A_1}{x-a_1} + \dfrac{A_2}{x-a_2} + \ldots + \dfrac{A_n}{x-a_n}$.

$$\therefore\ f(x) = \psi(x).a(x-a_1)(x-a_2)\ldots(x-a_n)$$
$$+ A_1a(x-a_2)\ldots(x-a_n) + \ldots$$

and since $\psi(x)$ is multiplied by *all* of the factors, that term will go out in every case, *i.e.* whether we put $x = a_1$, or a_2, or etc.

Hence the rule for finding A_1, A_2, etc., is the same as before.

Thus $f(a_1) = A_1a(a_1-a_2)\ldots(a_1-a_n)$; etc.

The advantage of this in practice is that, if we can write down the quotient by inspection, we do not require to find the remainder.

370. Another Form for A_1, etc.

Since $\phi'(x) = a(x-a_2)(x-a_3)\ldots(x-a_n)$ + terms containing $(x-a_1)$ as a factor,

$$\therefore\ \phi'(a_1) = a\,(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n).$$

$$\therefore\ \frac{A_1}{x-a_1} = \frac{f(a_1)}{\phi'(a_1)}\cdot\frac{1}{x-a_1};\ \text{and similarly for } A_2, \text{ etc.}$$

371. Examples.

Ex. 1. $\displaystyle\int \frac{10x^2-6x+1}{(x-1)(3x-2)(2x+3)}\,dx.$

(1) Writing $x = 1$ in all parts except $x - 1$, we get

$$\frac{10 - 6 + 1}{(x-1)(1)(5)} = \frac{1}{x - 1}.$$

(2) Writing $x = \frac{2}{3}$ in all parts except $3x - 2$, we get

$$\frac{\frac{40}{9} - 4 + 1}{(-\frac{1}{3})(3x - 2)(\frac{13}{3})} = -\frac{1}{3x - 2}.$$

(3) Writing $x = -\frac{3}{2}$ in all parts except $2x + 3$, we get

$$\frac{\frac{45}{2} + 9 + 1}{(-\frac{5}{2})(-\frac{13}{2})(2x + 3)} = \frac{2}{2x + 3}.$$

$$\therefore I = \int\left\{\frac{1}{x - 1} - \frac{1}{3x - 2} + \frac{2}{2x + 3}\right\}dx$$

$$= \log(x - 1) - \tfrac{1}{3}\log(3x - 2) + \log(2x + 3).$$

Ex. 2. $\displaystyle\int\frac{x^4 - x^3 + x^2 + 1}{(x - 1)(x + 2)(x - 4)}\,dx.$

Adopting the method of Art. 369, we note that since the numerator is one degree higher than the denominator, the quotient will be of the first degree, and will contain only two terms. We need only, therefore, take two terms of the numerator and denominator.

Thus $\dfrac{x^4 - x^3 \ldots}{x^3 - 3x^2 \ldots} = \dfrac{x^2 - x \ldots}{x - 3 \ldots} = x + 2 \ldots$

Putting $x = 1$ in all parts except $x - 1$, we have

$$\frac{1 - 1 + 1 + 1}{(x - 1)(3)(-3)} = -\frac{2}{9(x - 1)}.$$

Putting $x = -2$ in all parts except $x + 2$, we have

$$\frac{16 + 8 + 4 + 1}{(-3)(x + 2)(-6)} = \frac{29}{18(x + 2)}.$$

Putting $x = 4$ in all parts except $x - 4$, we have

$$\frac{256 - 64 + 16 + 1}{(3)(6)(x - 4)} = \frac{209}{18(x - 4)}.$$

$$\therefore \text{the fraction} = x + 2 - \frac{2}{9(x - 1)} + \frac{29}{18(x + 2)} + \frac{209}{18(x - 4)}.$$

$$\therefore I = \frac{x^2}{2} + 2x - \tfrac{2}{9}\log(x - 1) + \tfrac{29}{18}\log(x + 2) + \tfrac{209}{18}\log(x - 4).$$

NOTE.—Whenever possible, partial fractions should be obtained by inspection.

372. It will be seen that if two or more of the roots α_1, α_2, etc., coincide, then some of the quantities A_1, etc., will become infinite. Thus, if $\alpha_1 = \alpha_2$, then $A_1 = \infty$. For this reason the case of equal roots has to be considered specially, as below.

EXAMPLES LIX.

1. Integrate :—

(1) $\dfrac{x\,dx}{(x-3)(x-6)}$. (2) $\dfrac{2x-3}{(x-1)(x-2)}dx$. (3) $\dfrac{dx}{(ax-b)(bx-a)}$.

2. Integrate :—

(1) $\dfrac{dx}{(x-1)(x-2)(x-3)}$. (2) $\dfrac{x^2+1}{x(x^2-1)}dx$.

(3) $\dfrac{6x^2+8x-23}{(2x-1)(3x+2)(x-3)}dx$. (4) $\dfrac{(x+a)(x+b)}{x(x-a)(x-b)}dx$.

3. Integrate :—

(1) $\dfrac{x^2\,dx}{(x-1)(x-2)}$. (2) $\dfrac{x^3+3}{x^2-1}dx$. (3) $\dfrac{(x^2+a^2)^2\,dx}{(x^2-a^2)(x-2a)}$.

4. Integrate :—

(1) $\dfrac{1-\cos\theta}{\cos\theta\,(1+\cos\theta)}d\theta$. (2) $\dfrac{d\theta}{1-\cos^4\theta}$.

(3) $\dfrac{\cos\theta\,d\theta}{4-\cos^2\theta}$. (4) $\dfrac{d\theta}{\sin\theta\,(3+\cos^2\theta)}$.

(5) $\dfrac{1+x\sin x+\cos x}{x(1+\cos x)}dx$. (6) $\dfrac{\cos x+x\sin x}{x(x+\cos x)}dx$.

Answers.

1. (1) $2\log(x-6)-\log(x-3)$. (2) $\log(x-1)(x-2)$.

(3) $\dfrac{1}{a^2-b^2}\log\dfrac{bx-a}{ax-b}$.

2. (1) $\frac{1}{2}\log(x-1)-\log(x-2)+\frac{1}{2}\log(x-3)$. (2) $\log\dfrac{x^2-1}{x}$.

(3) $\log\dfrac{(2x-1)(x-3)}{3x+2}$. (4) $\log x+2\dfrac{a+b}{a-b}\log\dfrac{x-a}{x-b}$.

3. (1) $x-\log(x-1)+4\log(x-2)$. (2) $\frac{1}{2}x^2+2\log(x-1)-\log(x+1)$.
(3) $\frac{1}{2}x^2+2ax-2a^2\log(x-a)+\frac{2}{3}a^2\log(x+a)+\frac{25}{3}a^2\log(x-2a)$.

4. (1) $\log(\sec\theta+\tan\theta)-2\tan\frac{1}{2}\theta$. (2) $\frac{1}{2\sqrt{2}}\tan^{-1}\frac{\tan\theta}{\sqrt{2}}-\frac{1}{2}\cot\theta$.

(3) By partial fractions, $\frac{1}{\sqrt{3}}\left\{\tan^{-1}\left(\sqrt{3}\tan\frac{\theta}{2}\right)-\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right)\right\}$.

Otherwise, $I=\int\frac{\cos\theta\,d\theta}{3+\sin^2\theta}=\int\frac{dz}{3+z^2}$, which becomes $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\sin\theta\right)$. The two results are equal.

(4) $\frac{1}{4}\log\tan\frac{\theta}{2}-\frac{1}{4\sqrt{3}}\tan^{-1}\frac{\cos\theta}{\sqrt{3}}$. (5) $\log x-\log(1+\cos x)$.

(6) $\log x-\log(x+\cos x)$.

373. (B) Multiple Real Roots.

Let r of the roots of $\phi(x)=0$ coincide, so that

$$\phi(x)=a(x-a_1)^r(x-a_{r+1})\dots(x-a_n).$$

Then we shall express $f(x)/\phi(x)$ in the form

$$\frac{A_1}{x-a_1}+\frac{A_2}{(x-a_1)^2}+\dots\frac{A_r}{(x-a_1)^r}+\frac{A_{r+1}}{x-a_{r+1}}+\dots\frac{A_n}{x-a_n}.$$

In this case, as before, if we were to clear of fractions we should have on each side of the equation an expression of the $(n-1)$th degree, from which we could obtain n equations for finding A_1, etc. In practice, however, other methods are adopted, of which the following is perhaps the simplest. We give two numerical examples from which the general rule may be gathered.

Ex. 1. $\int\frac{(x+1)\,dx}{(x-1)^3(x-2)(x-3)}$.

Putting $x-1=y$, the fraction becomes $\frac{y+2}{y^3(y-1)(y-2)}$, which we shall put into the form

$$\frac{A_1}{y}+\frac{A_2}{y^2}+\frac{A_3}{y^3}+\frac{A_4}{y-1}+\frac{A_5}{y-2}.$$

Take $1/y^3$ outside, and divide $(y-1)(y-2)$ into $y+2$ by ordinary long division, but in *ascending* powers of y, until the remainder ***is just divisible by*** y^3, and we have

$$\frac{y+2}{y^3(y-1)(y-2)} = \frac{1}{y^3}\left\{1 + 2y + \tfrac{5}{2}y^2 + \frac{\tfrac{11}{2}y^3 - \tfrac{5}{2}y^4}{(1-y)(2-y)}\right\}$$

$$= \frac{1}{y^3} + \frac{2}{y^2} + \frac{5}{2y} + \frac{11-5y}{2(1-y)(2-y)}$$

$$= \frac{1}{y^3} + \frac{2}{y^2} + \frac{5}{2y} + \frac{3}{1-y} - \frac{1}{2(2-y)}$$ by the preceding rule in case (A),

$$= \frac{1}{(x-1)^3} + \frac{2}{(x-1)^2} + \frac{5}{2(x-1)} - \frac{3}{x-2} + \frac{1}{2(x-3)}.$$

Hence

$$I = -\frac{1}{2(x-1)^2} - \frac{2}{x-1} + \tfrac{5}{2}\log(x-1) - 3\log(x-2) + \tfrac{1}{2}\log(x-3).$$

The actual work may be made shorter if we observe that A_4 and A_5 can be found at the outset by case (A), so that in the process of long division we need only find the quotient, as the remainder will not be wanted. And the terms in the quotient will not go beyond y^2, one degree less than the y^3 of the denominator.

Ex. 2. $\displaystyle\int \frac{x^5\,dx}{(x-1)^2(x-2)^2}.$

Putting $x - 1 = y$, the fraction becomes

$$\frac{(1+y)^5}{y^2(1-y)^2} = \frac{1}{y^2}\cdot\frac{1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5}{1 - 2y + y^2}$$

$$= \frac{1}{y^2}\left\{1 + 7y + \frac{23y^2 + 3y^3 + 5y^4 + y^5}{(1-y)^2}\right\}$$

$$= \frac{1}{y^2} + \frac{7}{y} + \frac{23 + 3y + 5y^2 + y^3}{(1-y)^2}.$$

Putting $y - 1 = z$, the last fraction becomes

$$\frac{32 + 16z + 8z^2 + z^3}{z^2} = \frac{32}{z^2} + \frac{16}{z} + 8 + z,$$

and since $y = x - 1$, $z = x - 2$, we have

$$I = \int\left\{\frac{1}{(x-1)^2} + \frac{7}{x-1} + \frac{32}{(x-2)^2} + \frac{16}{x-2} + 8 + x - 2\right\}dx.$$

$$= -\frac{1}{x-1} + 7\log(x-1) - \frac{32}{x-2} + 16\log(x-2) + 6x + \tfrac{1}{2}x^2.$$

The remark at the end of Ex. 1 does not apply in this case.

Note.—The above method is also applicable to the case in which the roots are imaginary, but in practice the method of Art. 375 is preferable. See also note at end of Art. 371.

EXAMPLES LX.

Integrate :—

1. $\dfrac{dx}{x^2(x+1)}$.
2. $\dfrac{(x-2)dx}{(x-1)^2}$.
3. $\left(\dfrac{x+a}{x-a}\right)^3 dx$.
4. $\left(\dfrac{x+a}{x-a}\right)^2 \dfrac{dx}{x}$.
5. $\dfrac{2x^2\,dx}{(x-1)^2(x+1)}$.
6. $\dfrac{dx}{(x-2)^2(x-3)(x-1)}$.
7. $\dfrac{x^5\,dx}{x^3-3x+2}$.
8. $\dfrac{(8-7x)dx}{x(x-1)(x-2)^3}$.
9. $\dfrac{4dx}{(x^2-1)^2}$.
10. $\dfrac{4x\,dx}{(a^2-x^2)^3}$.
11. $\dfrac{32\,dx}{(x+1)^2(x-3)^3}$.
12. $\dfrac{x^5\,dx}{(x^2+3x+2)^2}$.

Answers.

1. $\log\dfrac{x+1}{x} - \dfrac{1}{x}$.
2. $\log(x-1) + \dfrac{1}{x-1}$.
3. $x + 6a\log(x-a) - \dfrac{12a^2}{x-a} - \dfrac{4a^3}{(x-a)^2}$.
4. $\log x - \dfrac{4a}{x-a}$.
5. $\frac{3}{2}\log(x-1) + \frac{1}{2}\log(x+1) - \dfrac{1}{x-1}$.
6. $\dfrac{1}{x-2} + \frac{1}{2}\log\dfrac{x-3}{x-1}$.
7. $\frac{1}{3}x^3 + 3x - \dfrac{1}{3(x-1)} + \frac{14}{9}\log(x-1) - \frac{32}{9}\log(x+2)$.
8. $\log\dfrac{x}{x-1} - \dfrac{1}{x-2} + \dfrac{3}{2(x-2)^2}$.
9. $\log\dfrac{x+1}{x-1} - \dfrac{2x}{x^2-1}$.
10. $\dfrac{1}{(a^2-x^2)^2}$. [Integrable at once if we put $x^2 = y$.]

11. $\dfrac{1}{x-3} - \dfrac{1}{(x-3)^2} + \dfrac{1}{2(x+1)} + \frac{3}{8}\log\dfrac{x-3}{x+1}.$

12. $\dfrac{1}{x+1} + 7\log(x+1) + \dfrac{32}{x+2} + 16\log(x+2) - 6x + \frac{1}{2}x^2.$

374. (C) Imaginary and Different Roots.

If $\phi(x)$ has real coefficients, the imaginary roots of $\phi(x) = 0$ will occur in conjugate pairs [*Hall and Knight's Higher Algebra*, Art. 543; or, *C. Smith*, Art. 446], so that the imaginary factors of $\phi(x)$ will also occur in conjugate pairs, the product of each pair being a quadratic expression with real coefficients. For instance, if $a \pm \beta i$ be a pair of roots, then the product of the corresponding factors will be

$$\begin{aligned}(x - a - \beta i)(x - a + \beta i) &= (x-a)^2 + \beta^2\\ &= x^2 - 2ax + a^2 + \beta^2,\end{aligned}$$

which is of the form $ax^2 + 2hx + b$.

We shall take the expression $\dfrac{A_1x + A_2}{ax^2 + 2hx + b}$ for the partial fraction corresponding to a pair of imaginary roots, and it will be seen that by so doing we shall still use n quantities A_1, etc., as before.

The method to be adopted is that of the first part of Art. 368, with certain modifications, however, by which the work can be shortened. We shall again explain by means of examples.

Ex. 1. $\displaystyle\int \frac{x^2-2}{(x+1)(x^2+1)}\,dx.$

Assume that $\dfrac{x^2-2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}.$

By case (A), we have at once $A = -\frac{1}{2}$; hence, clearing of fractions,

$$\begin{aligned}x^2 - 2 &= -\tfrac{1}{2}(x^2+1) + (x+1)(Bx+C)\\ &= (B - \tfrac{1}{2})x^2 + (B+C)x + C - \tfrac{1}{2} \quad . \quad . \quad . \quad (1)\end{aligned}$$

Equating coefficients of x^2 and x, $\therefore B - \frac{1}{2} = 1$, and $B + C = 0$.

$$\therefore B = \tfrac{3}{2}, \quad C = -\tfrac{3}{2}.$$

The latter, of course, satisfies the equation $-2 = C - \frac{1}{2}$, as it should do, since A has been determined.

$$\text{Hence } I = -\tfrac{1}{2}\int\frac{dx}{x+1} + \tfrac{3}{2}\int\frac{(x-1)dx}{x^2+1}$$
$$= -\tfrac{1}{2}\log(x+1) + \tfrac{3}{4}\log(x^2+1) - \tfrac{3}{2}\tan^{-1}x.$$

Otherwise, using imaginaries; since (1) is true for all values of x, real or imaginary, put $x^2 + 1 = 0$, or $x^2 = -1$.

$$\therefore\ -3 = -(B - \tfrac{1}{2}) + (B + C)x + C - \tfrac{1}{2}$$
$$= -B + C + (B + C)x.$$

But $x = \pm i$; hence equating real and imaginary parts,

$$B - C = 3;\quad B + C = 0;\quad \text{or } B = -C = \tfrac{3}{2}.$$

Ex. 2. $\displaystyle\int\frac{x^5dx}{(x^2+x+1)(x-2)(x-1)}$.

We may write at once by case (A) [see also Ex. 2, Art. 371].

$$\frac{x^5}{(x^2+x+1)(x-2)(x-1)} = x + 2 - \frac{1}{3(x-1)} + \frac{32}{7(x-2)} + \frac{Ax+B}{x^2+x+1}.$$
$$\therefore\ x^5 = M(x^2+x+1) + (Ax+B)(x-1)(x-2).$$

Put $x^2 + x + 1 = 0$; then, to find the simplest value of x^5, we have, multiplying by $x - 1$,

$$x^3 - 1 = 0, \text{ or } x^3 = 1;$$
$$\therefore\ x^5 = x^2 = -x - 1.$$
$$\therefore\ -x - 1 = 0 + (Ax + B)(x^2 - 3x + 2)$$
$$= Ax^3 + (B - 3A)x^2 + (2A - 3B)x + 2B$$
$$= A + (B - 3A)(-x - 1) + (2A - 3B)x + 2B$$
$$= (5A - 4B)x + 4A + B \quad \dots\dots\dots \quad (1)$$

x now being either of the roots of the equation $x^2 + x + 1 = 0$.

We shall show that we can still equate coefficients of x and 1; for suppose

$$lx + m = px + q, \text{ where } x = \alpha + \beta i,$$

then $\qquad l(\alpha + \beta i) + m = p(\alpha + \beta i) + q,$

or, $\qquad l\alpha + m + l\beta i = p\alpha + q + p\beta i;$

and equating real and imaginary parts, we easily get

$$l = p,\quad m = q.$$

Hence in (1) above

$$5A - 4B = -1, \quad 4A + B = -1,$$

$$\text{whence } A = -\tfrac{5}{21}, \quad B = -\tfrac{1}{21}.$$

$$\therefore I = \tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}\log(x-1) + \tfrac{32}{7}\log(x-2) - \tfrac{1}{21}\int\frac{5x+1}{x^2+x+1}dx,$$

and the integral

$$\int\frac{5x+1}{x^2+x+1}dx = \tfrac{5}{2}\int\frac{2x+1}{x^2+x+1}dx - \tfrac{3}{2}\int\frac{dx}{x^2+x+1}$$

$$= \tfrac{5}{2}\log(x^2+x+1) - \tfrac{3}{2}\int\frac{dx}{(x+\frac{1}{2})^2+(\sqrt{3}/2)^2}$$

$$= \tfrac{5}{2}\log(x^2+x+1) - \sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}};$$

whence I.

Ex. 3. $\displaystyle\int\frac{x^5-x^3+1}{x^6-1}dx = \int\frac{x^5-x^3+1}{(x+1)(x-1)(x^2+x+1)(x^2-x+1)}dx.$

Let $\dfrac{x^5-x^3+1}{x^6-1} = M + \dfrac{Ax+B}{x^2+x+1}$,

where M denotes all the other partial fractions.

$$\therefore x^5 - x^3 + 1 = M(x^6-1) + (Ax+B)(x^2-1)(x^2-x+1). \quad (1)$$

Put $x^2+x+1=0$, so that $x^3=1$, $x^2=-x-1$,

$$\therefore x^5 - x^3 + 1 = x^2 - 1 + 1 = x^2 = -x-1;$$

$\therefore$ in (1), by repeated substitution,

$$\begin{aligned} -x-1 &= (Ax+B)(-x-2)(-2x) \\ &= 2(Ax+B)(x^2+2x) = 2(Ax+B)(x-1) \\ &= 2\{Ax^2+(B-A)x-B\} \\ &= 2\{A(-x-1)+(B-A)x-B\} \\ &= 2\{(B-2A)x-(A+B)\}; \end{aligned}$$

$$\therefore B - 2A = -\tfrac{1}{2},\ A + B = \tfrac{1}{2}; \quad \text{whence } A = \tfrac{1}{3};\ B = \tfrac{1}{6}.$$

Similarly, we may write

$$x^5 - x^3 + 1 = N(x^6-1) + (Cx+D)(x^2-1)(x^2+x+1). \quad (2)$$

Put $x^2 - x + 1 = 0$, so that $x^3 = -1$, $x^2 = x - 1$;

$$\therefore\ x^5 - x^3 + 1 = -x^2 + 2 = -x + 3.$$

$\therefore$ in (2) by repeated substitution,

$$-x + 3 = (Cx + D)(x - 2)2x = -2(Cx + D)(x + 1)$$
$$= -2\{Cx^2 + (C + D)x + D\} = -2\{(2C + D)x + (D - C)\}$$
$$\therefore\ 2C + D = \tfrac{1}{2},\ C - D = \tfrac{3}{2};\ \text{whence } C = \tfrac{2}{3},\ D = -\tfrac{5}{6};$$

$$\therefore\ \text{the fraction} = \frac{1}{6(x-1)} - \frac{1}{6(x+1)} + \frac{2x+1}{6(x^2+x+1)} + \frac{4x-5}{6(x^2-x+1)}.$$

$$\therefore\ I = \tfrac{1}{6}\log\frac{x-1}{x+1} + \tfrac{1}{6}\log(x^2+x+1) + \tfrac{1}{3}\log(x^2-x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

Ex. 4. $\displaystyle\int\frac{dx}{x^4+1}.$

Since $x^4 + 1 = (x^2 + 1)^2 - 2x^2 = (x^2 + \sqrt{2}\,x + 1)(x^2 - \sqrt{2}x + 1)$,

assume $$\frac{1}{x^4+1} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}.$$

Clearing of fractions,

$$1 = (Ax + B)(x^2 - \sqrt{2}x + 1) + (Cx + D)(x^2 + \sqrt{2}x + 1).$$

Put $x^2 - \sqrt{2}x + 1 = 0$, $\therefore\ x^2 + \sqrt{2}x + 1 = 2\sqrt{2}x$;

$$\therefore\ 1 = (Cx + D)2\sqrt{2}x = 2\sqrt{2}Cx^2 + 2\sqrt{2}Dx$$
$$= 2\sqrt{2}C(\sqrt{2}x - 1) + 2\sqrt{2}Dx.$$
$$\therefore\ 2\sqrt{2}D + 4C = 0,\ \text{ or } D + \sqrt{2}C = 0,$$

and $$-2\sqrt{2}C = 1;\ \therefore\ C = -\frac{1}{2\sqrt{2}},\ D = \tfrac{1}{2}.$$

Similarly, putting $x^2 + \sqrt{2}x + 1 = 0$, we get

$$1 = (Ax + B)(-2\sqrt{2}x) = 2\sqrt{2}A(\sqrt{2}x + 1) - 2\sqrt{2}Bx,$$

whence $A = \dfrac{1}{2\sqrt{2}}$, $B = \tfrac{1}{2}$.

$$\therefore\ \frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\cdot\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{1}{2\sqrt{2}}\cdot\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}.$$

Now $$\int\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx = \tfrac{1}{2}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx + \frac{1}{\sqrt{2}}\int\frac{dx}{x^2+\sqrt{2}x+1}$$
$$= \tfrac{1}{2}\log(x^2 + \sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x + 1).$$

Similarly $\int \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx = \frac{1}{2}\log(x^2-\sqrt{2}x+1) - \tan^{-1}(\sqrt{2}x-1)$,

$$\therefore I = \frac{1}{2\sqrt{2}}\left\{\frac{1}{2}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1)\right\}$$

$$= \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2}$$

$$= \frac{1}{2\sqrt{2}}\left\{\tanh^{-1}\frac{\sqrt{2}x}{1+x^2} + \tan^{-1}\frac{\sqrt{2}x}{1-x^2}\right\}.$$

EXAMPLES LXI.

Integrate :—

1. $\dfrac{1+x+x^2}{x(1+x^2)}\,dx.$

2. $\dfrac{(x+1)dx}{x(1+x^2)}.$

3. $\dfrac{dx}{(x^2+1)(x-2)}.$

4. $\dfrac{x^4\,dx}{(x^2+1)(x-2)}.$

5. $\dfrac{x^3\,dx}{(1-x)^2(x^2+1)}.$

6. $\dfrac{x^5\,dx}{x^4-1}.$

7. $\dfrac{dx}{x^3-1}.$

8. $\dfrac{(x-a)^2\,dx}{x^3+a^3}.$

9. $\dfrac{(x+1)\,dx}{x^3(x^2+x+1)}.$

10. $\dfrac{(1-x)\,dx}{x^4+x^3+x^2+x}.$

11. $\dfrac{x\,dx}{x^4+x^2+1}.$

12. $\dfrac{a^3x^2\,dx}{x^6-a^6}.$

13. $\dfrac{a^3(x^2+2a^2)}{x^6-a^6}\,dx.$

14. $\dfrac{x^2+a^2}{x^4+a^4}\,dx.$

15. $\dfrac{dx}{4x^4+1}.$

16. $\dfrac{(1+\tan\theta)\,d\theta}{\tan\theta+\sec^2\theta}$, [put $\tan\theta = x$].

17. $\sqrt{\tan\theta}\,d\theta$, [put $\tan\theta = x^2$].

18. $\dfrac{\sqrt{x}\,dx}{1+x^2}.$

19. $\dfrac{\sqrt{x^2-1}}{(x^2+1)\sqrt{x}}\,dx$, [put $x = \sec\theta + \tan\theta$].

ANSWERS.

1. $\log x + \tan^{-1} x$. 2. $\log\dfrac{x}{\sqrt{x^2+1}}+\tan^{-1}x$. 3. $\frac{1}{5}\log\dfrac{x-2}{\sqrt{x^2+1}}-\frac{2}{5}\tan^{-1}x$.

4. $\frac{1}{2}x^2+2x+\frac{16}{5}\log(x-2)-\frac{1}{10}\log(x^2+1)-\frac{2}{5}\tan^{-1}x$.

5. $\dfrac{1}{2(1-x)}+\log(1-x)+\frac{1}{2}\tan^{-1}x$. 6. $\frac{1}{2}x^2+\frac{1}{4}\log\dfrac{x^2-1}{x^2+1}$.

7. $\frac{1}{3}\log\dfrac{x-1}{\sqrt{x^2+x+1}}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$.

8. $\frac{4}{3}\log(x+a)-\frac{1}{6}\log(x^2-ax+a^2)-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-a}{\sqrt{3}a}$.

9. $-\dfrac{1}{2x^2}-\log\dfrac{x}{\sqrt{1+x+x^2}}+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$. 10. $\log\dfrac{x}{1+x}-\tan^{-1}x$.

11. $-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{\sqrt{3}}{2x^2+1}=\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x^2+1}{\sqrt{3}}+C$, [put $x^2=y$].

12. $\frac{1}{6}\log\dfrac{x^3-a^3}{x^3+a^3}$, [put $x^3=y$]. 13. $\tanh^{-1}\dfrac{x}{a}+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{a\sqrt{3}x}{a^2-x^2}$.

14. $\dfrac{1}{\sqrt{2}a}\tan^{-1}\dfrac{a\sqrt{2}x}{a^2-x^2}$. 15. $\frac{1}{8}\log\dfrac{2x^2+2x+1}{2x^2-2x+1}+\frac{1}{4}\tan^{-1}\dfrac{2x}{1-2x^2}$.

16. $\theta+\log\sqrt{2+\sin 2\theta}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2\tan\theta+1}{\sqrt{3}}$.

17. $\dfrac{1}{2\sqrt{2}}\log\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\sqrt{2}x}{1-x^2}$.

$$=\frac{1}{\sqrt{2}}\left\{\tan^{-1}\frac{\sqrt{2}x}{1-x^2}-\tanh^{-1}\frac{\sqrt{2}x}{1+x^2}\right\},\text{ where } x^2=\tan\theta,$$

$$=\frac{1}{\sqrt{2}}\{\sin^{-1}\sqrt{\sin 2\theta}-\sinh^{-1}\sqrt{\sin 2\theta}\}.$$

18. Put $x=\tan\theta$, and see preceding example. 19. See Ex. 17.

375. (D) Multiple Imaginary Roots.

Suppose $\phi(x)=0$ to have r equal pairs of imaginary roots; then $\phi(x)$ will have r equal pairs of imaginary factors, *i.e.* will

contain $(ax^2+2hx+b)^r$ as a factor ($h^2 < ab$). In this case we may assume

$$\frac{f(x)}{\phi(x)} = \frac{A_1x+A_2}{ax^2+2hx+b} + \frac{A_3x+A_4}{(ax^2+2hx+b)^2} + \dots$$
$$+ \frac{A_{2r-1}x+A_{2r}}{(ax^2+2hx+b)^r} + \frac{A_{2r+1}}{x-a_{2r+1}} + \dots + \frac{A_n}{x-a_n},$$

where $A_{2r+1}, \dots A_n$ can be found as in Case (A).

*** 376.** **Ex. 1.** $\int \frac{x\,dx}{(x^2+1)^2(x-1)(x-2)}$.

Assume fraction $= M + \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$,

where $M = -\frac{1}{4(x-1)} + \frac{2}{25(x-2)}$.

$$\therefore x = M_1(x^2+1)^2 + \{(Ax+B)(x^2+1)+(Cx+D)\}(x-1)(x-2). \quad (1)$$

Put $x^2+1=0$;

$$\therefore x = (Cx+D)(x^2-3x+2) = -(Cx+D)(3x-1)$$
$$= -\{-3C+(3D-C)x-D\}.$$

$$\therefore \left.\begin{array}{r} C-3D=1 \\ 3C+D=0 \end{array}\right\} \quad \therefore C = \tfrac{1}{10};\ D = -\tfrac{3}{10}.$$

$$\therefore \text{in (1)},\ x = M_1(x^2+1)^2 + (Ax+B)(x^2+1)(x-1)(x-2)$$
$$+ \frac{x-3}{10}(x-1)(x-2)$$
$$= M_1(x^2+1)^2 + (Ax+B)(x^2+1)(x-1)(x-2)$$
$$+ \frac{x^3-6x^2+11x-6}{10}.$$

$$\therefore 0 = M_1(x^2+1)^2 + (Ax+B)(x^2+1)(x-1)(x-2)$$
$$+ \frac{x^3-6x^2+x-6}{10} \quad \dots\dots\dots \quad (2)$$

Dividing out by x^2+1 †, we have

$$0 = M_1(x^2+1) + (Ax+B)(x-1)(x-2) + \frac{x-6}{10}.$$

† Since the equation (2) is true for all values of x, it must be true when $x^2+1=0$; but this makes the first two terms vanish, in which case the third must also vanish. Hence x^2+1 must be a factor of that term. This is true generally.

Again, put $x^2 + 1 = 0$,

$$\therefore 0 = (Ax + B)(x^2 - 3x + 2) + \frac{x-6}{10}.$$

$$\therefore 10(Ax + B)(1 - 3x) + x - 6 = 0,$$

$$10\{3A + B + (A - 3B)x\} + x - 6 = 0,$$

whence $$\left.\begin{aligned} A - 3B &= -\tfrac{1}{10} \\ 3A + B &= \tfrac{6}{10} \end{aligned}\right\}; \quad \therefore A = \tfrac{17}{100},\ B = \tfrac{9}{100}.$$

$$\therefore \text{fraction} = M + \frac{17x + 9}{100(x^2 + 1)} + \frac{x - 3}{10(x^2 + 1)^2}.$$

$$\therefore I = -\tfrac{1}{4}\log(x - 1) + \tfrac{2}{25}\log(x - 2) + \tfrac{17}{200}\log(x^2 + 1)$$
$$+ \tfrac{9}{100}\tan^{-1}x + \tfrac{1}{10}\int\frac{(x - 3)dx}{(x^2 + 1)^2}.$$

Now $$\int\frac{x - 3}{(x^2 + 1)^2}dx = \int\frac{xdx}{(x^2 + 1)^2} - 3\int\frac{dx}{(x^2 + 1)^2}$$
$$= -\frac{1}{2(x^2 + 1)} - 3X, \text{ say,}$$

where $$X = \int\frac{dx}{(x^2 + 1)^2} = \int\frac{\sec^2\theta\, d\theta}{\sec^4\theta}, \text{ if } x = \tan\theta,$$
$$= \int\cos^2\theta\, d\theta = \tfrac{1}{2}\int(1 + \cos 2\theta)d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta)$$
$$= \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{2}\frac{x}{x^2 + 1}.$$

$$\therefore I = -\tfrac{1}{4}\log(x - 1) + \tfrac{2}{25}\log(x - 2) + \tfrac{17}{200}\log(x^2 + 1)$$
$$- \tfrac{3}{50}\tan^{-1}x - \frac{3x + 1}{20(x^2 + 1)}.$$

Ex. 2. $\displaystyle\int\frac{x^3 - x^2 - 2x + 1}{(x^2 + x + 1)^2}dx.$

Assume fraction $= \dfrac{Ax + B}{x^2 + x + 1} + \dfrac{Cx + D}{(x^2 + x + 1)^2}$.

$\therefore x^3 - x^2 - 2x + 1 = (Ax + B)(x^2 + x + 1) + Cx + D \quad . \quad . \ (1)$

Put $x^2 + x + 1 = 0$, or $x^3 = 1$;

$$\therefore 3 - x = Cx + D.$$

Substituting in (1) and transposing,

$$x^3 - x^2 - x - 2 = (Ax + B)(x^2 + x + 1).$$

Dividing out by x^2+x+1, $x-2=Ax+B$.

$$\therefore \text{ fraction} = \frac{x-2}{x^2+x+1} - \frac{x-3}{(x^2+x+1)^2}.$$

$$\therefore I = \int \frac{x-2}{x^2+x+1}\,dx - \int \frac{x-3}{(x^2+x+1)^2}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+1)-5}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{(2x+1)-7}{(x^2+x+1)^2}\,dx$$

$$= \frac{1}{2}\log(x^2+x+1) - \frac{5}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \frac{1}{2(x^2+x+1)} + \frac{7}{2}\int \frac{dx}{\{(x+\frac{1}{2})^2+\frac{3}{4}\}^2}.$$

Call the last integral X, and put $x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$.

$$\therefore X = \frac{7}{2}\int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{9}{16}\sec^4\theta} = \frac{14}{3\sqrt{3}}(\theta + \sin\theta\cos\theta). \quad \text{[See Ex. 1.]}$$

$$= \frac{14}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \frac{7}{6}\frac{2x+1}{x^2+x+1}.$$

$$\therefore I = \frac{1}{2}\log(x^2+x+1) - \frac{1}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \frac{7x+5}{3(x^2+x+1)}.$$

377. The method is the same for a higher number of coincident imaginary roots.

To integrate the general term, which we shall call

$$\frac{Ax+B}{(ax^2+2hx+b)^r}\,dx,$$

it may be written

$$\frac{\frac{A}{2a}(2ax+2h)\,dx}{(ax^2+2hx+b)^r} + \frac{\left(B-\frac{Ah}{a}\right)dx}{(ax^2+2hx+b)^r}.$$

The integral of the first term is $-\frac{A}{2a}\cdot\frac{1}{(r-1)(ax^2+2hx+b)^{r-1}}$ that of the second may be found by trigonometrical substitution as in Ex. 2, above, and then by the use of multiple angles; or by the method of the next chapter [Art. 407].

* EXAMPLES LXII.

Integrate :—

1. $\dfrac{x^2+x+1}{(x^2+1)^2}\,dx.$

2. $\dfrac{(x-1)(x^2+x+2)}{(x^2+1)^2}\,dx.$

3. $\dfrac{x^2\,dx}{(x^2+1)^2(x+2)}.$

4. $\dfrac{x^3\,dx}{(x^2-2x+2)^2}.$

5. $\dfrac{29x-17}{(x^2+1)^2(x-1)(x+2)}\,dx.$

6. $\dfrac{\cos^3\theta\,d\theta}{\cos\theta-\sin\theta}$, [put $\tan\theta = x$].

7. $\dfrac{x(x-1)(2x+1)dx}{(x^2-x+1)^2}.$

ANSWERS.

1. $\tan^{-1}x - \dfrac{1}{2(x^2+1)}.$

2. $\frac{1}{2}\log(x^2+1) - \dfrac{x}{x^2+1} - \tan^{-1}x.$

3. $\frac{4}{25}\log\dfrac{x+2}{\sqrt{x^2+1}} + \frac{3}{25}\tan^{-1}x - \dfrac{2x+1}{10(x^2+1)}.$

4. $\frac{1}{2}\log(x^2-2x+2) + 2\tan^{-1}(x-1) - \dfrac{x}{x^2-2x+2}.$

5. $\log\dfrac{(x-1)(x+2)}{x^2+1} + 5\tan^{-1}x + \dfrac{7+8x}{2(x^2+1)}.$

6. $\frac{1}{2}\theta - \frac{1}{4}\log(\cos\theta-\sin\theta) - \frac{1}{4}\cos\theta(\cos\theta-\sin\theta).$

7. $\log(x^2-x+1) + \dfrac{4}{3\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}} + \dfrac{5-4x}{3(x^2-x+1)}.$

CHAPTER XXV.

METHOD OF SUCCESSIVE REDUCTION.

378. In the following chapter we shall endeavour to show how certain forms involving high powers can (when otherwise not immediately integrable) be expressed in terms of forms the same as before, but of lower degree. The method adopted is Integration by Parts, or an equivalent to it; the formula obtained is called a *formula of reduction;* and the method in which we successively apply this formula is called the *method of successive reduction.* As this method is specially useful in the case of certain definite integrals, we shall first make a few necessary remarks concerning definite integration, going more fully into that subject in the next chapter.

379. Preliminary Remarks on Definite Integration.

Let $\phi(x) = f'(x)$, so that $\int\phi(x)dx = f(x)$, and

$$\int_a^b \phi(x)dx = f(b) - f(a) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(*a*) The first point to be observed is that, since x does not appear on the right of (1), $\int_a^b \phi(x)dx$ *is independent of the letter* x.

Hence, for example, $\int_a^b \phi(x)dx = \int_a^b \phi(y)dy$, since each is equal to $f(b) - f(a)$.

(*b*) Next, suppose we interchange the limits b and a; then

$$\int_b^a \phi(x)dx = f(a) - f(b) = -\{f(b) - f(a)\} = -\int_a^b \phi(x)dx,$$

$$\text{i.e.} \quad \int_b^a \phi(x)dx = -\int_a^b \phi(x)dx \, . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, *interchanging the limits changes the sign of the integral.*

(c) Again, it must not be forgotten that whenever the variable is changed by a substitution, *the limits must be changed as well.*

For example, in the integral $\int_0^1 \frac{dx}{(x+1)\sqrt{1-x^2}}$, if we put $x = \sin\theta$; then when $x = 1$, $\theta = \frac{1}{2}\pi$; and when $x = 0$, $\theta = 0$.

$$\text{Hence} \int_0^1 \frac{dx}{(x+1)\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta + 1}$$

$$= \left\{\tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)\right\}_0^{\frac{\pi}{2}} \text{ [by Ex. LVIII. 2 (2), p. 367]}$$

$$= 0 - (-1) = 1.$$

380. Integration of $x^n e^{-ax} dx$, n being a $+^{\text{ve}}$ Integer.

Integrating by parts,

$$\int x^n e^{-ax} dx = -\frac{1}{a} x^n e^{-ax} + \frac{n}{a}\int x^{n-1} e^{-ax} dx \quad . \quad . \quad (1)$$

and since $\int x^n e^{-ax} dx$ is expressed in terms of $\int x^{n-1} e^{-ax} dx$, which is of the same form, but one degree lower, we have a formula of reduction. The advantage of this formula is that we need not integrate again, since by changing n into $n-1$ we can express $\int x^{n-1} e^{-ax} dx$ in terms of $\int x^{n-2} e^{-ax} dx$; and so on, the operation being repeated until the power of x is reduced to zero.

Thus $\int x^n e^{-ax} dx$

$$= -\frac{1}{a} x^n e^{-ax} + \frac{n}{a}\left\{-\frac{1}{a} x^{n-1} e^{-ax} + \frac{n-1}{a}\int x^{n-2} e^{-ax} dx\right\}$$

$$= -\frac{1}{a} x^n e^{-ax} - \frac{n}{a^2} x^{n-1} e^{-ax} + \frac{n(n-1)}{a^2}\left\{-\frac{1}{a} x^{n-2} e^{-ax} + \frac{n-2}{a}\int x^{n-3} e^{-ax} dx\right\}$$

$$= -\frac{1}{a} e^{-ax}\left\{x^n + \frac{n}{a} x^{n-1} + \frac{n(n-1)}{a^2} x^{n-2}\right\} + \frac{n(n-1)(n-2)}{a^3}\int x^{n-3} e^{-ax} dx$$

$= \text{etc.}$

$$= -\frac{1}{a} e^{-ax}\left\{x^n + \frac{n}{a} x^{n-1} + \dots + \frac{n(n-1)\dots 3.2}{a^{n-1}} x\right\}$$

$$+ \frac{n(n-1)\dots 3.2.1}{a^n}\int e^{-ax} dx$$

$$= -\frac{1}{a} e^{-ax}\left\{x^n + \frac{n}{a} x^{n-1} + \frac{n(n-1)}{a^2} x^{n-2} + \dots + \frac{n!}{a^n}\right\}.$$

381. $\int_0^\infty x^n e^{-ax}dx$.

Since $\underset{x=\infty}{lt}\ x^n e^{-ax}=0$ [see Art. 153, Ex. 6, (4)], for $+^{\text{ve}}$ values of n and a, and $\underset{x=0}{lt}\ x^n e^{-ax}=0$, it follows, from equation (1) of the preceding article, that

$$\int_0^\infty x^n e^{-ax}dx=\frac{n}{a}\int_0^\infty x^{n-1}e^{-ax}dx$$

(and repeating the formula) $$=\frac{n(n-1)\dots 3.2.1}{a^n}\int_0^\infty e^{-ax}dx$$

$$=-\frac{n!}{a^{n+1}}\left[e^{-ax}\right]_0^\infty=\frac{n!}{a^{n+1}}.$$

382. Integration of $x^{m-1}(1-x)^{n-1}dx$, n being a $+^{\text{ve}}$ Integer.

Integrating by parts, we have

$$\int x^{m-1}(1-x)^{n-1}dx=\frac{x^m}{m}(1-x)^{n-1}+\frac{n-1}{m}\int x^m(1-x)^{n-2}dx \quad (1)$$

a formula of reduction in which the power of $1-x$ is lowered by 1, and that of x raised by 1.

By successive application we shall arrive at the integral $\int x^{m+n-2}dx$, which can be obtained at once.

We could also, of course, expand by the Binomial Theorem.

383. $\int_0^1 x^{m-1}(1-x)^{n-1}dx$.

Since in (1), $x^m(1-x)^{n-1}$ vanishes when $x=1$ and $x=0$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx=\frac{n-1}{m}\int_0^1 x^m(1-x)^{n-2}dx$$

$$=\frac{n-1}{m}\cdot\frac{n-2}{m+1}\int_0^1 x^{m+1}(1-x)^{n-3}dx$$

$$=\text{etc.}=\frac{(n-1)(n-2)\dots 2.1}{m(m+1)\dots(m+n-2)}\int_0^1 x^{m+n-2}dx,$$

that is, since $\int_0^1 x^{m+n-2}\,dx=\dfrac{1}{m+n-1}$,

$$= \frac{(n-1)!}{m(m+1)\dots(m+n-1)}$$

$$= \frac{(m-1)!(n-1)!}{(m+n-1)!}.$$

384. The condition, namely, that n is a $+^{ve}$ integer, is necessary, since otherwise the factor $(1-x)^{n-1}$ will not ultimately disappear; but m may be *any quantity*.

Suppose, however, that m is a $+^{ve}$ integer while n is any quantity. Put $1-x=y$, then

$$\int x^{m-1}(1-x)^{n-1}dx = -\int (1-y)^{m-1}y^{n-1}dy;$$

and the factor $(1-y)^{m-1}$ may be made to ultimately disappear by the same method as before.

Again, $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}dx = -\int_1^0 (1-y)^{m-1}y^{n-1}dy$ [Art. 379 (*c*).]

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy \quad \text{[Art. 379 (b).]}$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx. \quad \text{[Art. 379 (a).]}$$

Hence, if either m or n be a $+^{ve}$ integer, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^1 x^{n-1}(1-x)^{m-1}dx = \frac{(m-1)!(n-1)!}{(m+n-1)!};$$

this last result also follows from the symmetry of the answer, for m and n may be interchanged without altering its value.

If, however, m is not integral, the answer must be written in

the form $$\frac{(n-1)!}{(m+n-1)(m+n-2)\dots(m+1)m}.$$

Similarly, if n is not integral, it must be written

$$\frac{(m-1)!}{(m+n-1)(m+n-2)\dots(n+1)n}.$$

385. Integration of $\sin^n\theta\,d\theta$ and $\cos^n\theta\,d\theta$, n being integral.

We have $\int\sin^n\theta d\theta = -\int\sin^{n-1}\theta\, d(\cos\theta)$
(integrating by parts)

$$= -\sin^{n-1}\theta\cos\theta + (n-1)\int\sin^{n-2}\theta\cos^2\theta d\theta$$

(and, putting $\cos^2\theta = 1 - \sin^2\theta$)

$$= -\sin^{n-1}\theta\cos\theta + (n-1)\int(\sin^{n-2}\theta - \sin^n\theta)d\theta$$

$$\therefore\ I = -\sin^{n-1}\theta\cos\theta + (n-1)\int\sin^{n-2}\theta d\theta - (n-1)I$$

$\therefore$ transposing and dividing by n,

$$I = -\frac{1}{n}\sin^{n-1}\theta\cos\theta + \frac{n-1}{n}\int\sin^{n-2}\theta d\theta \quad . \quad . \quad (1)$$

a formula of reduction in which the degree is lowered by two.

If n be $-^{ve}$, $= -m$ say, then

$$\int\frac{d\theta}{\sin^m\theta} = \frac{1}{m}\frac{\cos\theta}{\sin^{m+1}\theta} + \frac{m+1}{m}\int\frac{d\theta}{\sin^{m+2}\theta}.$$

Now the L.H.S. has a lower $+^{ve}$ power of $\sin\theta$ than the R.H.S., but, as is always the case when the factor to be reduced is in the denominator, we may obtain a formula of reduction by transposition.

Thus, transposing, and dividing by $-\dfrac{m+1}{m}$, we have

$$\int\frac{d\theta}{\sin^{m+2}\theta} = -\frac{1}{m+1}\frac{\cos\theta}{\sin^{m+1}\theta} + \frac{m}{m+1}\int\frac{d\theta}{\sin^m\theta}.$$

Now change m into $n-2$, and this becomes

$$\int\frac{d\theta}{\sin^n\theta} = -\frac{1}{n-1}\frac{\cos\theta}{\sin^{n-1}\theta} + \frac{n-2}{n-1}\int\frac{d\theta}{\sin^{n-2}\theta}. \quad . \quad (2)$$

By successive reduction we shall arrive at the integral $\int\dfrac{d\theta}{\sin\theta}$ $= \log\tan\dfrac{\theta}{2}$, if n be odd; and at the integral $\int\dfrac{d\theta}{\sin^2\theta} = -\cot\theta$, if n be even.

We may also obtain the formula (2) by *beginning with the reduced integral* $\int \frac{d\theta}{\sin^{n-2}\theta}$. Or, we may work as follows :—

$$\int \frac{d\theta}{\sin^n \theta} = \int \frac{\sin^2\theta + \cos^2\theta}{\sin^n\theta} d\theta = \int \frac{d\theta}{\sin^{n-2}\theta} + \int \frac{\cos\theta\, d(\sin\theta)}{\sin^n\theta}$$

$$= \int \frac{d\theta}{\sin^{n-2}\theta} - \frac{1}{n-1}\frac{\cos\theta}{\sin^{n-1}\theta} - \frac{1}{n-1}\int \frac{d\theta}{\sin^{n-2}\theta}$$

$$= -\frac{1}{n-1}\frac{\cos\theta}{\sin^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\sin^{n-2}\theta} \text{ as before.}$$

Similarly $\int \cos^n\theta d\theta$ may be reduced; it can also be deduced from $\int \sin^n\theta\, d\theta$ by changing θ into $\frac{\pi}{2} - \theta$.

These two integrals are particular cases of the more general integral $\int \sin^m\theta \cos^n\theta\, d\theta$ given below.

386. $\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta$ **and** $\int_0^{\frac{\pi}{2}} \cos^n\theta d\theta$.

Referring to (1) above, since the first term on the right vanishes when $\theta = \frac{\pi}{2}$ and when $\theta = 0$,

$$\therefore \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta \quad . \quad . \quad . \quad (3)$$

First, *let* n *be even*; and reapply the formula by changing n into $n-2$; the above becomes

$$\frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4}\theta d\theta$$

$$= \text{etc.} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{(n-1)(n-3)\ldots3.1}{n(n-2)\ldots4.2}\frac{\pi}{2}.$$

Secondly, *let* n *be odd*, and we get

$$I = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin\theta d\theta$$

$$= \frac{(n-1)(n-3)\ldots4.2}{n(n-2)\ldots5.3}, \text{ since } \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = 1.$$

Again, change θ into $\frac{\pi}{2} - \theta$ [see Art. 379]; then

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = -\int_{\frac{\pi}{2}}^0 \cos^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta.$$

Hence the two given *definite* integrals are equal.

These results may be written thus :—

$$\int_0^{\frac{\pi}{2}} \sin^{2p} \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^{2p} \theta \, d\theta = \frac{(2p-1)(2p-3)\ldots 5.3.1}{2p(2p-2)\ldots 6.4.2} \frac{\pi}{2};$$

$$\int_0^{\frac{\pi}{2}} \sin^{2p+1} d\theta = \int_0^{\frac{\pi}{2}} \cos^{2p+1} \theta \, d\theta = \frac{2p(2p-2)\ldots 6.4.2}{(2p+1)(2p-1)\ldots 7.5.3}.$$

387. Examples.

Ex. 1. $\int x^3 e^{-x} \, dx = -x^3 e^{-x} + 3\int x^2 e^{-x} dx.$

$= -x^3 e^{-x} + 3\{-x^2 e^{-x} + 2\int x e^{-x} \, dx\}$ (diminishing numbers by 1)

$= -x^3 e^{-x} - 3x^2 e^{-x} + 3 \,.\, 2\{-x e^{-x} + \int e^{-x} \, dx\}$

$= -\epsilon^{-x}(x^3 + 3x^2 + 6x + 6).$

Ex. 2. $\int_0^1 x^4 \sqrt{1-x} \, dx.$

Since the given integral $= \int_0^1 \sqrt{x}(1-x)^4 dx$ [see Art. 384], we have

$$I = \int_0^1 \sqrt{x}(1-x)^4 dx = \left[\tfrac{2}{3} x^{\frac{3}{2}} (1-x)^4\right]_0^1 + \tfrac{2}{3}.4\int x^{\frac{3}{2}}(1-x)^3 dx$$

$$= \tfrac{2}{3}.4\int_0^1 x^{\frac{3}{2}}(1-x)^3 dx$$

$$= \tfrac{2}{3}.4.\tfrac{2}{5}.3\int_0^1 x^{\frac{5}{2}}(1-x)^2 \, dx = \tfrac{2}{3}.4.\tfrac{2}{5}.3.\tfrac{2}{7}.2\int_0^1 x^{\frac{7}{2}}(1-x) dx$$

$$= \tfrac{2}{3}.4.\tfrac{2}{5}.3.\tfrac{2}{7}.2.\tfrac{2}{9}.1\int_0^1 x^{\frac{9}{2}} \, dx$$

$$= \tfrac{2}{3}.\tfrac{2}{5}.\tfrac{2}{7}.\tfrac{2}{9}.\tfrac{2}{11}.4!, \text{ since } \int_0^1 x^{\frac{9}{2}} \, dx = \tfrac{2}{11}\left[x^{\frac{11}{2}}\right]_0^1 = \tfrac{2}{11}.$$

Ex. 3. $\int (x-2)^3 (x-3)^{\frac{3}{2}} dx$.

Put $x - 3 = y$;

then $$I = \int y^{\frac{3}{2}}(y+1)^3 dy = \tfrac{2}{5} y^{\frac{5}{2}}(y+1)^3 - \tfrac{2}{5}.3\int y^{\frac{5}{2}}(y+1)^2 dy$$
$$= \tfrac{2}{5} y^{\frac{5}{2}}(y+1)^3 - \tfrac{2}{5}.3\left\{\tfrac{2}{7} y^{\frac{7}{2}}(y+1)^2 - \tfrac{2}{7}.2\int y^{\frac{7}{2}}(y+1)\,dy\right\}$$
$$= \tfrac{2}{5} y^{\frac{5}{2}}(y+1)^3 - \tfrac{2}{5}.\tfrac{2}{7}.3y^{\frac{7}{2}}(y+1)^2 + \tfrac{2}{5}.\tfrac{2}{7}.3.2\left\{\tfrac{2}{9} y^{\frac{9}{2}}(y+1) - \tfrac{2}{9}.1\int y^{\frac{9}{2}} dy\right\}$$
$$= \tfrac{2}{5} y^{\frac{5}{2}}(y+1)^3 - \tfrac{2}{5}.\tfrac{2}{7}.3y^{\frac{7}{2}}(y+1)^2 + \tfrac{2}{5}.\tfrac{2}{7}.\tfrac{2}{9}.3.2y^{\frac{9}{2}}(y+1) - \tfrac{2}{5}.\tfrac{2}{7}.\tfrac{2}{9}.\tfrac{2}{11}.3!\,y^{\frac{11}{2}}, \text{ where } y = x - 3; \text{ etc.}$$

NOTE.—In this example I could be found more simply by expanding $(y+1)^3$. See Art. 331.

Ex. 4. $\int \sin^6\theta\, d\theta$ and $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta$.

Use s and c to denote $\sin\theta$ and $\cos\theta$ respectively. Then, since $dc = -s\,d\theta$, and $ds = c\,d\theta$, we have

$$\int s^6 d\theta = -\int s^5 dc = -s^5c + 5\int s^4c^2 d\theta \text{ (integrating by parts)}$$
$$= -s^5c + 5\int s^4 d\theta - 5\int s^6 d\theta \quad (\because c^2 = 1 - s^2)$$
$$= -\tfrac{1}{6}s^5c + \tfrac{5}{6}\int s^4 d\theta \text{ (by transposition and division)} \quad . \quad . \quad (1)$$
$$= -\tfrac{1}{6}s^5c + \tfrac{5}{6}\left[-\tfrac{1}{4}s^3c + \tfrac{3}{4}\int s^2 d\theta\right] \text{ (reapplying the formula)}$$
$$= -\tfrac{1}{6}s^5c - \tfrac{5}{6}.\tfrac{1}{4}s^3c + \tfrac{5}{6}.\tfrac{3}{4}\left[-\tfrac{1}{2}sc + \tfrac{1}{2}\theta\right] \text{ (reapplying the formula)}$$
$$= -\frac{1}{6}\sin^5\theta\cos\theta - \frac{5}{6.4}\sin^3\theta\cos\theta - \frac{5.3}{6.4.2}\sin\theta\cos\theta + \frac{5.3.1}{6.4.2}\theta.$$

From this it follows at once that $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{5.3.1}{6.4.2}\frac{\pi}{2}$.

Or (when the indefinite integral is not required) from (1)

$$\int_0^{\frac{\pi}{2}} s^6 d\theta = \frac{5}{6}\int_0^{\frac{\pi}{2}} s^4 d\theta = \frac{5.3}{6.4}\int_0^{\frac{\pi}{2}} s^2 d\theta = \frac{5.3.1}{6.4.2}\int_0^{\frac{\pi}{2}} d\theta = \frac{5.3.1}{6.4.2}\frac{\pi}{2}.$$

Ex. 5. $\int \frac{d\theta}{\cos^3\theta}$.

Beginning with the *reduced integral* (Art. 385), we have

$$\int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta\, d\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos^2\theta} - \int \sin\theta\left(\frac{-2}{\cos^3\theta}\right)(-\sin\theta)d\theta.$$
$$= \frac{\sin\theta}{\cos^2\theta} - 2\int \frac{1 - \cos^2\theta}{\cos^3\theta}d\theta = \frac{\sin\theta}{\cos^2\theta} - 2\int \frac{d\theta}{\cos^3\theta} + 2\int \frac{d\theta}{\cos\theta},$$

whence $$\int \frac{d\theta}{\cos^3\theta} = \frac{1}{2}\frac{\sin\theta}{\cos^2\theta} + \frac{1}{2}\int \frac{d\theta}{\cos\theta} = \frac{1}{2}\frac{\sin\theta}{\cos^2\theta} + \frac{1}{2}\log(\tan\theta + \sec\theta).$$

The definite integral $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos^3\theta}$ is evidently infinite; a similar result applies in the general case where n is a $-^{ve}$ *integer*.

EXAMPLES LXIII.

1. Evaluate the following definite integrals by changing the variable :—

(1) $\int_0^1 \frac{x\,dx}{(1+x)\sqrt{1-x^2}}$, [put $x = \sin\theta$].

(2) $\int_0^1 \frac{1-4x+2x^2}{\sqrt{2x-x^2}}\,dx$, [put $x = 1 - \sin\theta$].

(3) $\int_{2\sqrt{3}}^{\infty} \frac{dx}{x\sqrt{4+x^2}}$, [put $x = 2\tan\theta$].

(4) $\int_0^a \sqrt{a^2-x^2}\,dx$, [put $x = a\sin\theta$].

(5) $\int_0^a \sqrt{a^2+x^2}\,dx$, [put $x = a\tan\theta$].

(6) $\int_a^{2a} \sqrt{x^2-a^2}\,dx$, [put $x = a\sec\theta$].

(7) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$, [put $x = \cos\theta$.]

2. Integrate :—

(1) $e^x x^3\,dx$. (2) $\frac{x^5\,dx}{e^x}$. (3) $\sin^4\theta\,d\theta$.

(4) $\operatorname{cosec}^4\theta\,d\theta$. (5) $\operatorname{cosec}^6\theta\,d\theta$. (6) $\sin^5\theta\,d\theta$.

(7) $\operatorname{cosec}^5\theta\,d\theta$.

3. Evaluate :—

(1) $\int_0^\infty \frac{x^4\,dx}{e^x}$. (2) $\int_0^\infty \frac{x^5\,dx}{a^{2x}}$. (3) $\int_0^1 x^3(1-x)^{\frac{7}{2}}\,dx$.

(4) $\int_0^1 \left(\frac{x}{\sqrt{1-x}}\right)^5 dx$. (5) $\int_0^1 x^9(1-x^2)^{\frac{2}{3}}\,dx$.

(6) $\int_0^{\frac{\pi}{2}} \sin^4\theta\,d\theta$. (7) $\int_0^{\frac{\pi}{2}} \cos^6\theta\,d\theta$.

(8) $\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta.$ (9) $\int_0^{\frac{\pi}{2}} \cos^7\theta\, d\theta.$

(10) $\int_{-\infty}^0 x^n e^x\, dx.$ (11) $\int_0^1 (\log x)^n\, dx.$

4. Find a formula of reduction for $\int \frac{x^m\, dx}{(1+x)^{n+1}}$; and if $n > m$, show that

$$\int_0^\infty \frac{x^m\, dx}{(1+x)^{n+1}} = \frac{m!}{n(n-1)\dots(n-m)},\ m \text{ being a } +^{ve} \text{ integer.}$$

5. Show that

$$\int_0^\infty \frac{x^{m-1}\, dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}\, dx}{(1+x)^{m+n}} = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

if m and n are $+^{ve}$ integers; and is of that form if either m or n is a $+^{ve}$ integer.

6. Show that

$$\int x^n \cos ax\, dx = \frac{x^{n-1}}{a^2}(ax\sin ax + n\cos ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax\, dx.$$

Hence if $k = \pi/2a$, and ${}_nP_r = n(n-1)\dots(n-r+1)$, show that if n be even

$$\int_0^k x^n \cos ax\, dx = \frac{1}{a}\left\{k^n - \frac{{}_nP_2}{a^2}k^{n-2} + \frac{{}_nP_4}{a^4}k^{n-4} - \dots + (-1)^{n/2}\frac{n!}{a^n}\right\};$$

and if n be odd, the last term in the brackets is $(-1)^{\frac{n+1}{2}}\frac{n!}{a^n}$.

ANSWERS.

1. (1) $\frac{\pi}{2} - 1.$ (2) $0.$ (3) $\frac{1}{4}\log 3.$ (4) $\frac{\pi a^2}{4}.$

(5) $\frac{a^2}{2}\{\sqrt{2} + \log(1+\sqrt{2})\}.$ (6) $\frac{a^2}{2}\{2\sqrt{3} - \log(2+\sqrt{3})\}.$

(7) $\frac{\pi}{2} - 1.$

2. (1) $e^x(x^3 - 3x^2 + 6x - 6)$. (2) $-e^{-x}(x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120)$

(3) $-\frac{1}{4}s^3c - \frac{3}{8}sc + \frac{3}{8}\theta.$ (4) $-\frac{1}{3}\frac{c}{s^3} - \frac{2}{3}\frac{c}{s}.$ (5) $-\frac{1}{5}\frac{c}{s^5} - \frac{4}{15}\frac{c}{s^3} - \frac{8}{15}\frac{c}{s}.$

(6) $-\frac{1}{5}s^4c - \frac{4}{15}s^2c - \frac{8}{15}c.$ (7) $-\frac{1}{4}\frac{c}{s^4} - \frac{3}{8}\frac{c}{s^2} + \frac{3}{8}\log\tan\frac{\theta}{2}.$

3. (1) $4!$. (2) $\dfrac{5!}{(2\log a)^6}$. (3) $\dfrac{2^4.3!}{9.11.13.15}$. (4) $\dfrac{2^6.5!}{3.3.5.7}$.

(5) $\frac{1}{2}4!\,\dfrac{3^5}{17.14.11.8.5}$. (6) $\dfrac{3\pi}{16}$. (7) $\dfrac{5.3.1}{6.4.2}.\dfrac{\pi}{2}$. (8) $\frac{8}{15}$.

(9) $\dfrac{6.4.2}{7.5.3}$. (10) and (11) $(-1)^n n!$.

388. Integration of $\sin^m\theta\cos^n\theta\,d\theta$—when immediately integrable.—Before obtaining formulæ of reduction we shall consider the cases in which the expression is immediately integrable.

(A) *When either* m *or* n *is an odd* $+^{ve}$ *integer.*

For
$$\int\sin^m\theta\cos^n\theta\,d\theta = \int\sin^m\theta\cos^{n-1}\theta\,d(\sin\theta)$$
$$= \int\sin^m\theta\,(1-\sin^2\theta)^{\frac{n-1}{2}}\,d(\sin\theta)$$
$$= \int z^m(1-z^2)^{\frac{n-1}{2}}\,dz,$$

and, if n is an odd $+^{ve}$ integer, $\dfrac{n-1}{2}$ is integral, in which case $(1-z^2)^{\frac{n-1}{2}}$ can be expanded in a finite series, and the integral easily obtained.

Similarly, if m be an odd $+^{ve}$ integer.

Ex. 1.
$$\int\sin^{\frac{3}{2}}\theta\cos^5\theta\,d\theta = \int\sin^{\frac{3}{2}}\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$$
$$= \int z^{\frac{3}{2}}(1-z^2)^2\,dz = \int z^{\frac{3}{2}}(1-2z^2+z^4)dz$$
$$= \tfrac{2}{5}z^{\frac{5}{2}} - \tfrac{4}{9}z^{\frac{9}{2}} + \tfrac{2}{13}z^{\frac{13}{2}} = 2\sin^{\frac{5}{2}}\theta\,(\tfrac{1}{5} - \tfrac{2}{9}\sin^2\theta + \tfrac{1}{13}\sin^4\theta).$$

Ex. 2. $\int\sin^5\theta\,d\theta = \int(1-\cos^2\theta)^2\sin\theta\,d\theta = -\int(1-z^2)^2dz$, if $z=\cos\theta$; etc.

(B) *When* m + n = − 2r, *an even* $-^{ve}$ *integer.*

For
$$\int\sin^m\theta\cos^n\theta\,d\theta = \int\tan^m\theta\cos^{m+n}\theta\,d\theta = \int\tan^m\theta\sec^{-(m+n)}\theta\,d\theta$$
$$= \int\tan^m\theta\sec^{-(m+n+2)}\theta\sec^2\theta\,d\theta$$
$$= \int z^m(1+z^2)^{-\frac{1}{2}(m+n+2)}\,dz, \text{ if } z=\tan\theta.$$

This can be integrated by expansion if $-\frac{1}{2}(m+n+2) = 0$ or a $+^{ve}$ integer; *i.e.* if $m+n+2=0$ or an *even* $-^{ve}$ integer; *i.e.* if $m+n$ is an even $-^{ve}$ integer.

Similarly $\int \sin^m\theta\cos^n\theta\,d\theta = \int \cot^n\theta\,\mathrm{cosec}^{-(m+n+2)}\theta\,\mathrm{cosec}^2\theta\,d\theta$

$$= -\int z^n(1+z^2)^{-\frac{1}{2}(m+n+2)}\,dz, \text{ if } z=\cot\theta,$$

which is of the same form as above, and leads to the same result.

Cor.—$\int \frac{\sin^p\theta}{\cos^q\theta}\,d\theta$ and $\int\frac{\cos^p\theta}{\sin^q\theta}\,d\theta$ can be immediately integrated if q is greater than p by an even integer, whatever p and q may be.

Ex. 1. $\int\frac{\sin^3\theta}{\cos^9\theta}\,d\theta$. Here $m+n=3-9=-6$, an even $-^{ve}$ integer.

$$\therefore\ I=\int\tan^3\theta\sec^6\theta\,d\theta=\int\tan^3\theta(1+\tan^2\theta)^2\sec^2\theta\,d\theta$$

$$=\int z^3(1+2z^2+z^4)dz=\frac{z^4}{4}+\frac{z^6}{3}+\frac{z^8}{8}, \text{ where } z=\tan\theta.$$

Ex. 2. $\int\frac{\cos^{\frac{5}{3}}\theta}{\sin^{\frac{11}{3}}\theta}\,d\theta$. Here $m+n=-2$.

$$\therefore\ I=\int\cot^{\frac{5}{3}}\theta\,\mathrm{cosec}^2\theta\,d\theta=-\int z^{\frac{5}{3}}\,dz=-\tfrac{3}{8}\cot^{\frac{8}{3}}\theta.$$

Ex. 3. $\int\frac{d\theta}{\cos^{2n}\theta}$. Here $m+n=-2n$.

$$\therefore\ I=\int\sec^{2n-2}\theta\sec^2\theta\,d\theta=\int(1+\tan^2\theta)^{n-1}\sec^2\theta\,d\theta$$

$$=\int(1+z^2)^{n-1}\,d\theta,$$

which can be expanded in finite terms if n is a $+^{ve}$ integer.

Ex. 4. $\int\frac{d\theta}{\sin^2\theta\cos^4\theta}$. Here $m+n=-6$.

$$\therefore\ I=\int\frac{1}{\tan^2\theta}\sec^6\theta\,d\theta=\int\frac{1}{z^2}(1+z^2)^2\,dz$$

$$=\int\left(\frac{1}{z^2}+2+z^2\right)dz=-\frac{1}{z}+2z+\tfrac{1}{3}z^3$$

$$=-\cot\theta+2\tan\theta+\tfrac{1}{3}\tan^3\theta.$$

Ex. 5. $\int\frac{d\theta}{\sin^5\theta}=\frac{1}{2^5}\int\frac{d\theta}{\sin^5\frac{\theta}{2}\cos^5\frac{\theta}{2}}$, in which $m+n=-10$,

$$=\frac{1}{2^5}\int\frac{1}{\tan^5\frac{1}{2}\theta}\sec^{10}\frac{\theta}{2}\,d\theta=\frac{1}{2^4}\int\frac{(1+z^2)^4}{z^5}\,dz, \text{ if } z=\tan\tfrac{1}{2}\theta;\ \text{etc.}$$

(C) *When* m *and* n *are both even* $+^{ve}$ *integers, by the use of multiple angles.*

Although this method can be adopted when m and n are *not* both even, it is not necessary, since in that case the method (A) can be used.

We can show that $\sin^m\theta\cos^n\theta$ is expressible in terms of a series of sines and cosines of multiple angles of the first degree; but, unless m and n are simple, the method is complicated and that of reduction is preferable.

Ex. 1. $\int\sin^4\theta\cos^2\theta\,d\theta$.

Now $\sin^4\theta\cos^2\theta = \frac{1}{8}(2\sin^2\theta)^2\,2\cos^2\theta = \frac{1}{8}(1-\cos 2\theta)^2(1+\cos 2\theta)$

$$= \tfrac{1}{8}\left\{1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right\}(1+\cos 2\theta)$$

$$= \tfrac{1}{16}(3 - 4\cos 2\theta + \cos 4\theta)(1+\cos 2\theta)$$

$$= \tfrac{1}{16}(3 - \cos 2\theta - 4\cos^2 2\theta + \cos 4\theta + \cos 4\theta\cos 2\theta)$$

$$= \tfrac{1}{16}\{3 - \cos 2\theta - 2(1+\cos 4\theta) + \cos 4\theta + \tfrac{1}{2}(\cos 6\theta + \cos 2\theta)\}$$

$$= \tfrac{1}{32}(2 - \cos 2\theta - 2\cos 4\theta + \cos 6\theta)$$

$$\therefore\ I = \tfrac{1}{32}(2\theta - \tfrac{1}{2}\sin 2\theta - \tfrac{1}{2}\sin 4\theta + \tfrac{1}{6}\sin 6\theta).$$

Ex. 2. $\int\sin^6\theta\,d\theta$.

Since $2i\sin\theta = e^{\theta i} - e^{-\theta i}$,

$$\therefore\ -2^6\sin^6\theta = e^{6\theta i} - 6e^{5\theta i}e^{-\theta i} + 15e^{4\theta i}e^{-2\theta i} - 20e^{3\theta i}e^{-3\theta i} + \dots$$

$$= e^{6\theta i} - 6e^{4\theta i} + 15e^{2\theta i} - 20 + \dots$$

$$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20;$$

whence $\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$.

Or, $\sin^6\theta = \frac{1}{16}(4\sin^3\theta)^2 = \frac{1}{16}(3\sin\theta - \sin 3\theta)^2$

$$= \tfrac{1}{16}(9\sin^2\theta - 6\sin\theta\sin 3\theta + \sin^2 3\theta)$$

$$= \tfrac{1}{32}\{9(1-\cos 2\theta) - 6(\cos 2\theta - \cos 4\theta) + 1 - \cos 6\theta\}$$

$$= -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10) \text{ as before.}$$

$$\therefore\ I = -\tfrac{1}{32}(\tfrac{1}{6}\sin 6\theta - \tfrac{3}{2}\sin 4\theta + \tfrac{15}{2}\sin 2\theta - 10\theta).$$

EXAMPLES LXIV.

1. Integrate :—

(1) $\sin^3\theta\, d\theta.$ (2) $\cos^7\theta\, d\theta.$ (3) $\cos^2\theta\sin^3\theta\, d\theta.$

(4) $\dfrac{\sin^3\theta}{\cos^2\theta}\, d\theta.$ (5) $\dfrac{\cos^3\theta}{\sqrt{\sin\theta}}\, d\theta.$ (6) $\cot\theta\, d\theta.$

(7) $\tan^5\theta\, d\theta.$

2. Integrate :—

(1) $\dfrac{\sin^2\theta}{\cos^6\theta}\, d\theta.$ (2) $\dfrac{\cos^2\theta}{\sin^4\theta}\, d\theta.$ (3) $\dfrac{\sin^{\frac{3}{2}}\theta}{\cos^{\frac{15}{2}}\theta}\, d\theta.$

(4) $\dfrac{d\theta}{\cos^4\theta}.$ (5) $\operatorname{cosec}^6\theta\, d\theta.$ (6) $\dfrac{d\theta}{\sin\theta\cos\theta}.$

(7) $\dfrac{d\theta}{\sin^3\theta\cos^3\theta}.$ (8) $\dfrac{d\theta}{\sin^5\theta\cos\theta}.$ (9) $\dfrac{d\theta}{\sin^{\frac{1}{2}}\theta\cos^{\frac{3}{2}}\theta}.$

(10) $\dfrac{d\theta}{\sin^3\theta}.$

3. Integrate :—

(1) $\sin^2\theta\cos^2\theta\, d\theta.$ (2) $\sin^4\theta\, d\theta.$ (3) $\sin^2\theta\cos^4\theta\, d\theta.$

(4) $\cos^6\theta\, d\theta.$

ANSWERS.

NOTE.—$c = \cos\theta$, $s = \sin\theta$, $t = \tan\theta$, $k = \cot\theta$.

1. (1) $-c + \frac{1}{3}c^3.$ (2) $s - s^3 + \frac{3}{5}s^5 - \frac{1}{7}s^7.$ (3) $-\frac{1}{3}c^3 + \frac{1}{5}c^5.$

(4) $\sec\theta + \cos\theta.$ (5) $\frac{2}{5}\sqrt{s}(4 + c^2).$ (6) $\log s.$

(7) $\dfrac{1}{4c^4} - \dfrac{1}{c^2} - \log c.$

2. (1) $\frac{1}{3}t^3 + \frac{1}{5}t^5.$ (2) $-\frac{1}{3}k^3.$ (3) $2t^{\frac{5}{2}}(\frac{1}{5} + \frac{2}{9}t^2 + \frac{1}{13}t^4).$

(4) $t + \frac{1}{3}t^3.$ (5) $-k(1 + \frac{2}{3}k^2 + \frac{1}{5}k^4).$ (6) $\log t.$

(7) $-\frac{1}{2}k^2 + \frac{1}{2}t^2 + 2\log t.$ (8) $-\frac{1}{4}k^4 - k^2 + \log t.$

(9) $2\sqrt{t}.$ (10) $\frac{1}{2}\log\tan\dfrac{\theta}{2} - \dfrac{1}{2}\dfrac{c}{s^2}.$

3. (1) $\frac{1}{8}(\theta - \frac{1}{4}\sin 4\theta).$ (2) $\frac{1}{8}(3\theta - 2\sin 2\theta + \frac{1}{4}\sin 4\theta).$

(3) $\frac{1}{32}(2\theta + \frac{1}{2}\sin 2\theta - \frac{1}{2}\sin 4\theta - \frac{1}{6}\sin 6\theta).$

(4) $\frac{1}{32}(10\theta + \frac{15}{2}\sin 2\theta + \frac{3}{2}\sin 4\theta + \frac{1}{6}\sin 6\theta).$

389. Reduction of $\int \sin^m\theta \cos^n\theta\, d\theta$, m and n being $+^{ve}$ or $-^{ve}$ integers.

Although we can use integration by parts, the following method of differentiation (which is in reality the same process in disguise) is recommended in the general case and in certain particular cases. The former method is, perhaps, preferable in other cases.

We have

$$\frac{d}{d\theta}(\sin^m\theta\cos^n\theta) = m\sin^{m-1}\theta\cos^{n+1}\theta - n\sin^{m+1}\theta\cos^{n-1}\theta \quad . \quad (1)$$

$$\therefore \sin^m\theta\cos^n\theta = m\int\sin^{m-1}\theta\cos^{n+1}\theta\, d\theta - n\int\sin^{m+1}\theta\cos^{n-1}\theta\, d\theta \quad (2)$$

This gives a formula of reduction for whichever of the two integrals we choose to take to the left-hand side.

Comparing the integrals it will be seen that, if the power of $\sin\theta$ is increased by 2, that of $\cos\theta$ is diminished by 2, and *vice versâ*.

I. Transfer the *first* integral of (2) to the left, and we have

$$\int\sin^{m-1}\theta\cos^{n+1}\theta\, d\theta = \frac{1}{m}\sin^m\theta\cos^n\theta + \frac{n}{m}\int\sin^{m+1}\theta\cos^{n-1}\theta\, d\theta.$$

Now change $m-1$ into m, and $n+1$ into n,

$$\therefore \int\sin^m\theta\cos^n\theta\, d\theta$$

$$= \frac{1}{m+1}\sin^{m+1}\theta\cos^{n-1}\theta + \frac{n-1}{m+1}\int\sin^{m+2}\theta\cos^{n-2}\theta\, d\theta \quad . \quad (A)$$

II. Similarly, transferring the *second* integral of (2) to the left, and changing $m+1$ into m, and $n-1$ into n, we have

$$\int\sin^m\theta\cos^n\theta\, d\theta$$

$$= -\frac{1}{n+1}\sin^{m-1}\theta\cos^{n+1}\theta + \frac{m-1}{n+1}\int\sin^{m-2}\theta\cos^{n+2}\theta\, d\theta \,. \quad (B)$$

390. Again, in the R.H.S. of (1) we may either write $1-\cos^2\theta$ for $\sin^2\theta$ in the *higher* power [$(m+1)$th] of $\sin\theta$, or $1-\sin^2\theta$ for $\cos^2\theta$ in the *higher* power [$(n+1)$th] of $\cos\theta$. In this way we shall obtain four more equations.

First, then, $\frac{d}{d\theta}\sin^m\theta\cos^n\theta$

$$= m\sin^{m-1}\theta\cos^{n+1}\theta - n\sin^{m-1}\theta(1-\cos^2\theta)\cos^{n-1}\theta$$
$$= (m+n)\sin^{m-1}\theta\cos^{n+1}\theta - n\sin^{m-1}\theta\cos^{n-1}\theta.$$
$$\therefore \sin^m\theta\cos^n\theta$$
$$= (m+n)\int\sin^{m-1}\theta\cos^{n+1}\theta\,d\theta - n\int\sin^{m-1}\theta\cos^{n-1}\theta\,d\theta \quad . \quad (3)$$

III. Transfer the *first* integral of (3) to the left, change $m-1$ into m, and $n+1$ into n, and divide down by $m+n$ (which is unaltered by the change); then

$$\int\sin^m\theta\cos^n\theta\,d\theta$$
$$= \frac{1}{m+n}\sin^{m+1}\theta\cos^{n-1}\theta + \frac{n-1}{m+n}\int\sin^m\theta\cos^{n-2}\theta\,d\theta. \quad . \quad (C)$$

IV. Transfer the *second* integral of (3) to the left, change $m-1$ into m, and $n-1$ into n, and divide by $n+1$; then

$$\int\sin^m\theta\cos^n\theta\,d\theta$$
$$= -\frac{1}{n+1}\sin^{m+1}\theta\cos^{n+1}\theta + \frac{m+n+2}{n+1}\int\sin^m\theta\cos^{n+2}\theta\,d\theta. \quad (D)$$

Similarly, writing $1-\sin^2\theta$ for $\cos^2\theta$ in (1), we have

$$\frac{d}{d\theta}\sin^m\theta\cos^n\theta$$
$$= m\sin^{m-1}\theta\cos^{n-1}\theta(1-\sin^2\theta) - n\sin^{m+1}\theta\cos^{n-1}\theta$$
$$= m\sin^{m-1}\theta\cos^{n-1}\theta - (m+n)\sin^{m+1}\theta\cos^{n-1}\theta.$$
$$\therefore \sin^m\theta\cos^n\theta$$
$$= m\int\sin^{m-1}\theta\cos^{n-1}\theta\,d\theta - (m+n)\int\sin^{m+1}\theta\cos^{n-1}\theta\,d\theta. \quad . \quad (4)$$

V. Transfer the *first* integral of (4), and we shall have, after changing letters,

$$\int\sin^m\theta\cos^n\theta\,d\theta$$
$$= \frac{1}{m+1}\sin^{m+1}\theta\cos^{n+1}\theta + \frac{m+n+2}{m+1}\int\sin^{m+2}\theta\cos^n\theta\,d\theta. \quad (E)$$

VI. Transfer the *second* integral, and we shall have, similarly,

$$\int\sin^m\theta\cos^n\theta\,d\theta$$
$$= -\frac{1}{m+n}\sin^{m-1}\theta\cos^{n+1}\theta + \frac{m-1}{m+n}\int\sin^{m-2}\theta\cos^n\theta\,d\theta\,. \quad (F)$$

391. Summary and Remarks—Practical Method.—Each of these formulæ is useful according to the sign of m and n. The following table gives a comparison of the two integrals which have to be connected, and shows for what signs of m and n each formula is useful :—

	Power of $\sin\theta$ is	Power of $\cos\theta$ is	m	n
In (A) .	raised by 2	lowered by 2	−	+
In (B) .	lowered by 2	raised by 2	+	−
In (C) .	unaltered	lowered by 2	+	+
In (D) .	unaltered	raised by 2	−	−
In (E) .	raised by 2	unaltered	−	−
In (F) .	lowered by 2	unaltered	+	+

We can obtain the above six equations without making changes in m and n, and we shall show how we can connect any two of the above integrals at pleasure.

For convenience let $\int P d\theta = aQ + b\int R d\theta$ represent any one of the above equations (a and b being constant coefficients). Then Q is the expression to be differentiated at first. To find Q, having given P and R, it will be seen that *in every case the index of sin* x *or cos* x *is greater by one than the smaller*† (if unequal) *or common* (if equal) *index of the same function in P and R.*

Thus, supposing, as in (C), we wish to express $\int \sin^m\theta \cos^n\theta d\theta$ in terms of $\int \sin^m\theta \cos^{n-2}\theta\, d\theta$; then, to find Q, since the *smaller* power of $\cos\theta$ is the $(n-2)$th, and the *common* power of $\sin\theta$ is the mth,

$$\therefore Q = \sin^{m+1}\theta \cos^{n-1}\theta.$$

Hence $\dfrac{d}{d\theta}(\sin^{m+1}\theta\cos^{n-1}\theta)$

$$= (m+1)\sin^m\theta\cos^n\theta - (n-1)\sin^{m+2}\theta\cos^{n-2}\theta \ddagger$$
$$= (m+1)\sin^m\theta\cos^n\theta - (n-1)(1-\cos^2\theta)\sin^m\theta\cos^{n-2}\theta$$
$$= (m+n)\sin^m\theta\cos^n\theta - (n-1)\sin^m\theta\cos^{n-2}\theta.$$

† That is, *algebraically* smaller, whether m and n are $+^{ve}$ or $-^{ve}$.

‡ The indices involved in the two integrals to be connected are m, n, and -2. Since $\sin^{m+2}\theta$ *alone* has an index which is not included among these,

$\therefore$ integrating and transposing, we shall obtain the equation (C).

NOTE.—In no case are the powers of $\sin x$ and $\cos x$ *both* lowered or *both* raised.

392. Cases $\int \frac{d\theta}{\sin\theta\cos^n\theta}$ and $\int \frac{d\theta}{\sin^m\theta\cos\theta}$.

These may be treated by the above method: thus, using s and c, we shall connect $\int \frac{d\theta}{sc^n}$ with $\int \frac{d\theta}{sc^{n-2}}$, *i.e.* $\int s^{-1}c^{-n}\,d\theta$ with $\int s^{-1}c^{-n+2}\,d\theta$. Hence $Q = s^{-1+1}c^{-n+1} = c^{-n+1}$, and we have

$$\frac{d}{d\theta}c^{-n+1} = -(-n+1)c^{-n}s = (n-1)c^{-n}s^{-1}.s^2 \text{ (see footnote ‡, p. 403)}$$

$$= (n-1)c^{-n}s^{-1}(1-c^2) = (n-1)c^{-n}s^{-1} - (n-1)c^{-n+2}s^{-1}$$

$$\therefore \int s^{-1}c^{-n}\,d\theta = \frac{1}{n-1}c^{-n+1} + \int s^{-1}c^{-n+2}\,d\theta$$

$$\text{or} \int \frac{d\theta}{\sin\theta\cos^n\theta} = \frac{1}{(n-1)\cos^{n-1}\theta} + \int \frac{d\theta}{\sin\theta\cos^{n-2}\theta}.$$

Otherwise, $$\int \frac{d\theta}{sc^n} = \int \frac{s^2+c^2}{sc^n}\,d\theta = \int \frac{s\,d\theta}{c^n} + \int \frac{d\theta}{sc^{n-2}}$$

$$= \frac{1}{n-1}\cdot\frac{1}{c^{n-1}} + \int \frac{d\theta}{sc^{n-2}}, \text{ as before.}$$

Similarly for the other integral.

393. Examples.

Ex. 1. $\int \sin^4\theta\cos^6\theta\,d\theta$.

To reduce this, since $\cos^6\theta$ has the higher power, we shall express this in terms of $\int \sin^4\theta\cos^4\theta\,d\theta$ [Art. 391, note]; or, $P = s^4c^6$, and $R = s^4c^4$.

Hence $Q = s^5c^5$, and we have

$$\frac{d}{d\theta}s^5c^5 = 5s^4c^6 - 5s^6c^4;$$

---

we know that here the change ($\sin^2\theta = 1 - \cos^2\theta$) has to be made. This point is extremely useful; and as the rule is invariable, except in (A) and (B) where no change is necessary, it serves as a test for the accuracy of the work.

and altering s^6, since it is the only factor of the R.H.S. which does not appear in P or R (see footnote ‡, p. 403), the above

$$= 5s^4c^6 - 5s^4(1-c^2)c^4 = 10s^4c^6 - 5s^4c^4.$$

$$\therefore \int s^4c^6\,d\theta = \tfrac{1}{10}s^5c^5 + \tfrac{5}{10}\int s^4c^4\,d\theta, \text{ the formula of reduction} . \quad (a)$$

Repeating it, we get

$$\tfrac{1}{10}s^5c^5 + \tfrac{5}{10}[\tfrac{1}{8}s^5c^3 + \tfrac{3}{8}\int s^4c^2\,d\theta]$$
$$= \tfrac{1}{10}s^5c^5 + \tfrac{5}{10}\cdot\tfrac{1}{8}s^5c^3 + \tfrac{5}{10}\cdot\tfrac{3}{8}[\tfrac{1}{6}s^5c + \tfrac{1}{6}\int s^4\,d\theta]$$
$$= \tfrac{1}{10}s^5c^5 + \tfrac{5}{10}\cdot\tfrac{1}{8}s^5c^3 + \tfrac{5}{10}\cdot\tfrac{3}{8}\cdot\tfrac{1}{6}s^5c + \tfrac{5}{10}\cdot\tfrac{3}{8}\cdot\tfrac{1}{6}\int s^4\,d\theta \quad . \quad (1)$$

The latter integral may be obtained by ordinary integration by parts, as in Art. 387, Ex. 4; *or by the present method*, thus:—

Since $P = s^4$, $R = s^2$, $\therefore\ Q = s^3c$, since s^4 and s^2 are equivalent to s^4c^0 and s^2c^0 respectively.

Hence $\dfrac{d}{d\theta}s^3c = 3s^2c^2 - s^4 = 3s^2(1-s^2) - s^4$ (see footnote ‡, above)

$$= 3s^2 - 4s^4.$$

$$\therefore \int s^4\,d\theta = -\tfrac{1}{4}s^3c + \tfrac{3}{4}\int s^2\,d\theta = -\tfrac{1}{4}s^3c + \tfrac{3}{4}[-\tfrac{1}{2}sc + \tfrac{1}{2}\int d\theta]$$
$$= -\tfrac{1}{4}s^3c - \tfrac{3}{8}sc + \tfrac{3}{8}\theta.$$

Or, again, we might have used multiple angles, thus:—

$$\int s^4\,d\theta = \tfrac{1}{4}\int(1-\cos 2\theta)^2\,d\theta = \tfrac{1}{4}\int\left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \text{etc.}$$

Substituting this result in the above expression (1), the given integral is obtained.

NOTE.—The powers s and c in the given integral are both even and $+^{\text{ve}}$; had either or both of them been odd, we should have used the method of Art. 388 (A).

Ex. 2. $\displaystyle\int\frac{\sin^8\theta\,d\theta}{\cos^5\theta}.$

Here $I = \int s^8c^{-5}\,d\theta$, and we shall express it in terms of $\int s^6c^{-3}d\theta$;

or $$P = s^8c^{-5},\ R = s^6c^{-3}.$$

$$\therefore\ Q = s^7c^{-4},$$

and we have $$\frac{d}{d\theta}s^7c^{-4} = 7s^6c^{-3} + 4s^8c^{-5};$$

and comparing, we find that the indices on the R.H.S. agree with those in P and R. Hence no change is necessary. [Art. 391, footnote.]

$$\therefore \int s^8c^{-5}\,d\theta = \tfrac{1}{4}s^7c^{-4} - \tfrac{7}{4}\int s^6c^{-3}\,d\theta$$
$$= \tfrac{1}{4}s^7c^{-4} - \tfrac{7}{4}\{\tfrac{1}{2}s^5c^{-2} - \tfrac{5}{2}\int s^4c^{-1}\,d\theta\}$$
$$= \tfrac{1}{4}s^7c^{-4} - \tfrac{7}{4}\cdot\tfrac{1}{2}s^5c^{-2} + \tfrac{7}{4}\cdot\tfrac{5}{2}\int s^4c^{-1}\,d\theta.$$

Now connect $\int s^4c^{-1}\,d\theta$ with $\int s^2c^{-1}\,d\theta$; $\therefore\ Q = s^3$,

and $\dfrac{d}{d\theta}s^3 = 3s^2c = 3s^2c^{-1}\,.\,c^2$ [see footnote ‡, p. 403] $= 3s^2c^{-1}(1-s^2)$

$$= 3s^2c^{-1} - 3s^4c^{-1}.$$

$$\therefore \int s^4c^{-1}\,d\theta = -\tfrac{1}{3}s^3 + \int s^2c^{-1}\,d\theta$$

$$= -\tfrac{1}{3}s^3 - s + \int c^{-1}\,d\theta, \text{ reapplying the formula,}$$

$$= -\tfrac{1}{3}s^3 - s + \log(\tan\theta + \sec\theta).$$

$$\therefore I = \tfrac{1}{4}s^7c^{-4} - \tfrac{7}{4}\cdot\tfrac{1}{2}s^5c^{-2} - \tfrac{7}{4}\cdot\tfrac{5}{2}\cdot\tfrac{1}{3}s^3 - \tfrac{7}{4}\cdot\tfrac{5}{2}s + \tfrac{7}{4}\cdot\tfrac{5}{2}\log(\tan\theta + \sec\theta)$$

Ex. 3. $\displaystyle\int\frac{d\theta}{\sin^4\theta\cos^5\theta}$.

Here $I = \int s^{-4}c^{-5}\,d\theta$, and we shall express it in terms of $\int s^{-4}c^{-3}\,d\theta$.

$$\therefore\ Q = s^{-3}c^{-4}, \text{ and we have}$$

$$\frac{d}{d\theta}s^{-3}c^{-4} = -3s^{-4}c^{-3} + 4s^{-2}c^{-5}$$

$$= -3s^{-4}c^{-3} + 4s^{-4}(1-c^2)c^{-5} \text{ [see footnote ‡, p. 403]}$$

$$= -7s^{-4}c^{-3} + 4s^{-4}c^{-5}.$$

$$\therefore \int s^{-4}c^{-5}\,d\theta = \tfrac{1}{4}s^{-3}c^{-4} + \tfrac{7}{4}\int s^{-4}c^{-3}\,d\theta$$

$$= \tfrac{1}{4}s^{-3}c^{-4} + \tfrac{7}{4}\left[\tfrac{1}{2}s^{-3}c^{-2} + \tfrac{5}{2}\int s^{-4}c^{-1}\,d\theta\right]$$

$$= \tfrac{1}{4}s^{-3}c^{-4} + \tfrac{7}{8}s^{-3}c^{-2} + \tfrac{35}{8}\int s^{-4}c^{-1}\,d\theta.$$

The latter integral may be obtained by making $Q = s^{-3}$; or as follows:—

$$s^{-4}c^{-1} = \frac{1}{s^4c} = \frac{s^2+c^2}{s^4c} = \frac{1}{s^2c} + \frac{c}{s^4} = \frac{s^2+c^2}{s^2c} + \frac{c}{s^4} = \frac{1}{c} + \frac{c}{s^2} + \frac{c}{s^4};$$

$$\therefore \int s^{-4}c^{-1}\,d\theta = gd^{-1}\theta - \frac{1}{s} - \frac{1}{3s^3}.$$

$$\therefore I = \tfrac{1}{4}s^{-3}c^{-4} + \tfrac{7}{8}s^{-3}c^{-2} - \tfrac{35}{24}s^{-3} - \tfrac{35}{8}s^{-1} + \tfrac{35}{8}gd^{-1}\theta.$$

394. $\displaystyle\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta$, m and n being $+^{\text{ve}}$ integers.

We may notice that if either m or n is a $-^{\text{ve}}$ integer the definite integral is infinite, at least when taken between the limits $\frac{1}{2}\pi$ and 0; we shall therefore only use equations (C) and (F) (*q.v.*).

It will be seen that the first term on the R.H.S. of each equation vanishes at both limits.

There are four cases, according as m and n are even or odd.

(1) Let $m = 2p$; $n = 2q$.

Then from (C), $\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2q}\theta\, d\theta = \frac{2q-1}{2p+2q}\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2q-2}\theta\, d\theta$

$$= \text{etc.} = \frac{(2q-1)(2q-3)\ldots 5.3.1}{(2p+2q)(2p+2q-2)\ldots(2p+2)}\int_0^{\frac{\pi}{2}} \sin^{2p}\theta\, d\theta.$$

Again in (F), let $n = 0$, and $m = 2p$;

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{2p}\theta\, d\theta = \frac{2p-1}{2p}\int_0^{\frac{\pi}{2}} \sin^{2p-2}\theta\, d\theta$$

$$= \frac{(2p-1)(2p-3)\ldots 5.3.1}{2p(2p-2)\ldots 6.4.2}\int_0^{\frac{\pi}{2}} d\theta$$

and since $\int_0^{\frac{\pi}{2}} d\theta = \pi/2$,

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2q}\theta\, d\theta = \frac{1.3.5\ldots(2p-1).1.3.5\ldots(2q-1)}{2.4.6\ldots(2p+2q)}\,\frac{\pi}{2}.$$

We should have got the same result had we begun with (F).

(2) Let $m = 2p$, $n = 2q+1$.

Then from (C),

$$\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2q+1}\theta\, d\theta = \frac{2q}{2p+2q+1}\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2q-1}\theta\, d\theta$$

$$= \text{etc.} = \frac{2q(2q-2)\ldots 6.4.2}{(2p+2q+1)(2p+2q-1)\ldots(2p+3)}\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos\theta\, d\theta$$

But $\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos\theta\, d\theta$ is immediately integrable, and

$$= \left[\frac{\sin^{2p+1}\theta}{2p+1}\right]_0^{\frac{\pi}{2}} = \frac{1}{2p+1};$$

$$\therefore I = \frac{2q(2q-2)\ldots 6.4.2}{(2p+2q+1)\ldots(2p+1)},$$

which may be written

$$\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2q+1}\theta\, d\theta = \frac{1.3.5\ldots(2p-1).2.4.6\ldots 2q}{1.3.5\ldots(2p+2q+1)}.$$

We might have used equation (F), but since $\cos\theta$ is raised to an *odd* power, it is evidently better to reduce this factor, as thereby we ultimately obtain a result which is immediately integrable, as above.

Similarly, we have

$$(3)\quad \int_0^{\frac{\pi}{2}} \sin^{2p+1}\theta\cos^{2q}\theta\,d\theta = \frac{2.4.6\ldots 2p\,.1.3.5\ldots(2q-1)}{1.3.5\ldots(2p+2q+1)}$$

$$\text{and } (4)\quad \int_0^{\frac{\pi}{2}} \sin^{2p+1}\theta\cos^{2q+1}\theta\,d\theta = \frac{2.4.6\ldots 2p\,.2.4.6\ldots 2q}{2.4.6\ldots(2p+2q+2)}.$$

It will be seen that these four cases may be included in the formula,

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta = \frac{(m-1)(m-3)\ldots\times(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots}\times 1 \text{ or } \frac{\pi}{2},$$

the latter factor $(\pi/2)$ being adopted only when *both* m and n are *even*.

395. Examples.

Ex. 1. $\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\,d\theta.$

From Art. 393, Ex. 1 (a), we have

$$\int_0^{\frac{\pi}{2}} s^4c^6\,d\theta = \tfrac{5}{10}\int_0^{\frac{\pi}{2}} s^4c^4\,d\theta = \tfrac{5}{10}\cdot\tfrac{3}{8}\cdot\tfrac{1}{6}\int_0^{\frac{\pi}{2}} s^4\,d\theta.$$

To express $\int s^4\,d\theta$ in terms of $\int s^2\,d\theta$, $Q = s^3c$;

$$\therefore\ \frac{d}{d\theta}s^3c = 3s^2c^2 - s^4 = 3s^2(1-s^2) - s^4 = 3s^2 - 4s^4;$$

$$\therefore\ \int_0^{\frac{\pi}{2}} s^4\,d\theta = \tfrac{3}{4}\int_0^{\frac{\pi}{2}} s^2\,d\theta = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2};$$

$$\therefore\ I = \frac{5.3.1.3.1}{10.8.6.4.2}\,\frac{\pi}{2}.$$

Ex. 2. $\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^5\theta\,d\theta.$

To express $\int s^7c^5\,d\theta$ in terms of $\int s^5c^5\,d\theta$, $Q = s^6c^6$;

$$\therefore\ \frac{d}{d\theta}s^6c^6 = 6s^5c^7 - 6s^7c^5 = 6s^5(1-s^2)c^5 - 6s^7c^5 = 6s^5c^5 - 12s^7c^5.$$

$$\therefore\ \int_0^{\frac{\pi}{2}} s^7c^5\,d\theta = \tfrac{6}{12}\int s^5c^5\,d\theta = \tfrac{6}{12}\cdot\tfrac{4}{10}\cdot\tfrac{2}{8}\int_0^{\frac{\pi}{2}} sc^5\,d\theta.$$

But $$\int_0^{\frac{\pi}{2}} sc^5\,d\theta = \left[-\tfrac{1}{6}c^6\right]_0^{\frac{\pi}{2}} = \tfrac{1}{6}.$$

$\therefore\ I = \dfrac{6.4.2.1}{12.10.8.6}$, which may be written $\dfrac{6.4.2.4.2}{12.10.8.6.4.2}$; thus agreeing with the formula.

396. Eulerian Integrals.

The case in which m and n are *any* quantities cannot be discussed here. But, in connection with this, we may call attention to two very important definite integrals, partly discussed in Arts. 381 and 383, and in terms of which the integral $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta$ can be expressed. They are as follows:—

(1) $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, called the *First Eulerian Integral*, and denoted by $B(m,n)$; m and n being any quantities.

(2) $\int_0^\infty e^{-x}x^{n-1}\,dx$, called the *Second Eulerian Integral*, and denoted by $\Gamma(n)$, n being any quantity. It is better known, however, as the *Gamma Function.*

Obviously neither of these functions, when evaluated, involve x. It is beyond the scope of this work to discuss these integrals, but we may state that a simple relation exists between $B(m,n)$ and $\Gamma(n)$; in consequence of which, results involving either of them are usually expressed in terms of the latter.

It will be seen that if we put $\sin^2\theta = x$ in the integral $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta$, it becomes $\frac{1}{2}\int_0^1 x^{\frac{m-1}{2}}(1-x)^{\frac{n-1}{2}}\,dx = \frac{1}{2}\int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{1}{2}B(p,q)$, where $p = \dfrac{m+1}{2}$, $q = \dfrac{n+1}{2}$; and this again can be expressed in gamma functions.

Similarly, the definite integrals, corresponding to the indefinite integrals discussed in the remaining portion of this chapter, can all be expressed in gamma functions, provided that the limits be properly chosen.

It may be observed that, for general values of n, $\Gamma(n)$ cannot be expressed algebraically except as an infinite series. It therefore resembles such a function as $\log x$, and is one of the so-called Higher Transcendental Functions. As in the case of the logarithmic function, its properties have

been investigated, and tables of values corresponding to different values of n have been compiled for reference.

EXAMPLES LXV.

1. Integrate by reduction :—

(1) $\dfrac{\sin^2\theta}{\cos^3\theta}d\theta.$ (2) $\dfrac{\cos^4\theta}{\sin\theta}d\theta.$ (3) $\dfrac{d\theta}{\sin^2\theta\cos\theta}.$

(4) $\dfrac{d\theta}{\sin\theta\cos^4\theta}.$ (5) $\dfrac{d\theta}{\sin^5\theta\cos^2\theta}.$ (6) $\dfrac{\cos^4\theta}{\sin^3\theta}d\theta.$

(7) $\dfrac{\sin^6\theta}{\cos^5\theta}d\theta.$ (8) $\sin^2\theta\cos^6\theta\,d\theta.$ (9) $\sin^6\theta\cos^4\theta\,d\theta.$

2. Evaluate :—

(1) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^5\theta\cos^3\theta\,d\theta.$ (2) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^8\theta\cos^6\theta\,d\theta.$

(3) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^7\theta\cos^{10}\theta\,d\theta.$ (4) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^{12}\theta\cos^{10}\theta\,d\theta.$

3. Integrate by the *shortest* method :—

(1) $\dfrac{\cos^3\theta}{\sin^2\theta}d\theta.$ (2) $\dfrac{\sin^4\theta}{\cos^8\theta}d\theta.$ (3) $\dfrac{\sin^8\theta}{\cos^4\theta}d\theta.$

(4) $\sin^2\theta\cos^4\theta\,d\theta.$ (5) $\sin^6\theta\cos^2\theta\,d\theta.$ (6) $\dfrac{\sin^3\theta}{\cos^5\theta}d\theta.$

(7) $\dfrac{d\theta}{\sin^3\theta\cos\theta}.$ (8) $\dfrac{d\theta}{\sin^3\theta\cos^2\theta}.$ (9) $\tan^4\theta\,d\theta.$

4. Show that $\int\tan^n\theta\,d\theta = \dfrac{\tan^{n-1}\theta}{n-1} - \int\tan^{n-2}\theta\,d\theta.$

Hence find (1) $\int\tan^5\theta\,d\theta$ and (2) $\int\tan^6\theta\,d\theta.$

5. Show that (1) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta = \int_0^{\frac{\pi}{2}}\sin^n\theta\cos^m\theta\,d\theta.$

(2) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\cos^{2n+1}\theta\,d\theta = \frac{(n!)^2}{2(2n+1)!}.$

6. Show that

(1) $\displaystyle\int_0^1 x^m(1-x)^n\,dx = 2\int_0^{\frac{\pi}{2}}\sin^{2m+1}x\cos^{2n+1}x\,dx = \frac{m!\,n!}{(m+n+1)!}.$

(2) $\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{7}{2}}\,dx = \dfrac{7.5.3.1.5.3.1}{14.12.10.8.6.4.2}\,\pi.$

(3) $\int_0^1 x^4\sqrt{1-x^2}\,dx = \pi/32.$

7. Show that

(1) $\int\theta\cos^3\theta\,d\theta = \theta s(1-\frac{1}{3}s^2) + \frac{2}{3}c + \frac{1}{9}c^3;$

(2) $\int\sin^3\theta\log\tan\theta\,d\theta = (\frac{1}{3}c^3 - c)\log\tan\theta - \frac{1}{3}c + \frac{2}{3}\log\tan\frac{1}{2}\theta;$

where $s = \sin\theta$, and $c = \cos\theta$.

8. Show that when m and n are $+^{ve}$ integers,

$$B(m, n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!};\quad \Gamma(n) = (n-1)!$$

Answers.

1. (1) $\frac{1}{2}sc^{-2} - \frac{1}{2}\log(\tan\theta+\sec\theta)$. (2) $\frac{1}{3}c^3 + c + \log\tan\frac{1}{2}\theta$.

(3) $-\dfrac{1}{s} + \log(\tan\theta+\sec\theta)$. (4) $-\frac{1}{3}c^{-3} + c^{-1} + \log\tan\frac{1}{2}\theta$.

(5) $-\frac{1}{4}s^{-4}c^{-1} - \frac{5}{8}s^{-2}c^{-1} + \frac{15}{8}c^{-1} + \frac{15}{8}\log\tan\frac{1}{2}\theta$.

(6) $-\frac{1}{2}s^{-2}c^3 - \frac{3}{2}c - \frac{3}{2}\log\tan\frac{1}{2}\theta$. (7) $\frac{1}{4}s^5c^{-4} - \frac{5}{8}s^3c^{-2} - \frac{15}{8}s + \frac{15}{8}\mathrm{gd}^{-1}\theta$

(8) $\frac{1}{8}s^3c^5 + \dfrac{5}{8.6}s^3c^3 + \dfrac{5.3}{8.6.4}s^3c + \dfrac{5.3.1}{8.6.4.2}(\theta - sc)$.

(9) $-\frac{1}{10}s^5c^5 - \dfrac{5}{10.8}s^3c^5 - \dfrac{5.3}{10.8.6}sc^5 + \dfrac{5.3.1}{10.8.6}\left\{\frac{1}{4}c^3s + \dfrac{3}{4.2}cs + \dfrac{3}{4.2}\theta\right\}$.

2. (1) $\frac{1}{24}$. (2) $\dfrac{7.5.3.1.5.3.1}{14.12\ldots2}\dfrac{\pi}{2}$. (3) $\dfrac{6.4.2}{17.15.13.11}$. (4) $\dfrac{11.9\ldots1.9.7\ldots1}{22.20\ldots4.2}\cdot\dfrac{\pi}{2}$.

3. (1) $-(s+s^{-1})$. (2) $\frac{1}{5}t^5+\frac{1}{7}t^7$. (3) $\frac{1}{3}s^7c^{-3} - \frac{7}{3}s^5c^{-1} - \frac{35}{12}s^3c - \frac{35}{8}sc + \frac{35}{8}\theta$.

(4) $\frac{1}{6}s^3c^3 + \frac{1}{8}s^3c - \frac{1}{16}sc + \frac{1}{16}\theta$. (5) $-\frac{1}{3}c^3 + \frac{2}{5}c^5 - \frac{1}{7}c^7$. (6) $\frac{1}{4}\tan^4\theta$.

(7) $-\dfrac{1}{2s^2} + \log\tan\theta$. (8) $-\frac{1}{2}s^{-2}c^{-1} + \frac{3}{2}c^{-1} + \frac{3}{2}\log\tan\frac{1}{2}\theta$. (9) $\frac{1}{3}t^3 - t + \theta$.

4. (1) $\frac{1}{4}t^4 - \frac{1}{2}t^2 - \log c$. (2) $\frac{1}{5}t^5 - \frac{1}{3}t^3 + t - \theta$.

397. Integration of $x^m(ax+b)^n\,dx$—When immediately integrable.

There are three cases in which the above expression is immediately integrable :—

(A) *When* n *is a* $+^{ve}$ *integer.*

For then $(ax+b)^n$ can be expanded into a *finite* series, and the resulting expression integrated term by term.

(B) *When* m *is a* $+^{ve}$ *integer.*

Put $$ax+b=y; \quad \therefore x=\frac{1}{a}(y-b).$$

Then, neglecting constants,

$$\int x^m(ax+b)^n\,dx \propto \int y^n(y-b)^m\,dy;$$

and by case (A), this is immediately integrable if m be a $+^{ve}$ integer.

(C) *When* $m+n+2=0$ *or a* $-^{ve}$ *integer.*

Put $$x=\frac{1}{y}; \quad \therefore dx=-\frac{1}{y^2}dy.$$

$$\therefore \int x^m(ax+b)^n\,dx = -\int\frac{1}{y^m}\cdot\left(\frac{a+by}{y}\right)^n\frac{dy}{y^2} = -\int\frac{(a+by)^n}{y^{m+n+2}}\,dy$$

$$= -\int y^{-(m+n+2)}(a+by)^n\,dy;$$

and by case (B) this is immediately integrable when $-(m+n+2)$ is a $+^{ve}$ integer (including zero).

Hence $m+n+2$ must $=0$ or a $-^{ve}$ integer.

398. If in the given integral we put $ax=-b\sin^2\theta$, it becomes

$$\int\left(-\frac{b}{a}\right)^m\sin^{2m}\theta.(b\cos^2\theta)^n.\left(-\frac{b}{a}\cdot 2\sin\theta\cos\theta\,d\theta\right)$$

$$\propto \int\sin^{2m+1}\theta\cos^{2n+1}\theta\,d\theta;$$

and the condition (C) of the last article follows from Art. 388 (B).

399. Examples.

Ex. 1. $\int x^{\frac{2}{3}}(1+x)^2\,dx = \int(x^{\frac{2}{3}} + 2x^{\frac{5}{3}} + x^{\frac{8}{3}})\,dx = \frac{3}{5}x^{\frac{5}{3}} + \frac{3}{4}x^{\frac{8}{3}} + \frac{3}{11}x^{\frac{11}{3}}.$

Ex. 2. $\int x^2(2x-1)^{\frac{5}{2}}\,dx.$

Put $2x-1=y$; $\therefore\ x = \frac{1}{2}(y+1)$; $dx = \frac{1}{2}dy.$

$$\therefore\ I = \tfrac{1}{8}\int y^{\frac{5}{2}}(y+1)^2\,dy = \tfrac{1}{8}\int(y^{\frac{5}{2}} + 2y^{\frac{7}{2}} + y^{\frac{9}{2}})\,dy$$

$$= \tfrac{1}{8}(\tfrac{2}{7}y^{\frac{7}{2}} + \tfrac{4}{9}y^{\frac{9}{2}} + \tfrac{2}{11}y^{\frac{11}{2}}) = \tfrac{1}{4}y^{\frac{7}{2}}(\tfrac{1}{7} + \tfrac{2}{9}y + \tfrac{1}{11}y^2)$$

where $y = 2x-1$.

Ex. 3. Let $m = -\frac{11}{3}$; $n = -\frac{4}{3}$;

$$\therefore\ m+n+2 = -3, \text{ a } -^{ve} \text{ integer.}$$

Hence $\int \frac{dx}{x^{\frac{11}{3}}(2x+1)^{\frac{4}{3}}}$ is immediately integrable if we put $x = \frac{1}{y}$.

$$\therefore\ I = \int \frac{-\frac{1}{y^2}\,dy}{y^{-\frac{11}{3}}(2+y)^{\frac{4}{3}}\,y^{-\frac{4}{3}}} = -\int y^3(2+y)^{-\frac{4}{3}}\,dy$$

$$= -\int(z-2)^3 z^{-\frac{4}{3}}\,dz \text{ (if } 2+y=z) = \text{etc.}$$

400. Integration of $x^m(ax^2+b)^n\,dx$ — When immediately integrable.

This integral can be reduced to the preceding by putting $x^2 = y$;

$$\therefore\ dx = \frac{1}{2\sqrt{y}}\,dy.$$

Hence

$$\int x^m(ax^2+b)^n\,dx = \int y^{\frac{m}{2}}(ay+b)^n\frac{dy}{2\sqrt{y}} = \frac{1}{2}\int y^{\frac{m-1}{2}}(ay+b)^n\,dy,$$

and, by Art. 397, this is immediately integrable when

(A) n *is a* $+^{ve}$ *integer.*

(B) m *is an odd* $+^{ve}$ *integer.*

(C) $\dfrac{m-1}{2} + n + 2 = 0$ or a $-^{ve}$ integer; *i.e.* when $m + 2n + 3 = 0$ or an *even* $-^{ve}$ integer, *i.e.* m + 2n + 1 *is an even* $-^{ve}$ *integer.*

Ex. $\displaystyle\int\frac{x^{\frac{2}{3}}\,dx}{(2x^2-3)^{\frac{17}{6}}}$. Here $m + 2n + 1 = -4$

Put $x^2 = y$; then

$$I = \tfrac{1}{2}\int\frac{y^{\frac{1}{3}}}{(2y-3)^{\frac{17}{6}}}\frac{dy}{y^{\frac{1}{2}}} = \tfrac{1}{2}\int\frac{dy}{y^{\frac{1}{6}}(2y-3)^{\frac{17}{6}}}.$$

Put $y = \dfrac{1}{z}$; $\therefore\ I = \tfrac{1}{2}\displaystyle\int\frac{z^{\frac{1}{6}}\,.\,z^{\frac{17}{6}}}{(2-3z)^{\frac{17}{6}}}\left(-\frac{dz}{z^2}\right)$

$$= -\tfrac{1}{2}\int\frac{z\,dz}{(2-3z)^{\frac{17}{6}}}.$$

Put $2 - 3z = v$; $\therefore\ z = \frac{1}{3}(2 - v),\ dz = -\frac{1}{3}dv.$

$$\therefore\ I = -\tfrac{1}{2}\cdot\tfrac{1}{3}(-\tfrac{1}{3})\int\frac{(2-v)dv}{v^{17/6}} = \tfrac{1}{9}\int\frac{dv}{v^{17/6}} - \tfrac{1}{18}\int\frac{dv}{v^{11/6}}$$

$$= \tfrac{1}{9}(-\tfrac{6}{11})\frac{1}{v^{11/6}} - \tfrac{1}{18}(-\tfrac{6}{5})\frac{1}{v^{5/6}}$$

$$= \frac{1}{v^{11/6}}\left\{-\frac{2}{33} + \frac{v}{15}\right\} = \frac{11v - 10}{165v^{11/6}}.$$

But $v = 2 - 3z = 2 - \dfrac{3}{y} = 2 - \dfrac{3}{x^2} = \dfrac{2x^2 - 3}{x^2}$;

$$\therefore\ 11v - 10 = \frac{12x^2 - 33}{x^2}.$$

$$\therefore\ I = \frac{1}{165}\,\frac{12x^2 - 33}{x^2}\left(\frac{x^2}{2x^2 - 3}\right)^{\frac{11}{6}} = \frac{1}{55}\,\frac{x^{\frac{5}{3}}(4x^2 - 11)}{(2x^2 - 3)^{11/6}}.$$

NOTE—In actual work it is perhaps more convenient to put first $x = 1/y$, and afterwards $y^2 = z$.

401. Integration of $x^m(ax^p + b)^n\,dx$—When immediately integrable.

Let $x^p = y$, $\therefore\ x = y^{\frac{1}{p}}$; $dx = \dfrac{1}{p}y^{\frac{1}{p}-1}dy = \dfrac{1}{p}y^{\frac{1-p}{p}}dy.$

Then

$$\int x^m(ax^p + b)^n\,dx = \frac{1}{p}\int y^{\frac{m}{p}}(ay + b)^n\,y^{\frac{1-p}{p}}\,dy = \frac{1}{p}\int y^{\frac{m-p+1}{p}}(ay + b)^n\,dy.$$

Hence, by Art. 397, this is immediately integrable when

(A) n *is a* $+^{ve}$ *integer.*

(B) m+1 *is a* $+^{ve}$ *multiple of* p.

(C) $\dfrac{m-p+1}{p}+n+2$, i.e. $\dfrac{m+1}{p}+n+1=0$ or a $-^{ve}$ integer,

i.e. *when* $\dfrac{m+1}{p}+n$ *is a* $-^{ve}$ *integer.*

Ex. 1. $\int x^{\frac{5}{3}}\sqrt{x^{\frac{2}{3}}+1}\,dx.$

Put $x^{\frac{2}{3}}=y$; $\therefore x=y^{\frac{3}{2}},\ dx=\frac{3}{2}y^{\frac{1}{2}}\,dy.$

$$\therefore I=\tfrac{3}{2}\int y^{\frac{5}{2}}\sqrt{y+1}\,.\,y^{\frac{1}{2}}\,dy=\tfrac{3}{2}\int y^3\sqrt{y+1}\,dy,$$

which can be integrated by putting $y+1=z$.

Ex. 2. Suppose $m=-\frac{5}{6},\ n=-\frac{4}{3},\ p=\frac{1}{2}$; then $\dfrac{m+1}{p}+n=\frac{1}{3}-\frac{4}{3}$ $=-1$, a $-^{ve}$ integer; hence, for example,

$$\int\frac{dx}{x^{\frac{5}{6}}(2x^{\frac{1}{2}}-1)^{\frac{4}{3}}}\ \text{will be immediately integrable.}$$

Put $x^{\frac{1}{2}}=y$; $\therefore dx=2y\,dy$, and $I=2\displaystyle\int\frac{y\,dy}{y^{\frac{5}{3}}(2y-1)^{\frac{4}{3}}}=2\int\frac{dy}{y^{\frac{2}{3}}(2y-1)^{\frac{4}{3}}}.$

Put $y=\dfrac{1}{z}$; $\therefore I=-2\displaystyle\int\frac{z^{\frac{2}{3}}.z^{\frac{4}{3}}}{(2-z)^{\frac{4}{3}}}\cdot\frac{dz}{z^2}=-2\int\frac{dz}{(2-z)^{\frac{4}{3}}}$

$$=2\int\frac{dv}{v^{\frac{4}{3}}},\ \text{if } v=2-z,$$

$$=-\frac{6}{v^{\frac{1}{3}}}.$$

But $\qquad v=2-z=2-\dfrac{1}{y}=2-\dfrac{1}{x^{\frac{1}{2}}}=\dfrac{2x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}},$

$$\therefore I=-6\frac{x^{\frac{1}{6}}}{(2x^{\frac{1}{2}}-1)^{\frac{1}{3}}}.$$

402. Reduction of $\int x^m(ax+b)^n dx$, $\int x^m(ax^2+b)^n\,dx$, and $\int x^m(ax^p+b)^n\,dx$.

Since the first two integrals are particular cases of the third, we shall only consider that one.

As in the case of $\int \sin^m \theta \cos^n \theta \, d\theta$, *and by a similar rule* (*q.v.*) we can obtain six formulæ of reduction; but in the integral $\int x^m(ax^p+b)^n\,dx$, *the index* m *is increased or diminished by* p, *and the index* n *by* 1.

Putting $ax^p + b = X$, the six integrals in terms of which $\int x^m X^n \, dx$ can be expressed are

$$\int x^{m+p}X^{n-1}dx, \quad \int x^{m-p}X^{n+1}dx, \quad \int x^m X^{n-1}dx, \quad \int x^m X^{n+1}dx,$$
$$\int x^{m+p}X^n dx, \quad \int x^{m-p}X^n dx.$$

Ex. 1. *Express* $\int x^m(ax^p + b)^n\,dx$ *in terms of* $\int x^m(ax^p + b)^{n-1}dx$.

Turning to Art. 391, $Q = x^{m+1}(ax^p + b)^n$.

$$\therefore \frac{d}{dx} \cdot x^{m+1}(ax^p + b)^n = (m + 1)x^m(ax^p + b)^n + apnx^{m+p}(ax^p + b)^{n-1}$$

$= [\text{See footnote ‡, Art. 391}]\,(m+1)x^m(ax^p+b)^n+pn\,.\,ax^p\,.\,x^m(ax^p+b)^{n-1}$

$= (m + 1)x^m(ax^p + b)^n + pn\{(ax^p + b) - b\}x^m(ax^p + b)^{n-1}$

$= (m + pn + 1)x^m(ax^p + b)^n - bpn\,.\,x^m(ax^p + b)^{n-1}.$

Transposing and integrating, we have

$$\int x^m X^n\,dx = \frac{1}{m + pn + 1} \cdot x^{m+1}X^n + \frac{bpn}{m + pn + 1}\int x^m X^{n-1}\,dx.$$

Ex. 2. *Express* $\int x^m(ax^p + b)^n\,dx$ *in terms of* $\int x^{m-p}(ax^p + b)^n\,dx$.

Here $Q = x^{m-p+1}(ax^p + b)^{n+1}$.

$$\therefore \frac{d}{dx} \cdot x^{m-p+1}(ax^p + b)^{n+1}$$

$= (m - p + 1)x^{m-p}(ax^p + b)^{n+1} + ap(n + 1)x^m(ax^p + b)^n$

$= [\text{footnote}]\,(m-p+1)x^{m-p}(ax^p+b)(ax^p+b)^n+ap(n+1)x^m(ax^p+b)^n$

$= \{a(m-p+1)+ap(n+1)\}x^m(ax^p+b)^n+b(m-p+1)x^{m-p}(ax^p+b)^n$

$= a(m + pn + 1)x^m X^n + b(m - p + 1)x^{m-p}X^n.$

$$\therefore \int x^m X^n dx = \frac{1}{a(m + pn + 1)}\,x^{m-p+1}X^{n+1} - \frac{b(m - p + 1)}{a(m + pn + 1)}\int x^{m-p}X^n dx.$$

NOTE.—By putting $p = 1$ and 2 respectively, we can deduce the corresponding formulæ of reduction for the first two integrals at the head of this article.

403. Each of the three above integrals can be converted into the form $\int \sin^m \theta \cos^n \theta \, d\theta$. Thus, if a and b be both $+^{ve}$ in the integral $\int x^m(ax^p+b)^n \, dx$, put $ax^p = b\tan^2\theta$, or $x=(b/a)^{1/p}\tan^{2/p}\theta$. Then neglecting constant coefficients, we have

$$I \propto \int \tan^{2m/p}\theta \,.\, (b\sec^2\theta)^n \,.\, \tan^{\frac{2}{p}-1}\theta \,.\, \sec^2\theta d\theta,$$

and this evidently reduces to the above form, as stated.

Similarly, if a be $+^{ve}$, and b $-^{ve}$, put $ax^p = b\sec^2\theta$, and
if a be $-^{ve}$, and b $+^{ve}$, put $ax^p = -b\sin^2\theta$.

Having converted the integral into this form, we may employ reduction, as in Art. 391. In working certain examples this is frequently the simpler method.

NOTE.—It should be borne in mind that the method of reduction is only used when other methods fail.

404. Examples.

Ex. 1. $\displaystyle\int \frac{dx}{x^n(a^2-x^2)^{\frac{1}{2}}}$, n being a $+^{ve}$ integer.

Using the same notation as before,

$$P = x^{-n}(a^2-x^2)^{-\frac{1}{2}},\ R = x^{-n+2}(a^2-x^2)^{-\frac{1}{2}}$$

(the power of x being increased by 2; see Art. 402).

$\therefore\ Q = x^{-n+1}(a^2-x^2)^{\frac{1}{2}}$, and

$$\begin{aligned}\frac{d}{dx}Q &= (-n+1)x^{-n}(a^2-x^2)^{\frac{1}{2}} - x^{-n+2}(a^2-x^2)^{-\frac{1}{2}}\\ &= (-n+1)x^{-n}(a^2-x^2)^{-\frac{1}{2}}(a^2-x^2) - R\\ &= a^2(-n+1)P - (-n+1)R - R\\ &= -(n-1)a^2P + (n-2)R.\end{aligned}$$

$$\therefore\ \int P\,dx = -\frac{1}{(n-1)a^2}Q + \frac{n-2}{(n-1)a^2}\int R\,dx;$$

or

$$\int x^{-n}(a^2-x^2)^{-\frac{1}{2}}\,dx = -\frac{1}{(n-1)a^2}x^{-n+1}(a^2-x^2)^{\frac{1}{2}} + \frac{n-2}{(n-1)a^2}\int x^{-n+2}(a^2-x^2)^{-\frac{1}{2}}\,dx$$

which is the formula of reduction.

Hence I will be ultimately expressed in terms of $\int\frac{dx}{x\sqrt{a^2-x^2}}$ or $\int\frac{dx}{\sqrt{a^2-x^2}}$, according as n is odd or even.

As an example let $n=3$.

Then
$$\int\frac{dx}{x^3\sqrt{a^2-x^2}}=-\frac{1}{2a^2}\frac{\sqrt{a^2-x^2}}{x^2}+\frac{1}{2a^2}\int\frac{dx}{x\sqrt{a^2-x^2}}$$
$$=-\frac{1}{2a^2}\frac{\sqrt{a^2-x^2}}{x^2}-\frac{1}{2a^3}\cosh^{-1}\frac{a}{x}\quad\text{[Art. 319]}$$
$$=-\frac{1}{2a^2}\frac{\sqrt{a^2-x^2}}{x^2}-\frac{1}{2a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}.$$

[See also Art. 351.]

Otherwise:—Put $x=a\sin\theta$;
$$\therefore\int\frac{dx}{x^3\sqrt{a^2-x^2}}=\frac{1}{a^3}\int\frac{d\theta}{\sin^3\theta}.$$

Here $P=s^{-3}$, $R=s^{-1}$; $\therefore Q=s^{-2}c$, and
$$\frac{d}{d\theta}s^{-2}c=-2s^{-3}c^2-s^{-1}=-2s^{-3}(1-s^2)-s^{-1}=-2s^{-3}+s^{-1}.$$
$$\therefore\int s^{-3}d\theta=-\tfrac{1}{2}s^{-2}c+\tfrac{1}{2}\int s^{-1}d\theta=-\frac{\cos\theta}{2\sin^2\theta}+\tfrac{1}{2}\log\tan\tfrac{1}{2}\theta$$
$$=-\frac{\cos\theta}{2\sin^2\theta}+\tfrac{1}{2}\log\frac{1-\cos\theta}{\sin\theta}=-\frac{a\sqrt{a^2-x^2}}{2x^2}+\tfrac{1}{2}\log\frac{a-\sqrt{a^2-x^2}}{x}.$$
$$\therefore\frac{1}{a^3}\int\frac{d\theta}{\sin^3\theta}=\text{etc., as before.}$$

Ex. 2. $\int x^3(x^2-a^2)^{\frac{n}{2}}\,dx$.

Since x^3 has an *odd* index, we have
$$I=\tfrac{1}{2}\int x^2(x^2-a^2)^{\frac{n}{2}}\,d(x^2)=\tfrac{1}{2}\int y(y-a^2)^{\frac{n}{2}}\,dy,\text{ if }y=x^2;$$
and integrating by parts, this
$$=\frac{1}{n+2}y(y-a^2)^{\frac{n+2}{2}}-\frac{1}{n+2}\int(y-a^2)^{\frac{n+2}{2}}dy=\text{etc.}$$

Otherwise:—Put $y-a^2=z$.

405. Other Forms. We shall conclude this chapter, and with it the subject of indefinite integration, by giving as briefly as possible a few other forms requiring reduction.

406. $\int (ax^2 + b)^n \, dx$.

This is really of the form $\int x^m (ax^p + b)^n \, dx$, where $m = 0$ and $p = 2$.

Hence by Art. 402, Ex. 1, we shall have, putting $ax^2 + b = X$,

$$\int X^n \, dx = \frac{1}{2n+1} xX^n + \frac{2nb}{2n+1} \int X^{n-1} \, dx.$$

Or, we have independently, noting that $dX/dx = 2ax$,

$$P = x^0 X^n, \quad R = x^0 X^{n-1}, \quad \therefore Q = xX^n.$$

Now $\frac{d}{dx} xX^n = X^n + nxX^{n-1} \frac{dX}{dx} = X^n + 2anx^2 X^{n-1}$

$$= X^n + 2n(X - b)X^{n-1} = (2n+1)X^n - 2nbX^{n-1}$$

$$\therefore \int X^n \, dx = \text{etc., as before.}$$

Note.—If n be $-^{ve}$, we must express $\int X^n \, dx$ in terms of $\int X^{n+1} \, dx$.

407. $\int (ax^2 + 2hx + b)^n dx$.

We have $ax^2 + 2hx + b = a\left\{\left(x + \frac{h}{a}\right)^2 + \frac{ab - h^2}{a^2}\right\} = ay^2 + b'$,

if $$y = \frac{ax + h}{a}, \text{ and } b' = \frac{ab - h^2}{a};$$

$$\therefore I = \int (ay^2 + b')^n dy = \int Y^n dy \text{ say,}$$

$$= \frac{1}{2n+1} yY^n + \frac{2nb'}{2n+1} \int Y^{n-1} \, dy, \text{ by Art. 406,}$$

$$= \frac{1}{2n+1} \frac{ax+h}{a} (ax^2 + 2hx + b)^n + \frac{2n}{2n+1} \frac{ab - h^2}{a} \int (ax^2 + 2hx + b)^{n-1} dx.$$

See also preceding *Note*.

***408.** $\int x^m (ax^2 + 2hx + b)^n \, dx$.

Putting X for $ax^2 + 2hx + b$, we shall express $\int x^m X^n dx$ in terms of *both* $\int x^{m-1} X^n dx$ and $\int x^{m-2} X^n dx$.

We have $\frac{d}{dx} x^{m-1} X^{n+1}$ [cf. Rule, Art. 391].

$$= (m-1)x^{m-2} X^{n+1} + (n+1)x^{m-1} X^n \frac{dX}{dx}$$

$$= (m-1)x^{m-2}(ax^2 + 2hx + b)X^n + 2(n+1)x^{m-1}(ax + h)X^n$$

$$= (m + 2n + 1)ax^m X^n + 2(m+n)hx^{m-1} X^n + (m-1)bx^{m-2} X^n.$$

Whence, transposing and integrating,

$$\int x^m X^n dx = \frac{x^{m-1}X^{n+1}}{(m+2n+1)a} - \frac{2(m+n)}{m+2n+1}\frac{h}{a}\int x^{m-1}X^n dx$$
$$-\frac{m-1}{m+2n+1}\frac{b}{a}\int x^{m-2}X^n dx.$$

By repeating the formula, the given integral will ultimately depend on $\int xX^n\,dx$ and $\int X^n\,dx$.

The latter may be obtained by the last article. For the former we have, noting that $\frac{dX}{dx} = 2ax+2h$,

$$\int x(ax^2+2hx+b)^n dx = \frac{1}{2a}\int(2ax+2h)X^n dx - \frac{h}{a}\int X^n dx$$
$$= \frac{1}{2a}\frac{X^{n+1}}{n+1} - \frac{h}{a}\int X^n dx,$$

the integral on the right being obtained as before.

If m be $-^{ve}$ we must express $\int x^m X^n dx$ in terms of $\int x^{m+1}X^n dx$ and $\int x^{m+2}X^n dx$; and by repeating the formula we shall ultimately arrive at the integrals $\int \frac{X^n}{x}\,dx$ and $\int X^n dx$.

The integral $\quad \displaystyle\int \frac{X^n}{x}dx = \int\frac{(ax^2+2hx+b)^n}{x}dx$

$$= \int\frac{ax^2+2hx+b}{x}(ax^2+2hx+b)^{n-1}dx$$
$$= a\int xX^{n-1}dx + 2h\int X^{n-1}dx + b\int\frac{X^{n-1}}{x}\,dx,$$

and, the first two integrals being obtainable as above, we have a formula of reduction.

In all these cases the integrals may be ultimately obtained when n is integral or a multiple of $\frac{1}{2}$.

***409.** $\displaystyle\int\frac{dx}{(a+b\cos x)^n}$.

The following method will be more easily remembered by noting that, *in the numerator*, (1) $\sin^2 x + \cos^2 x$ is written for 1; (2) whenever we come across $\sin x$ we integrate by parts; (3)

whenever we come across $\cos x$ or $\cos^2 x$ we express it in terms of $a + b\cos x$.

Thus $\cos x = \frac{1}{b}.b\cos x = \frac{1}{b}(a + b\cos x) - \frac{a}{b} = \frac{X-a}{b}$ say.

$$\therefore \cos^2 x = \frac{X^2 - 2aX + a^2}{b^2}; \text{ also } dX = -b\sin x\,dx.$$

Hence $\displaystyle\int\frac{dx}{X^n} = \int\frac{\sin^2 x\,dx}{X^n} + \int\frac{\cos^2 x\,dx}{X^n}$

$$= -\frac{1}{b}\int\frac{\sin x\,dX}{X^n} + \frac{1}{b^2}\int\frac{X^2 - 2aX + a^2}{X^n}dx$$

$$= \frac{1}{(n-1)b}\frac{\sin x}{X^{n-1}} - \frac{1}{(n-1)b}\int\frac{\cos x\,dx}{X^{n-1}} + \frac{1}{b^2}\int\frac{dx}{X^{n-2}} - \frac{2a}{b^2}\int\frac{dx}{X^{n-1}} + \frac{a^2}{b^2}\int\frac{dx}{X^n};$$

$$\therefore \left(1 - \frac{a^2}{b^2}\right)\int\frac{dx}{X^n}$$

$$= \frac{1}{(n-1)b}\frac{\sin x}{X^{n-1}} - \frac{1}{(n-1)b^2}\int\frac{X-a}{X^{n-1}}dx + \frac{1}{b^2}\int\frac{dx}{X^{n-2}} - \frac{2a}{b^2}\int\frac{dx}{X^{n-1}}$$

$$= \frac{1}{(n-1)b}\frac{\sin x}{X^{n-1}} + \left(1 - \frac{1}{n-1}\right)\frac{1}{b^2}\int\frac{dx}{X^{n-2}} + \left(\frac{1}{n-1} - 2\right)\frac{a}{b^2}\int\frac{dx}{X^{n-1}};$$

$$\therefore \frac{b^2 - a^2}{b^2}I_n = \frac{1}{(n-1)b}\frac{\sin x}{X^{n-1}} + \frac{n-2}{n-1}\frac{1}{b^2}I_{n-2} - \frac{2n-3}{n-1}\frac{a}{b^2}I_{n-1};$$

$$\therefore I_n = \frac{1}{n-1}\frac{b}{b^2-a^2}\frac{\sin x}{X^{n-1}} + \frac{n-2}{n-1}\frac{1}{b^2-a^2}I_{n-2} - \frac{2n-3}{n-1}\frac{a}{b^2-a^2}I_{n-1};$$

or $\displaystyle\int\frac{dx}{(a+b\cos x)^n}$

$$= \frac{1}{n-1}\frac{b}{b^2-a^2}\frac{\sin x}{(a+b\cos x)^{n-1}} - \frac{2n-3}{n-1}\frac{a}{b^2-a^2}\int\frac{dx}{(a+b\cos x)^{n-1}}$$
$$+ \frac{n-2}{n-1}\frac{1}{b^2-a^2}\int\frac{dx}{(a+b\cos x)^{n-2}};$$

and by repeating the formula, the given integral will ultimately depend on $\displaystyle\int\frac{dx}{a+b\cos x}$, which has been considered above (Art 363).

410. The subject, as we leave it, is very incomplete; but it may be pointed out that the preceding methods lead up to the integration of the general expression $f(x, \sqrt{X})\,dx$, where f denotes a rational (integral or fractional) algebraical function of x and $\sqrt{X}$, and $X = ax^2 + 2hx + b$.

We have been obliged to omit the integration of

$$\frac{(px+q)\,dx}{(lx^2+2mx+n)\sqrt{ax^2+2hx+b}}$$

and kindred forms, but when this is done we can show that by rationalization, the method of partial fractions, and reduction it is possible for $f(x, \sqrt{X})$ to be integrated in *finite* terms involving only the ordinary transcendental functions.

When X is of a degree higher than the second, it cannot be so expressed (except in special cases), but necessitates the introduction of fresh transcendental functions. [See *Williamson's Diff. Calc.*, Art. 28.]

411. We may remark, finally, that the method of expansion can be adopted in the case of the integral $\int f(x)\,dx$, provided that $f(x)$ is capable of being expanded in a convergent series of ascending or descending powers of x.

Thus $$\int \frac{dx}{\sqrt{1-x^3}} = \int\left(1 + \frac{1}{2}x^3 + \frac{1.3}{2.4}x^6 + \dots\right)dx$$
$$= x + \frac{1}{2}\cdot\frac{x^4}{4} + \frac{1.3}{2.4}\cdot\frac{x^7}{7} + \dots,$$

which is true so long as $x < 1$.

EXAMPLES LXVI.

1. Integrate by the method of Art. 397:—

(1) $\sqrt{x}(1-x)^2\,dx$. (2) $x^2\sqrt{1-x}\,dx$. (3) $\dfrac{dx}{x^2(2x-3)^2}$.

(4) $\dfrac{dx}{x^{\frac{1}{3}}(1+x)^{\frac{5}{3}}}$. (5) $\dfrac{x^{\frac{5}{3}}\,dx}{(2x+3)^{\frac{11}{3}}}$.

2. Show that $(ax+b)^m(cx+d)^n\,dx$ is immediately integrable if either m or n is a $+^{\text{ve}}$ integer, or if $m+n+2=0$, or a $-^{\text{ve}}$ integer.

3. Integrate :—

(1) $(2x-3)^2(x-2)^{\frac{3}{2}}\,dx$. (2) $\dfrac{(x-1)^{\frac{2}{3}}\,dx}{(x+1)^{\frac{11}{3}}}$.

(3) $\dfrac{dx}{(x^2-1)^{\frac{3}{2}}}$.

4. Integrate by the method of Art. 400 :—

(1) $\dfrac{(a^2+x^2)^2}{x^2\sqrt{x}}\,dx$. (2) $x^3(a^2-x^2)^{\frac{1}{2}}\,dx$. (3) $\dfrac{dx}{x^2\sqrt{1+x^2}}$.

(4) $\dfrac{dx}{x^4(2x^2-1)^{\frac{5}{8}}}$. (5) $\dfrac{(1-x^2)^{\frac{1}{2}}\,dx}{x^{\frac{11}{2}}}$.

5. Integrate by the method of Art. 401 :—

(1) $x^9(x^5+2)^{\frac{1}{2}}\,dx$. (2) $\dfrac{dx}{\sqrt{x-x^{\frac{2}{3}}}}$. (3) $\dfrac{dx}{x^{\frac{1}{3}}(2x^{\frac{1}{2}}-1)^{\frac{7}{3}}}$.

(4) $x^{\frac{1}{2}}(2x^{-\frac{3}{2}}-3)^{\frac{1}{3}}dx$. (5) $\dfrac{dx}{(2x^{\frac{6}{5}}+3)^{\frac{11}{6}}}$. (6) $\dfrac{x^{\frac{4}{3}}\,dx}{(2+x^{\frac{2}{3}})^{\frac{11}{2}}}$.

6. Show that

$$\int_0^{\frac{b}{a}} x^m(ax-b)^n\,dx = -\frac{na}{m+1}\int_0^{\frac{b}{a}} x^{m+1}(ax-b)^{n-1}\,dx;$$

and if either m or n be a $+^{\text{ve}}$ integer, show that the value of the integral is either $(-1)^n\dfrac{m!\,n!}{(m+n+1)!}\dfrac{b^{m+n+1}}{a^{m+1}}$, or of that form.

7. Show that if $ax^2+b=X$,

(1) $\int x^mX^n dx = \dfrac{1}{(m+2n+1)a}x^{m-1}X^{n+1} - \dfrac{m-1}{m+2n+1}\dfrac{b}{a}\int x^{m-2}X^n dx.$

(2) $\int x^mX^n dx = \dfrac{1}{m+2n+1}x^{m+1}X^n + \dfrac{2nb}{m+2n+1}\int x^mX^{n-1}dx.$

(3) $\int x^mX^n dx = \dfrac{1}{(m+1)b}x^{m+1}X^{n+1} - \dfrac{m+2n+3}{m+1}\dfrac{a}{b}\int x^{m+2}X^n dx.$

Show also that

(4) $\displaystyle\int\frac{x^m dx}{(a^2+x^2)^n} = -\frac{1}{2(n-1)}\frac{x^{m-1}}{(a^2+x^2)^{n-1}} + \frac{m-1}{2(n-1)}\int\frac{x^{m-2}dx}{(a^2+x^2)^{n-1}}$

8. Integrate by reduction :—

(1) $\dfrac{x^4 dx}{\sqrt{a^2-x^2}}$. (2) $\dfrac{dx}{x(a^2-x^2)^{\frac{3}{2}}}$. (3) $\dfrac{dx}{x^2\sqrt{x^2+a^2}}$.

(4) $\dfrac{dx}{x^3\sqrt{x^2-a^2}}$. (5) $\dfrac{x^4 dx}{(a^2+x^2)^{\frac{5}{2}}}$. (6) $\dfrac{x^6}{(a^2+x^2)^3}$.

(7) $\dfrac{dx}{x(x^2+2)^2}$. (8) $x^2(x^2+a^2)^{\frac{3}{2}}\,dx$.

Can any of these be integrated immediately?

9. Verify the examples in the preceding question by trigonometrical substitution.

10. Show that if $ax^p + b = X$,

(1) $\int x^m X^n dx = \dfrac{1}{(m+1)b}x^{m+1}X^{n+1} - \dfrac{m+np+p+1}{m+1}\dfrac{a}{b}\int x^{m+p}X^n dx.$

(2) $\int x^m X^n dx = -\dfrac{1}{bp(n+1)}x^{m+1}X^{n+1} + \dfrac{m+np+p+1}{bp(n+1)}\int x^m X^{n+1} dx.$

11. Show that if n be a $+^{\text{ve}}$ integer,

(1) $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+m^2)^n} = \frac{\pi}{m^{2n-1}}\cdot\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}$;

(2) $\displaystyle\int_{-\infty}^{\infty}\frac{x\,dx}{(a+bx+cx^2)^n} = \frac{-b}{c^{n+1}m^{2n-1}}\cdot\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}\frac{\pi}{2}$;

where $m^2 = \dfrac{4ac-b^2}{4c^2}$, the roots of the quadratic $a+bx+cx^2=0$ being imaginary. [See Answer.]

ANSWERS.

1. (1) $2x^{\frac{3}{2}}(\frac{1}{3} - \frac{2}{5}x + \frac{1}{7}x^2)$. (2) $-\frac{2}{105}(8+12x+15x^2)(1-x)^{\frac{3}{2}}$.

(3) $\dfrac{3-4x}{9x(2x-3)} - \dfrac{4}{27}\log\dfrac{2x-3}{x}$. (4) $\dfrac{2}{3}\cdot\dfrac{(2x+3)\sqrt{x}}{(1+x)^{\frac{3}{2}}}$.

(5) $\dfrac{2(4x+27)}{567}\cdot\dfrac{x^{\frac{7}{2}}}{(2x+3)^{\frac{9}{2}}}$.

3. (1) $\frac{2}{315}(140x^2-380x+263)(x-2)^{\frac{5}{2}}$. (2) $\dfrac{3}{160}\cdot\dfrac{(x-1)^{\frac{5}{3}}}{(x+1)^{\frac{8}{3}}}(3x+13)$.

(3) $-\dfrac{x}{\sqrt{x^2-1}}$.

4. (1) $\frac{2}{15}\cdot\frac{1}{x^{\frac{3}{2}}}(3x^4+30x^2a^2-5a^4)$. (2) $-\frac{1}{15}(2a^2+3x^2)(a^2-x^2)^{\frac{3}{2}}$.

(3) $-\frac{\sqrt{1+x^2}}{x}$. (4) $\frac{3(2x^2-1)^{\frac{1}{6}}}{x^{\frac{1}{3}}}$. (5) $-\frac{2}{45}(5+4x^2)\cdot\frac{(1-x^2)^{\frac{5}{4}}}{x^{\frac{9}{2}}}$.

5. (1) $\frac{2}{75}(3x^5-4)(x^5+2)^{\frac{3}{2}}$. (2) $2(x^{\frac{1}{3}}+2)\sqrt{x^{\frac{1}{3}}-1}$.

(3) $-\frac{3}{2}\cdot\frac{x^{\frac{4}{3}}}{(2x^{\frac{1}{3}}-1)^{\frac{4}{3}}}$. (4) $-\frac{1}{21}(2x^{\frac{3}{2}}+1)(2-3x^{\frac{3}{2}})^{\frac{4}{3}}$.

(5) $\frac{1}{33}\cdot\frac{x(4x^{\frac{6}{5}}+11)}{(2x^{\frac{6}{5}}+3)^{\frac{11}{6}}}$. (6) $\frac{1}{42}\,\frac{x^{\frac{7}{3}}(9+x^{\frac{2}{3}})}{(2+x^{\frac{2}{3}})^{\frac{9}{2}}}$.

8. (1) $-\frac{1}{8}x(2x^2+3a^2)\sqrt{a^2-x^2}+\frac{3}{8}a^4\sin^{-1}\frac{x}{a}$.

(2) $\frac{1}{a^2(a^2-x^2)^{\frac{1}{2}}}-\frac{1}{a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}$. (3) $-\frac{\sqrt{x^2+a^2}}{a^2x}$.

(4) $\frac{1}{2a^2}\frac{\sqrt{x^2-a^2}}{x^2}+\frac{1}{2a^3}\sec^{-1}\frac{x}{a}$. (5) $-\frac{x(4x^2+3a^2)}{3(x^2+a^2)^{\frac{3}{2}}}+\sinh^{-1}\frac{x}{a}$.

(6) $-\frac{1}{4}\frac{x^5}{X^2}-\frac{5}{8}\frac{x^3}{X}+\frac{15}{8}x-\frac{15}{8}a\tan^{-1}\frac{x}{a}$, where $X=a^2+x^2$.

(7) $\frac{1}{4(x^2+2)}+\frac{1}{8}\log\frac{x^2}{x^2+2}$.

(8) $\frac{1}{6}xX^{\frac{5}{2}}-\frac{a^2}{24}xX^{\frac{3}{2}}-\frac{a^4}{16}xX^{\frac{1}{2}}-\frac{a^6}{16}\sinh^{-1}\frac{x}{a}$, where $X=a^2+x^2$.

Yes; (3) and (7) by Art. 400 (C).

11. (2) $I=\frac{1}{c^n}\int_{-\infty}^{\infty}\frac{x\,dx}{\{(x+b/2c)^2+m^2\}^n}$; put $x+b/2c=y$, or $m\tan\theta$.

CHAPTER XXVI.

DEFINITE INTEGRATION.

412. Definite Integration and Summation—Strict Proof of the Formula $\int_a^b f'(x)\,dx = f(b) - f(a)$.

We shall suppose $f(x)$ and its d.c.'s to be finite and continuous for values of x between a and b.

Using Lagrange's form of remainder, so as to avoid the question of convergency, we have—

$$f(x+h) - f(x) = hf'(x) + \frac{h^2}{2}f''(x+\theta h), \quad . \quad . \quad (1)$$

which is true provided that $x + h$ is not $> b$, nor $x < a$.

Now let the total increment $b - a$ be divided into n equal increments, h; n being very large and therefore h very small.

Then
$$nh = b - a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let x have in turn the n values,

$$a,\ a+h,\ a+2h, \ldots\ b-2h,\ b-h;$$

then substituting these values in (1), but in reverse order, we shall have the n equations,

$$\left.\begin{aligned}
f(b) \quad - f(b-h) &= hf'(b-h) + \frac{h^2}{2}f''(b-h+\theta h) \\
f(b-h) - f(b-2h) &= hf'(b-2h) + \frac{h^2}{2}f''(b-2h+\theta h) \\
\text{etc.} \quad &= \quad \text{etc.} \\
f(x+h) - f(x) \quad &= hf'(x) + \frac{h^2}{2}f''(x+\theta h) \\
\text{etc.} \quad &= \quad \text{etc.} \\
f(a+2h) - f(a+h) &= hf'(a+h) + \frac{h^2}{2}f''(a+h+\theta h) \\
f(a+h) - f(a) \quad &= hf'(a) + \frac{h^2}{2}f''(a+\theta h)
\end{aligned}\right\} \quad . \quad (3)$$

where $f'(b-h)$ is the value of $f'(x)$ when $b-h$ is put for x *after* differentiation; and so on.

Adding these together we have

$$f(b)-f(a)=\sum_{x=a}^{x=b-h} hf'(x)+\frac{h^2}{2}\sum_{x=a}^{x=b-h} f''(x+\theta h). \quad . \quad . \quad (4)$$

Consider the last term of (4). Since $f''(x+\theta h)$ is finite (hyp.), let us suppose α to be its *greatest* value in (4).

Then $$\frac{h^2}{2}\Sigma f''(x+\theta h)<\frac{h^2}{2}\Sigma\alpha<\frac{h^2}{2}.(\alpha+\alpha+\ldots \text{ to } n \text{ terms})$$

$$\text{i.e.} <\frac{h^2}{2}.n\alpha, \text{ i.e.} <\frac{1}{2}(b-a)\alpha.h \text{ by (2).}$$

Hence, when h is made indefinitely small, this term will be negligible, and, in the limit, will vanish with h.

Writing dx for h, we have, proceeding to the limit,

$$f(b)-f(a)=\underset{h=0}{lt}\sum_{a}^{b-h} hf'(x)=\sum_{a}^{b} f'(x)\,dx,$$

which is written $\int_a^b f'(x)\,dx$.

413. Other Forms of Statement.

Let the general value of x be $a+rh$, or $a+r\frac{b-a}{n}$, by (2).

Then $hf'(x)=\frac{b-a}{n}f'\left(a+r\frac{b-a}{n}\right)$; and from (3), neglecting the terms in h^2, we have the following statement:—

The limit, when $n=\infty$, of the series

$$\frac{b-a}{n}\left\{f'(a)+f'\left(a+\frac{b-a}{n}\right)+\ldots+f'\left(a+r\frac{b-a}{n}\right)+\ldots+f'(b)\right\} \quad (5)$$

is $f(b)-f(a)$.

This may be written, $$\underset{n=\infty}{lt}\frac{b-a}{n}\sum_{r=0}^{r=n} f'\left(a+r\frac{b-a}{n}\right)=f(b)-f(a).$$

Another and a simpler statement may be obtained by putting

$h = \frac{1}{n}$ in (1), so that the number of equations in (3) will be $(b-a)n$ instead of n. The series will then be

$$\frac{1}{n}\left\{f'(a) + f'\left(a+\frac{1}{n}\right) + f'\left(a+\frac{2}{n}\right) + \ldots + f'\left(b-\frac{1}{n}\right) + f'(b)\right\} \dagger \quad (7)$$

the limit of which, when $n = \infty$, is $f(b) - f(a)$.

Again, (7) may be written

$$\frac{1}{n}\left\{f'\left(\frac{na}{a}\right) + f'\left(\frac{na+1}{n}\right) + f'\left(\frac{na+2}{n}\right) + \ldots + f'\left(\frac{r}{n}\right) + \ldots + f'\left(\frac{nb}{n}\right)\right\}$$

which $= \sum_{r=na}^{r=nb} \frac{1}{n} f'\left(\frac{r}{n}\right)$, where r increases by increments of 1 from na to nb.

Hence
$$\mathop{lt}_{n=\infty} \sum_{r=na}^{r=nb} \frac{1}{n} f'\left(\frac{r}{n}\right) = f(b) - f(a) \quad . \quad . \quad . \quad . \quad (8)$$

414. Converse Problem.

Conversely, given the series (5); put $a + r\frac{b-a}{n} = x$, then the increment of x is $\frac{b-a}{n}$; call this dx. The limits of x being a and b, the series $= \mathop{lt}_{n=\infty} \Sigma_a^b f'(x)dx = \int_a^b f'(x)dx = f(b) - f(a)$.

Or, since r/n is a variable quantity, which increases from 0 to 1 by infinitesimal increments, $1/n$, we may put $r/n = y$, and $1/n = dy$, the limits of y being 0 and 1.

Hence the series (5) $= (b-a)\int_0^1 f'\{a + (b-a)y\}dy = f(b) - f(a)$;
which we can also obtain from the equation

$$\int_a^b f'(x)dx = f(b) - f(a),$$

by putting $x = a + (b-a)y$.

---

† Strictly, the last term should be $f'\left(b - \frac{1}{n}\right)$; but as we make an infinitesimal error (only) in introducing $f'(b)$, the statement which follows will still be true.

Or, again, *given the series* (7), we may take as the general term $f'\left(a+\frac{r}{n}\right)$, and calling $\frac{r}{n}=x$, noting that dx is still $\frac{1}{n}$, we have

$$lt\,\Sigma\frac{1}{n}f'\left(a+\frac{r}{n}\right)=\int_0^{b-a}f'(a+x)dx=\left[f(a+x)\right]_0^{b-a}=f(b)-f(a).$$

415. Examples.

Ex. 1. *Find* $\underset{n=\infty}{lt}\left\{\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{r}+\dots+\frac{1}{2n}\right\}$

The general term $=\frac{1}{r}=\frac{1}{n}\cdot\frac{1}{r/n}$† $=\frac{1}{x}dx.$

The limits of r are n and $2n$; $\therefore$ the limits of $\frac{r}{n}$, or x, are 1 and 2.

Hence the series $=\int_1^2\frac{dx}{x}=\left\{\log x\right\}_1^2=\log_e 2.$

Ex. 2. $\underset{n=\infty}{lt}\left\{\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\dots+\frac{1}{nb}\right\}.$

As before, the denominators increase by 1; hence the general term is $\frac{1}{r}$, *where r increases by increments of* 1 from na to nb;

$$\therefore\ \frac{r}{n}\text{ or } x \text{ increases by } \frac{1}{n}\ (\text{or } dx) \text{ from } a \text{ to } b$$

Hence the series $=\int_a^b\frac{dx}{x}=\log\frac{b}{a}.$

Or, we might take $\frac{1}{na+r}$ for the general term, where r increases by increments of 1 from 0 to $nb-na$;

$$\therefore\ \frac{r}{n}\text{ or } x \text{ increases by } \frac{1}{n}\ (\text{or } dx) \text{ from } 0 \text{ to } b-a.$$

Hence the series $=lt\,\Sigma\frac{1}{na+r}=lt\,\Sigma\frac{1}{n}\frac{1}{a+\frac{r}{n}}$

$$=\int_0^{b-a}\frac{dx}{a+x}=\left\{\log(a+x)\right\}_0^{b-a}=\log\frac{b}{a}\text{ as before.}$$

† It must be clearly understood that $1/n$ *is* the increment of r/n, otherwise we could not put $r/n=x$ and $1/n=dx$.

Ex. 3. Find $\underset{n=\infty}{lt}\left\{\frac{1}{n}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\dots+\frac{1}{2n}\right\}$.

The general term $=\frac{n}{n^2+r^2}=\frac{1}{n}\frac{1}{1+\frac{r^2}{n^2}}=\frac{dx}{1+x^2}$.

Also $\frac{1}{n}=\frac{n}{n^2}=\frac{n}{n^2+0}$; $\frac{1}{2n}=\frac{n}{n^2+n^2}$; hence the limits of r are 0 and n, and those of x are 0 and 1.

Hence series $=\int_0^1\frac{dx}{1+x^2}=\left\{\tan^{-1}x\right\}_0^1=\frac{\pi}{4}$.

Ex. 4. Find $\underset{n=\infty}{lt}\left\{\frac{1}{n}+\frac{n}{n^2+1}+\frac{n}{n^2+2}+\dots+\frac{1}{n+1}\right\}$.

The general term $=\frac{n}{n^2+r}$ (the increment of r being 1)

$$=\frac{1}{n}\cdot\frac{1}{1+\frac{r}{n^2}}=\frac{1}{n}\cdot\frac{1}{1+\frac{r}{n}\cdot\frac{1}{n}}=\frac{dx}{1+x\,dx}=dx,$$

the error being infinitesimal.

Also $\frac{1}{n+1}=\frac{n}{n^2+n}$; hence the limits of $\frac{r}{n}$, or x, are 0 and 1.

$$\therefore \text{ series }=\int_0^1 dx=1.$$

Ex. 5. Find

$$\underset{n=\infty}{lt}\left\{\frac{1}{2n}+\frac{1}{\sqrt{n+1}(\sqrt{n}+\sqrt{n+1})}+\frac{1}{\sqrt{n+2}(\sqrt{n}+\sqrt{n+2})}+\dots\right.$$
$$\left.+\frac{1}{\sqrt{4n-1}(\sqrt{n}+\sqrt{4n-1})}+\frac{1}{6n}\right\}.$$

The general term $=\frac{1}{\sqrt{r}(\sqrt{n}+\sqrt{r})}=\frac{1}{n}\frac{1}{\sqrt{\frac{r}{n}}\left(1+\sqrt{\frac{r}{n}}\right)}=\frac{dx}{\sqrt{x}(1+\sqrt{x})}$;

and r increases by increments of 1 from n to $4n$.

$$\therefore \text{ series }=\int_1^4\frac{dx}{\sqrt{x}(1+\sqrt{x})}=\left[2\log(1+\sqrt{x})\right]_1^4=2\log_e\tfrac{3}{2}.$$

416. In certain cases integration can be performed by actual algebraical or trigonometrical summation.

Ex. 1. $\int_0^1 x\,dx.$

Put $x = r/n$, $dx = 1/n$ [see Art. 415, Ex. 1, footnote].

Then the series is $\frac{1}{n}\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\dots+\frac{n}{n}\right\}$

$$=\frac{1}{n^2}(1+2+3+\dots+n)=\frac{1}{n^2}\cdot\frac{n(n+1)}{2}=\frac{n+1}{2n},$$

the limit of which, when $n=\infty$, is $\frac{1}{2}$.

And $$\int_0^1 x\,dx=\left[\tfrac{1}{2}x^2\right]_0^1=\tfrac{1}{2}, \text{ as before.}$$

Ex. 2. $\int_0^{\frac{\pi}{2}} \cos x\,dx.$

Put $x = r/n$, $dx = 1/n$.

Then the series is $\frac{1}{n}\left\{\cos\frac{1}{n}+\cos\frac{2}{n}+\dots+\cos\frac{\pi n}{2n}\right\}$,

the number of terms being $\pi n/2$; hence, by Trigonometry, this

$$=\frac{1}{n}\cdot\frac{\cos\frac{1}{2}\left(\frac{1}{n}+\frac{\pi n}{2n}\right)\sin\left(\frac{\pi n}{2}\frac{1}{2n}\right)}{\sin\frac{1}{2n}}=\frac{1}{n}\,\frac{\cos\frac{\pi n+2}{4n}\cdot\sin\frac{\pi}{4}}{\sin\frac{1}{2n}},$$

the limit of which is $2\cos\frac{\pi}{4}\sin\frac{\pi}{4}=\sin\frac{\pi}{2}=1$.

Otherwise, putting $x=\frac{r}{n}\frac{\pi}{2}$, the increment dx is $\frac{1}{n}\frac{\pi}{2}$.

Hence the series is $\frac{1}{n}\frac{\pi}{2}\left[\cos\frac{1}{n}\frac{\pi}{2}+\cos\frac{2}{n}\frac{\pi}{2}+\dots+\cos\frac{n}{n}\frac{\pi}{2}\right]$; etc.

Also $\int_0^{\frac{\pi}{2}}\cos x\,dx=\left[\sin x\right]_0^{\frac{\pi}{2}}=1$, as before.

417. Definite Integrals.

In Art. 379, three statements were made which we repeat here for convenience :—

(1) $\int_a^b \phi(x)\,dx=\int_a^b\phi(y)\,dy$, *i.e.* is independent of x or y.

(2) $\int_b^a \phi(x)\,dx=-\int_a^b\phi(x)\,dx.$

(3) When the variable is changed, the limits must be changed accordingly.

We shall now add a few more propositions.

418. To prove that $\int_a^b \phi(x)\,dx = \int_c^b \phi(x)\,dx + \int_a^c \phi(x)\,dx.$

This follows easily, for $\int_a^b \phi(x)\,dx = f(b) - f(a)$ [see Art. 379.]

$$= \{f(b) - f(c)\} + \{f(c) - f(a)\} = \int_c^b \phi(x)\,dx + \int_a^c \phi(x)\,dx.$$

419. To prove that $\int_0^a \phi(x)\,dx = \int_0^a \phi(a - x)\,dx.$

Put $x = a - y$, then the limits of y are a and 0; while $dx = -dy$.

$$\therefore \int_0^a \phi(x)\,dx = -\int_a^0 \phi(a - y)\,dy = \int_0^a \phi(a - y)\,dy = \int_0^a \phi(a - x)\,dx.$$

420. To prove that $\int_0^{\frac{\pi}{2}} \phi(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \phi(\cos x)\,dx.$

This follows at once from the preceding article, for—

$$\int_0^{\frac{\pi}{2}} \phi(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \phi\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}dx = \int_0^{\frac{\pi}{2}} \phi(\cos x)\,dx.$$

Cor.—$\int_0^{\frac{\pi}{2}} \phi(\sin x, \cos x)\,dx = \int_0^{\frac{\pi}{2}} \phi(\cos x, \sin x)\,dx.$

Ex. 1. $\int_0^{\frac{\pi}{2}} \frac{\sin x\,dx}{\sin x + \cos x} = \int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{\cos x + \sin x}.$

But if $a = b$, then each $= \frac{1}{2}(a + b)$ . . . . (1)

$$\therefore I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx = \frac{\pi}{4}.$$

Ex. 2. $\displaystyle\int_0^{\frac{\pi}{2}}\frac{x\,dx}{\sin x+\cos x}=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)dx}{\sin x+\cos x}$, by the last article,

$$=\frac{1}{2}\cdot\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{dx}{\sin x+\cos x}\text{ by (1) above,}$$

Now $\displaystyle\int\frac{dx}{\sin x+\cos x}=\int\frac{\sec^2\frac{x}{2}\,dx}{1+2\tan\frac{x}{2}-\tan^2\frac{x}{2}}$

$$=2\int\frac{dz}{1+2z-z^2},\text{ if } z=\tan\frac{x}{2},$$

$$=2\int\frac{dz}{2-(z-1)^2}=2\cdot\frac{1}{\sqrt{2}}\tanh^{-1}\frac{z-1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+z-1}{\sqrt{2}-z+1};$$

and when $x=\frac{\pi}{2}$, $z=1$; when $x=0$, $z=0$.

$$\therefore I=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{dx}{\sin x+\cos x}=\frac{\pi}{4\sqrt{2}}\left[0-\log_e\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$=\frac{\pi}{4\sqrt{2}}\log_e\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{\pi}{2\sqrt{2}}\log_e(\sqrt{2}+1).$$

In this example the *indefinite* integral cannot be obtained, except by expansion.

421. To prove that $\displaystyle\int_{-a}^{a}\phi(x)\,dx=0$ or $\displaystyle 2\int_0^a\phi(x)\,dx$, according as $\phi(x)$ is an Odd or an Even Function.

(1) Let $\phi(x)$ be an odd function, *i.e.* let $\phi(-x)=-\phi(x)$.

Then $\displaystyle\int_{-a}^{a}\phi(x)\,dx=\int_0^a\phi(x)\,dx+\int_{-a}^{0}\phi(x)\,dx$ [Art. 418].

Put $x=-y$ in the second integral; then, since $y=a$ when $x=-a$,

$$\int_{-a}^{0}\phi(x)\,dx=\int_a^0\phi(-y)\,d(-y)=-\int_a^0\phi(-y)\,dy=\int_a^0\phi(y)\,dy,\text{ by hyp.,}$$

$$=-\int_0^a\phi(y)\,dy=-\int_0^a\phi(x)\,dx.$$

$$\therefore\int_{-a}^{a}\phi(x)\,dx=0\text{ when }\phi(x)\text{ is an odd function.}$$

(2) Let $\phi(x)$ be an even function, *i.e.* let $\phi(-x) = \phi(x)$; then we can similarly show that $\int_{-a}^{a} \phi(x)\,dx = 2\int_0^a \phi(x)\,dx$.

Ex. 1. $\int_{-a}^{a} \sin x\,dx = 0$, and $\int_{-a}^{a} \cos x\,dx = 2\int_0^a \cos x\,dx = 2\sin a$.

Ex. 2. $\int_{-a}^{a} x^2 \tan^{-1} x\,dx = 0$, the function being odd.

Ex. 3. $\int_{-1}^{1} (1+x)^2 \tan^{-1} x\,dx = \int_{-1}^{1} (1+x^2)\tan^{-1} x\,dx + 2\int_{-1}^{1} x\tan^{-1} x\,dx.$

The first integral vanishes by case (1); the second integral

$$= 4\int_0^1 x\tan^{-1} x\,dx, \text{ by case (2).}$$

Now, $\int x\tan^{-1} x\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$, integrating by parts,

$$= \tfrac{1}{2}x^2\tan^{-1} x - \tfrac{1}{2}(x - \tan^{-1} x) = \tfrac{1}{2}(1+x^2)\tan^{-1} x - \tfrac{1}{2}x.$$

$$\therefore\ I = 4\int_0^1 x\tan^{-1} x\,dx = \left[2(1+x^2)\tan^{-1} x - 2x\right]_0^1 = \pi - 2.$$

422. The following example, which has often been quoted, will still further illustrate the use of these propositions :—

Ex. $\int_0^{\frac{\pi}{2}} \log\sin x\,dx = \int_0^{\frac{\pi}{2}} \log\cos x\,dx$ [Art. 420]

$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} (\log\sin x + \log\cos x)\,dx$$

$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin x\cos x\,dx$$

$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin 2x\,dx - \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \log 2\,dx$$

$$= \tfrac{1}{4}\int_0^{\frac{\pi}{2}} \log\sin 2x\,d(2x) - \frac{\pi}{4}\log 2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put y for $2x$ and change the limits, then

$$\int_0^{\frac{\pi}{2}} \log\sin 2x\,d(2x) = \int_0^{\pi} \log\sin y\,dy = \int_0^{\pi} \log\sin x\,dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} \log\sin x\,dx + \int_0^{\frac{\pi}{2}} \log\sin x\,dx \quad . \quad . \quad (2)$$

Put $x = \frac{\pi}{2} + y$ in the first integral on the right of (2),

$$\therefore \int_{\frac{\pi}{2}}^{\pi} \log \sin x \, dx = \int_0^{\frac{\pi}{2}} \log \cos y \, dy = \int_0^{\frac{\pi}{2}} \log \sin y \, dy \text{ [Art. 420]}$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence from (2) and (3),

$$\int_0^{\frac{\pi}{2}} \log \sin 2x \, d(2x) = 2I.$$

$$\therefore \text{ in (1), } I = \tfrac{1}{2}I - \frac{\pi}{4}\log 2.$$

$$\therefore I = -\frac{\pi}{2}\log_e 2.$$

EXAMPLES LXVII.

1. Find the limit, when $n = \infty$, of the series:—

(1) $\frac{1}{n^3}(1 + 4 + 9 + \ldots + n^2)$, by two methods.

(2) $\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots + \frac{1}{8n}$.

(3) $\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{3n}} + \ldots + \frac{1}{n}$.

(4) $\frac{1}{n}\left\{\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \sin\frac{(n-1)\pi}{2n}\right\}$, by two methods.

(5) $\frac{n+2}{n^2+1} + \frac{n+4}{n^2+4} + \frac{n+6}{n^2+9} + \ldots$ to n terms.

(6) $\frac{1}{n} + \frac{1}{\sqrt{(n+1)(n-1)}} + \frac{1}{\sqrt{(n+2)(n-2)}} + \ldots + \frac{1}{\sqrt{(2n-1).1}}$.

(7) $\frac{1+n}{3n^2+1} + \frac{2+n}{3n^2+2} + \frac{3+n}{3n^2+3} + \ldots$ to n terms.

(8) $\frac{n}{(n+1)\sqrt{2n+1}} + \frac{n}{(n+2)\sqrt{2(2n+2)}} + \frac{n}{(n+3)\sqrt{3(2n+3)}} + \ldots$

$+ \frac{n}{2n\sqrt{n(3n)}}$.

(9) $\dfrac{1}{n}+\dfrac{\sqrt{n}}{n\sqrt{n}+1}+\dfrac{\sqrt{n}}{n\sqrt{n}+2\sqrt{2}}+\dfrac{\sqrt{n}}{n\sqrt{n}+3\sqrt{3}}+\dots+\dfrac{1}{9n}$.

(10) $\dfrac{\sqrt{n}}{n^2+1^2}+\dfrac{\sqrt{2n}}{n^2+2^2}+\dfrac{\sqrt{3n}}{n^2+3^2}+\dots+\dfrac{\sqrt{(n-1)n}}{n^2+(n-1)^2}+\dfrac{1}{2n}$.

2. Evaluate the following definite integrals:—

(1) $\displaystyle\int_0^{\frac{\pi}{2}} e^{-x}\cos^2 x\,dx$.

(2) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x}$.

(3) $\displaystyle\int_0^1\frac{x^3\,dx}{(x+1)(4-x^2)}$.

(4) $\displaystyle\int_0^1 \tan^{-1}\sqrt{x}\,dx$.

(5) $\displaystyle\int_0^1\frac{\sqrt{1-x}}{3+x}\,dx$.

(6) $\displaystyle\int_0^1 (x\log x)^n\,dx$.

(7) $\displaystyle\int_1^\infty\left(\frac{\log x}{x}\right)^n dx$.

(8) $\displaystyle\int_0^\pi (1+\cos x)^n\,dx$.

(9) $\displaystyle\int_1^2\frac{x^3+2}{(x+1)(x^2+2)}\,dx$

(10) $\displaystyle\int_1^{\sqrt{3}}\frac{x^2(x+1)\,dx}{(x^2+1)^{\frac{3}{2}}}$.

(11) $\displaystyle\int_1^\infty\frac{\cot^{-1}x\,dx}{x^2(1+x^2)}$.

(12) $\displaystyle\int_{\pi/4}^{\pi/2}\sqrt{\cot\theta}\,d\theta$.

3. Prove that $\displaystyle\int_a^b\phi(x)\,dx=\int_a^b\phi(a+b-x)\,dx$.

Hence show, without integration, that

$$\int_{1-a}^{1+a}\frac{(1-x)^5\,dx}{\sqrt{2x-x^2}}=0\ (a<1),$$

and generally that

$$\int_{1-a}^{1+a}\frac{(1-x)^{2n+1}\,dx}{(2x-x^2)^m}=0,\ n \text{ being integral.}$$

4. Prove that $\displaystyle\int_a^b\phi(x)\psi'(x)dx=-\int_a^b\psi(x)\phi'(x)dx+\phi(b)\psi(b)-\phi(a)\psi(a)$.

5. Prove that $\displaystyle\int_{-1}^1\frac{x^2\sin^{-1}x\,dx}{\sqrt{1-x^2}}=0$, and that $\displaystyle\int_{-1}^1\frac{x\sin^{-1}x\,dx}{\sqrt{1-x^2}}=2$.

6. Prove by the method of Art. 420, that

(1) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta=\pi/4$.

(2) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+\tan\theta}=\pi/4$.

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\theta}}{\sqrt{\sin\theta}+\sqrt{\cos\theta}}\,d\theta = \pi/4.$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\,dx}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\log_e(\sqrt{2}+1).$

(5) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x-\sin x}{\cos x+\sin x}\,dx = 0.$

7. Show that $\displaystyle\int_a^b \phi(x)dx = \frac{a-b}{\alpha-\beta}\int_\alpha^\beta \phi\left(\frac{a-b}{\alpha-\beta}x+\frac{ab-\beta a}{\alpha-\beta}\right)dx.$

8. If m and n be $+^{ve}$ integers, and $n > m$, prove that

$$\int_0^1 x^m \frac{d^n}{dx^n}\{x^n(1-x)^n\} = 0.$$

9. If $f(x) = f(a+x)$, prove that

$\displaystyle\int_{ma}^{na} f(x)dx = (n-m)\int_0^a f(x)dx$, and give a geometrical illustration.

Hence find $\displaystyle\int_{3\pi}^{5\pi} \sin^6 x\,dx.$

10. Prove that $\displaystyle\int_0^\pi x\log\sin x\,dx = -\frac{\pi^2}{2}\log_e 2$, by using the result of Art. 422.

11. Show that the infinite series

$$1-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{17}+\dots$$

can be expressed in the form $\displaystyle\int_0^1 \frac{1-x^4}{1-x^6}\,dx$, and hence deduce its value.

Answers.

1. (1) $\frac{1}{3}$. (2) $\frac{3}{8}$. (3) 2. (4) $\frac{2}{\pi}$. (5) $\frac{\pi}{4}+\log_e 2$. (6) $\frac{\pi}{2}$. (7) $\frac{1}{2}$.

(8) $\frac{\pi}{3}$. (9) $\frac{\pi}{\sqrt{3}}-\frac{1}{3}\log_e 3$. (10) $\frac{\pi}{2\sqrt{2}}+\frac{1}{\sqrt{2}}\log_e(\sqrt{2}-1)$.

2. (1) $\frac{1}{5}(3-2e^{-\frac{1}{2}\pi})$. (2) $\frac{\pi}{3\sqrt{3}}$. (3) $2\log_e 3-\frac{5}{3}\log_e 2-1$. (4) $\frac{\pi}{2}-1$.

(5) $2(\log_e 3-1)$. (6) $(-1)^n\frac{n!}{(n+1)^{n+1}}$. (7) $\frac{n!}{(n-1)^{n+1}}$.

(8) $\dfrac{1.3.5\ldots(2n-1)}{n!}\pi$. (9) $1+\frac{1}{3}\log_e 3-\log_e 2-\dfrac{\sqrt{2}}{3}\tan^{-1}\dfrac{1}{2\sqrt{2}}$.

(10) $\frac{1}{2}(5-\sqrt{3}-2\sqrt{2})+\log_e\dfrac{2+\sqrt{3}}{1+\sqrt{2}}$.

(11) $\dfrac{\pi}{4}\left(1-\dfrac{\pi}{8}\right)-\frac{1}{2}\log_e 2$. (12) $\dfrac{\pi}{2\sqrt{2}}+\dfrac{1}{\sqrt{2}}\log_e(\sqrt{2}-1)$.

9. $4.\dfrac{5.3.1}{6.4.2}.\dfrac{\pi}{2}$.

11. $\dfrac{\pi}{2\sqrt{3}}$.

CHAPTER XXVII.

AREAS AND LENGTHS OF PLANE CURVES.

423. Areas in Cartesian Coordinates.

We have already shown, in Art. 328, that the area bounded by the curve $y = f(x)$, Ox, and the two lines $x = a$, $x = b$, is given by $\int_a^b y\,dx$, or $\int_a^b f(x)dx$.

We now add further examples.

424. Area of Circle.

Let the equation be $x^2 + y^2 = a^2$, or $y = \sqrt{a^2 - x^2}$

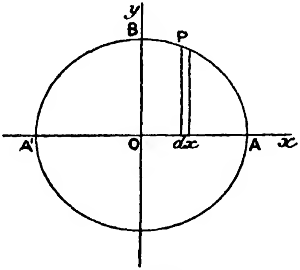

FIG. 98.

Then since the upper and lower limits of x are respectively a and $-a$, the area of ABA'

$$= \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$$

$$= \frac{a^2}{2}[\sin^{-1}(1) - \sin^{-1}(-1)] = \frac{a^2}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{a^2}{2}\pi.$$

The area of the whole circle is therefore πa^2.

Or, putting $x = a\sin\theta$,

$$\therefore \int_{-a}^{a}\sqrt{a^2-x^2}\,dx = a^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \frac{a^2}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{a^2}{2}\pi \text{ as before.}$$

[*The parabola* has been considered in Art. 328.]

425. Area of Ellipse.

Since $y = \frac{b}{a}\sqrt{a^2-x^2}$, the whole area is evidently

$$2\int_{-a}^{a} y\,dx = 2\,.\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = 2\frac{b}{a}\cdot\frac{a^2}{2}\pi = \pi ab.$$

426. Area of Cycloid.

The equations referred to a cusp as origin, and the base as axis of x are (Art. 308)—

$$x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta).$$

Now, $y = 0$ when $\cos\theta = 1$, *i.e.* when $\theta = 0$, or 2π.

Hence, we may express $y\,dx$ in terms of θ, and integrate between the limits 0 and 2π.

$$\therefore \text{ area from cusp to cusp} = a^2\int_0^{2\pi}(1-\cos\theta)(1-\cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi}\{1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\}\,d\theta = a^2\big[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\big]_0^{2\pi}$$

$= 3a^2\pi.$†

† It is useful to remember that

$$\int_0^{\pi}\cos\theta\,d\theta = 0, \quad \int_0^{\pi}\cos 2\theta\,d\theta = 0, \ldots \int_0^{\pi}\cos n\theta\,d\theta = 0, \ n \text{ being integral.}$$

Also that $\int_0^{\frac{\pi}{2}}\cos 2\theta\,d\theta = 0$, $\int_0^{\frac{\pi}{2}}\cos 4\theta\,d\theta = 0$, ... $\int_0^{\frac{\pi}{2}}\cos 2n\theta\,d\theta = 0$;

and, more generally, $\int_0^{m\pi}\cos n\theta\,d\theta = 0$, if mn is integral.

427. Oblique Coordinates.

When the axes of coordinates are oblique, the area of an element $PMNQ$ is $y\,dx \sin\omega$, where $\omega = \angle yOx$. Hence the area between the curve, Ox, and two ordinates parallel to Oy and given by $x = a$, $x = b$, is

$$\sin\omega \int_a^b y\,dx.$$

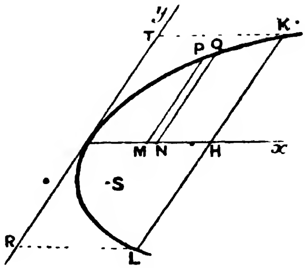

FIG. 99.

Ex. In the parabola $y^2 = 4a'x$ (where $a' = SO$) the area of the segment KOL, in which $OH = h$, $KH = k$, is

$$2\sin\omega \int_0^h y\,dx = 4\sqrt{a'}\sin\omega \int_0^h x^{\frac{1}{2}}\,dx = \tfrac{8}{3}\sqrt{a'}\,.\,h^{\frac{3}{2}}\sin\omega$$
$$= \tfrac{4}{3}\,hk\sin\omega$$
$$(\because\ k = 2\sqrt{a'h})$$
$$= \tfrac{2}{3}\ (\text{parallelogram } TKLR).$$

428. Case in which the Curve crosses the Axis of x.

The element of area $y\,dx$ is $+^{ve}$ so long as y is $+^{ve}$; that is, at least, if we consider dx as always $+^{ve}$ (or the curve traced from left to right).

If the curve cross the axis of x, then every one of the elements $y\,dx$ will be $-^{ve}$ for that part of the curve which is below the axis.

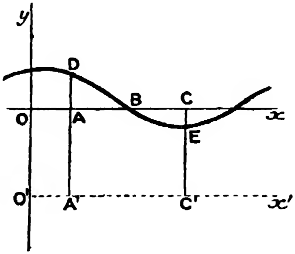

FIG. 100.

Hence, in the figure, if OA and OC be the limits of integration, the sum of all the elements between B and C will be $-^{ve}$, and therefore the *algebraical* sum of the elements between A and C will be the area ADB − area BEC; or $\int_{OA}^{OC} y\,dx = ADB - BEC$.

Hence, if we require the sum of the two areas, we must integrate each part separately and add *arithmetically*.

This does not apply to the case in which the curve crosses Oy, since neither y nor dx thereby become $-^{ve}$.

429. Geometrical Proof.

Otherwise, let X denote the area between DBE, Ox, and AD, CE. Then, if Ox move downwards parallel to itself into the position $O'x'$, the new value of X (say X') will be $DA'C'E$.

Now let $O'x'$ move back again to Ox, and X' will be diminished by the rectangle $A'C'CA$ (the base $A'C'$ having swept out, or cut off, this area).

$$\therefore X = X' - A'C'CA = (A'C'EBA + ABD) - (A'C'EBA + EBC)$$
$$= ABD - EBC \text{ as before.}$$

430. Application to Odd Functions.

We have seen that $\int_{-a}^{a} \phi(x)dx = 0$, if $\phi(x)$ be an *odd* function.

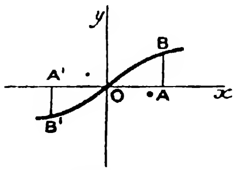

FIG. 101.

The graph of an odd function is such that on turning the page upside down the figure is unaltered in shape. It will also pass through the origin; † and if $A'O = OA = a$, the areas $A'OB'$, OAB are equal and opposite, their algebraical sum being zero.

Ex. $y = \frac{x}{a}\sqrt{a^2 - x^2}$.

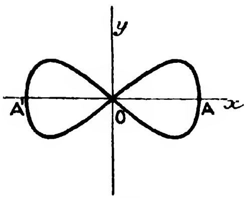

FIG. 102.

The curve is as in the second figure, and is doubly symmetrical, since the equation may be written $x^4 = a^2(x^2 - y^2)$, in which the powers of x and y are both even. Since y is an odd function of x, the algebraical value of the whole area is zero.

The arithmetical value of the area, regarded as the sum of two loops, is four times that portion in the first quadrant, and therefore

$$= 4\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\,dx = \frac{4}{a}\left[-\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \tfrac{4}{3}a^2.$$

† Unless the equation breaks up into two distinct rational factors, *e.g.* $(y - mx)^2 = c^2$.

431. Area between two Branches—Area of Closed Curve.

First, let EA and FB cut off an area $ECDF$ bounded by either two branches of the same curve or by two different curves. In either case the area $ECDF = EABF - CABD$.

Next, let $GCDHFE$ be either a closed curve or an area inter-

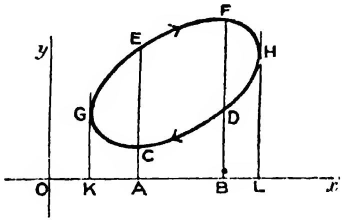

FIG. 103.

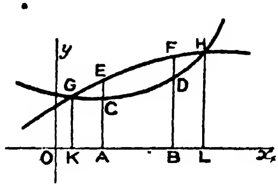

FIG. 104.

cepted between two different curves. Then, if GK and HL are the extreme ordinates,

$$\text{Area} = GKLHE - GKLHC.$$

In the case of the closed curve, if we suppose the curve traced out in the direction of the arrows, dx will be $+^{ve}$ for the higher, and $-^{ve}$ for the lower branch, so that the area may be regarded as the *algebraical sum* of the two areas mentioned.

Adopting this convention, *the expression* $\int y\,dx$ *integrated from point to point all round the curve gives the area*; or, if $OK = a$, $OL = b$, the area $= \int_a^b y_1\,dx + \int_b^a y_2\,dx$, y_1 being the greater value of y, and y_2 the smaller value.

The fact that dx is $-^{ve}$ in the second integral is shown by the limits, the superior (a) being *less* than the inferior (b).

For practical purposes, however, the form $\int_a^b y_1\,dx - \int_a^b y_2\,dx$ is used.

432. Examples.

Ex. 1. *Find the area of the oval* $y = x^2 \pm \sqrt{(x-1)(2-x)}$.

The curve is evidently an oval, for (1) x must lie between 1 and 2 for real values of y; (2) y cannot become infinite for finite values of x; (3) $dy/dx = \infty$, when $x = 1$ or 2; (4) y has two distinct values between $x = 1$ and $x = 2$, at which points they coincide.

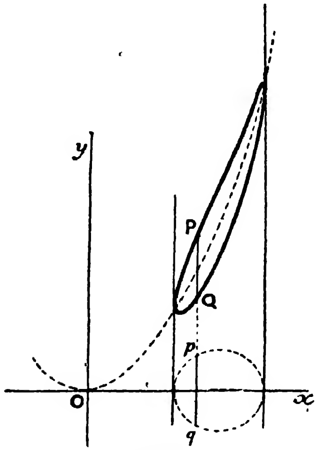

FIG. 105.

The curve may be traced by means of the auxiliary parabola $y = x^2$, and the circle $y = \sqrt{(x-1)(2-x)}$. [See Ex. 6, Art. 279.]

Again, since $y_1 = x^2 + \sqrt{(x-1)(2-x)}$, and $y_2 = x^2 - \sqrt{(x-1)(2-x)}$; the area

$$= \int_1^2 \{x^2 + \sqrt{(x-1)(2-x)}\}\,dx - \int_1^2 \{x^2 - \sqrt{(x-1)(2-x)}\}\,dx$$

$$= 2\int_1^2 \sqrt{(x-1)(2-x)}\,dx \ . \quad (a)$$

Put $x = \sin^2\theta + 2\cos^2\theta$ [Art. 348] $= 1 + \cos^2\theta$;

$\therefore\ dx = -2\cos\theta\sin\theta\,d\theta$, the limits of θ being $\pi/2$ and 0;

and $(x-1)(2-x) = \cos^2\theta\sin^2\theta$.

$$\therefore\ \text{area} = -4\int_{\frac{\pi}{2}}^{0} \cos^2\theta\sin^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \sin^2 2\theta\,d\theta$$

$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 4\theta)\,d\theta = \tfrac{1}{4}\pi. \quad \text{[Footnote, Art. 426.]}$$

This is also apparent since, if we draw the line $PQpq$ parallel to Oy, $PQ = pq$; hence every element of area of the oval is equal to the corresponding element of the circle. The whole area is therefore equal to that of the circle, which can easily be shown to be $\frac{1}{4}\pi$, its radius being $\frac{1}{2}$. The integral (a) points to the same thing, as will be seen if we find the area of the circle by means of its equation

$$y = \sqrt{(x-1)(2-x)}.$$

Ex. 2. *Find the area of the segment cut off from the parabola* $y^2 = 4ax$, *by the line* $2x - 3y + 4a = 0$.

To find where the line cuts the curve, we have

$$y^2 = 2a \cdot 2x = 2a(3y - 4a);$$

$$\therefore\ y^2 - 6ay + 8a^2 = 0, \text{ whence } y = 4a, \text{ or } 2a, \text{ and}$$

$$\therefore\ x = 4a, \text{ or } a \text{ accordingly.}$$

For the parabola, $\quad y_1\,dx = 2\sqrt{a}\sqrt{x}\,dx;$

and, for the line, $\quad y_2\,dx = \frac{2}{3}(x + 2a)\,dx.$

Hence, taking $4a$ and a for the limits of x,

$$\text{Area} = \int_a^{4a} [2\sqrt{a}\sqrt{x} - \tfrac{2}{3}(x + 2a)]dx = \left[\tfrac{4}{3}\sqrt{a}\,x^{\frac{3}{2}} - \tfrac{1}{3}x^2 - \tfrac{4}{3}ax\right]_a^{4a}$$

$$= (\tfrac{32}{3} - \tfrac{16}{3} - \tfrac{16}{3})a^2 - (\tfrac{4}{3} - \tfrac{1}{3} - \tfrac{4}{3})a^2 = \tfrac{1}{3}a^2.$$

NOTE.—The area of the trapezium cut off by the line $2x - 3y + 4a = 0$, Ox, and the lines $x = 4a$, $x = a$, might have been found without integration, and the result taken from $\int_a^{4a} 2\sqrt{a}\sqrt{x}\,dx$.

433. Closed Curve containing a Node.

According to the above rule, the area of the adjoining figure

$= CDEFGBA - CKEHGBA$

$= CDEK - EHGF$,

taking away the common part $CKEFGBA$.

Hence $\int y\,dx$, *integrated all round the curve, is the arithmetical difference of the areas on either side of the node.*

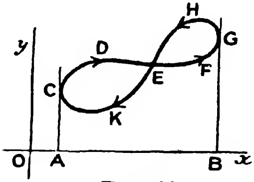

FIG. 106.

434. Sectorial Areas.

Let $P(x, y)$ and $Q(x + dx, y + dy)$ be two infinitely near points on a curve.

Draw the ordinates and other lines as in the figure.

Then the area of the elementary sector

$$OPQ = \triangle OQL - \triangle OPM - \text{fig. } PQLM$$

$$= \tfrac{1}{2}(y + dy)(x + dx) - \tfrac{1}{2}yx - (y\,dx + \text{infinitesimals of the 2nd and higher orders})$$

$$= \tfrac{1}{2}(x\,dy - y\,dx) \text{ ultimately.}$$

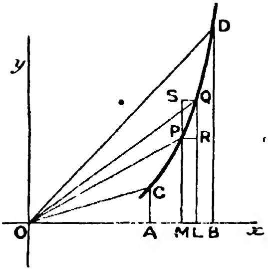

FIG. 107.

Hence the area of the sector swept out by OP as P moves from C to D

$$= \tfrac{1}{2}\int_a^b (x\,dy \sim y\,dx), \text{ if } OB = b,\ OA = a.$$

Otherwise:—Regarding PQ as an infinitesimal arc, we make an infinitesimal error in taking it as the diagonal of the rectangle RS.

Taking moments, about O, of forces represented by $\overline{PR}$, $\overline{PS}$, and their resultant $\overline{PQ}$, we have

moment of $\overline{PQ}$ = moment of $\overline{PS}$ − moment of $\overline{PR}$,

or $\triangle OPQ = \triangle OPS - \triangle OPR = \tfrac{1}{2}(OM.PS - PM.PR) = \tfrac{1}{2}(x\,dy - y\,dx)$.

NOTE.—We evidently have, for the $\triangle OPQ$, $\tfrac{1}{2}(x\,dy - y\,dx)$ or $\tfrac{1}{2}(y\,dx - x\,dy)$, according as the moment of $\overline{PQ}$ about O is +^ve^ or −^ve^.

Ex. 1. *Find the area of the sector of the curve* $ay = a\sqrt{ax + a^2} + x^2$, *bounded by radii drawn to the points for which* $x = -a$, *and* $x = a$.

We have $y = \sqrt{a}\sqrt{x + a} + x^2/a$; $\therefore \dfrac{dy}{dx} = \dfrac{\sqrt{a}}{2\sqrt{x + a}} + \dfrac{2x}{a}$;

$$\therefore\ y\,dx - x\,dy = \left\{\sqrt{a}\sqrt{x + a} + x^2/a - \frac{\sqrt{a}\,x}{2\sqrt{x + a}} - \frac{2x^2}{a}\right\}dx$$

$$= \left\{\frac{\sqrt{a}(x + 2a)}{2\sqrt{x + a}} - \frac{x^2}{a}\right\}dx$$

$$= \left\{\frac{\sqrt{a}(x + a) + \sqrt{a}\cdot a}{2\sqrt{x + a}} - \frac{x^2}{a}\right\}dx = \left(\frac{\sqrt{a}}{2}\sqrt{x + a} + \frac{a\sqrt{a}}{2\sqrt{x + a}} - \frac{x^2}{a}\right)dx.$$

$$\therefore\ \tfrac{1}{2}\int(y\,dx - x\,dy) = \tfrac{1}{2}\int_{-a}^{a}\left(\frac{\sqrt{a}}{2}\sqrt{x + a} + \frac{a\sqrt{a}}{2\sqrt{x + a}} - \frac{x^2}{a}\right)dx$$

$$= \left[\frac{\sqrt{a}}{6}(x + a)^{\frac{3}{2}} + \frac{a\sqrt{a}}{2}\sqrt{x + a} - \frac{1}{6}\frac{x^3}{a}\right]_{-a}^{a}$$

$$= \left(\frac{2\sqrt{2}}{6} + \frac{\sqrt{2}}{2} - \frac{2}{6}\right)a^2 = \frac{5\sqrt{2} - 2}{6}\,a^2.$$

The curve may be traced by means of the two auxiliary parabolas $y = \sqrt{ax} + a^2$ and $ay = x^2$.

We have taken the branch corresponding to the $+^{ve}$ value of the root, $\sqrt{ax + a^2}$; if we take that corresponding to the $-^{ve}$ value, we shall have $\frac{-5\sqrt{2} - 2}{6}a^2$; or $\frac{5\sqrt{2} + 2}{6}a^2$, numerically. See preceding note.

Ex. 2. In the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; [Fig. 94, Art. 308]

$$y\,dx - x\,dy = a^2\{(1 - \cos\theta)^2 - (\theta - \sin\theta)\sin\theta\}d\theta$$
$$= a^2(2 - 2\cos\theta - \theta\sin\theta)d\theta.$$

$\therefore$ area of a sector is given by

$$\tfrac{1}{2}a^2\int(2 - 2\cos\theta - \theta\sin\theta)d\theta = a^2[\theta - \sin\theta + \tfrac{1}{2}(\theta\cos\theta - \sin\theta)] + C$$
$$= \tfrac{1}{2}a^2(2\theta - 3\sin\theta + \theta\cos\theta) + C.$$

The area of the curve between two cusps

$$= \tfrac{1}{2}a^2[2\theta - 3\sin\theta + \theta\cos\theta]_0^{2\pi} = 3a^2\pi, \text{ as before.}$$

435. Polar Coordinates.

In the last article we have used cartesian coordinates for finding sectorial areas: they are, however, usually found by means of polars.

We shall show that the area between two consecutive radii and the curve $r = f(\theta)$ is $\frac{1}{2}r^2 d\theta$.

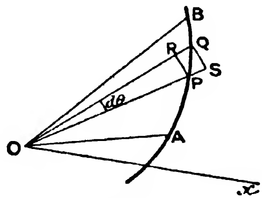

Fig. 108.

Let $P(r, \theta)$ and $Q(r + dr, \theta + d\theta)$ be two infinitely near points on the curve. Join OP, OQ, and let PR, QS be arcs of circles having their centre at O.

Then area $OSQ > OPQ > OPR$; or, since the area of a sector of a *circle* is $\frac{1}{2}(\text{radius})^2 \times$ (circular measure of included angle),

$$\tfrac{1}{2}(r + dr)^2 d\theta > OPQ > \tfrac{1}{2}r^2 d\theta.$$

But $\frac{1}{2}(r + dr)^2 d\theta = \frac{1}{2}r^2 d\theta +$ higher orders of infinitesimals. Hence, sector $OPQ = \frac{1}{2}r^2 d\theta$ ultimately.

If $\angle AOx = \alpha$, $\angle BOx = \beta$, then area $AOB = \frac{1}{2}\int_\alpha^\beta r^2 d\theta.$

Since r^2 is always $+^{\text{ve}}$, the sectorial area will be $+^{\text{ve}}$ so long as θ is increasing.

NOTE.—In the figure of Art. 434, x and y both increase with θ, so that the moment of $\overline{PQ}$ about O is $+^{\text{ve}}$; therefore $r^2d\theta = xdy - ydx$, each being twice the sector OPQ.

436. Area of a Closed Curve.

If the curve be traced in the direction of the arrows (see Fig. 109), we must first integrate from α to β, using the greater value of r, and then back from β to α, using the smaller value.

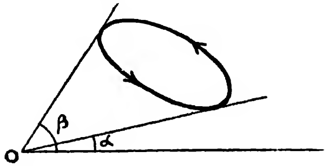

FIG. 109.

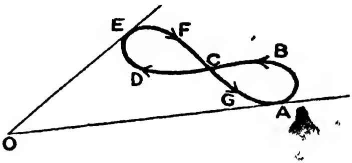

FIG. 110.

The algebraical sum, or the arithmetical difference, of the two results gives the area.

Or, $$\text{area} = \tfrac{1}{2}\int_\alpha^\beta r_1^2 d\theta + \tfrac{1}{2}\int_\beta^\alpha r_2^2\, d\theta,\ r_1 > r_2.$$

In the case of the figure eight given, we can show that the algebraical area

$$= CGAB - CDEF, \text{ as before.}$$

437. Examples.

Ex. 1. *Find the area of the cardioide* $r = a(1 - \cos\theta)$.

The curve can be traced completely and continuously as θ increases from 0 to 2π.

$$\text{Hence the whole area } = \tfrac{1}{2}a^2\int_0^{2\pi}(1 - \cos\theta)^2\, d\theta$$

$$= \tfrac{1}{2}a^2\int_0^{2\pi}\{1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\}\, d\theta$$

$$= \tfrac{3}{2}\pi a^2.$$

Ex. 2. Find the area of the lemniscate $r^2 = a^2 \cos 2\theta$.

The limits of θ are $-\frac{\pi}{4}$ and $\frac{\pi}{4}$.

The area of one loop is therefore $\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\, d\theta$

$$= \frac{a^2}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\, d\theta = \frac{a^2}{4}\Big[\sin 2\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a^2}{2}.$$

In tracing the curve, we observe that, for a given value of θ, r has equal and opposite values.

If we trace the curve in the direction of the arrows, starting from O, then for the first loop, r is $+^{ve}$, while θ varies from $-\frac{\pi}{4}$ to $+\frac{\pi}{4}$; for the second loop, r is $-^{ve}$, while θ passes back from $+\frac{\pi}{4}$ to $-\frac{\pi}{4}$.

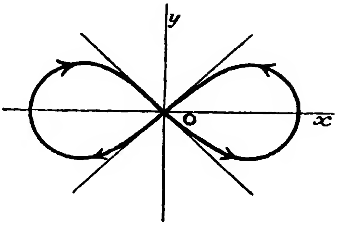

Fig. 111

The algebraical area is therefore

$$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\, d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} (-r)^2\, d\theta, \text{ which is evidently zero.}$$

438. Elliptic Sector.

To find the area of a sector of the ellipse, the pole being at its centre; we have, for the cartesian equation,

$$b^2x^2 + a^2y^2 = a^2b^2;$$

and turning to polars this becomes

$$r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2,$$

$\therefore$ area of sector AOP

$$= \frac{1}{2}\int_0^\theta r^2 d\theta = \frac{1}{2}a^2b^2\int_0^\theta \frac{d\theta}{b^2\cos^2\theta + a^2\sin^2\theta} = \frac{1}{2}a^2b^2\int_0^\theta \frac{\sec^2\theta\, d\theta}{b^2 + a^2\tan^2\theta}$$

$$= \frac{1}{2}ab\tan^{-1}\left(\frac{a}{b}\tan\theta\right).$$

But if ϕ be the eccentric angle of the point P, whose coordinates are (x, y), or (r, θ), then

$$\therefore \quad r\cos\theta = x = a\cos\phi; \quad r\sin\theta = y = b\sin\phi;$$

$$\therefore \tan\theta = \frac{b}{a}\tan\phi; \quad \therefore \tan^{-1}\left(\frac{a}{b}\tan\theta\right) = \phi.$$

Hence area of sector $AOP = \frac{1}{2}ab\phi$.

Otherwise, by geometry:—

Draw the auxiliary circle; then, comparing the sector OPA with the sector OpA, we have

$$\frac{RN}{rN} = \frac{PL}{pL} = \frac{b}{a}; \quad \text{also } \frac{QM}{qM} = \frac{b}{a}.$$

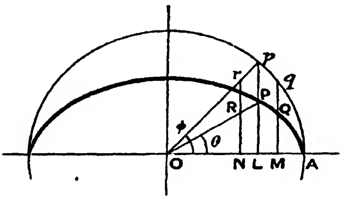

FIG. 112.

Hence every ordinate of the sector of the ellipse is b/a times the corresponding ordinate of the sector of the circle; hence, also, every element of area $y\,dx$ of the sector of the ellipse will be b/a times the corresponding element of the circular sector.

$$\therefore \text{area } AOP = \frac{b}{a} \times (\text{area } AOp) = \frac{b}{a}\cdot\frac{1}{2}a^2\phi = \tfrac{1}{2}ab\phi, \text{ as before.}$$

Otherwise, since $x = a\cos\phi$, $y = b\sin\phi$.

$$\therefore x\,dy - y\,dx = ab(\cos^2\phi + \sin^2\phi)d\phi = ab\,d\phi.$$

$$\therefore \tfrac{1}{2}\int_0^\phi (x\,dy - y\,dx) = \tfrac{1}{2}ab\phi, \text{ as before.}$$

Cor.—If $\phi = 2\pi$, the sector becomes the whole ellipse, whose area is thus πab.

439. Hyperbolic Sector.

In the last article change b^2 into $-b^2$; then, for the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, we shall have

Area of sector AOP

$$= \tfrac{1}{2}\int_0^\theta r^2\,d\theta = \tfrac{1}{2}a^2b^2\int_0^\theta \frac{\sec^2\theta\,d\theta}{a^2b^2 - a^2\tan^2\theta} = \tfrac{1}{2}ab\tanh^{-1}\left(\frac{a}{b}\tan\theta\right).$$

But, using the hyperbolic functions [see Art. 68],

$$r\cos\theta = x = a\cosh u;\quad r\sin\theta = y = b\sinh u;$$

$$\therefore \tan\theta = \frac{b}{a}\tanh u;\quad \text{or } \tanh^{-1}\left(\frac{a}{b}\tan\theta\right) = u.$$

Hence, area of sector $= \frac{1}{2}abu$.

Otherwise, since $x = a\cosh u$, $y = b\sinh u$,

$$\therefore x\,dy - y\,dx = ab(\cosh^2 u - \sinh^2 u)du = ab\,du:$$

$$\therefore \text{area} = \tfrac{1}{2}\int_0^u (x\,dy - y\,dx) = \tfrac{1}{2}abu.$$

Or, again, putting $x = a\sec\phi$, $y = b\tan\phi$, so that $\phi = \operatorname{gd} u$, we have

$$x\,dy - y\,dx = ab(\sec^3\phi - \sec\phi\tan^2\phi)d\phi = ab\sec\phi\,d\phi.$$

$$\therefore \text{area} = \tfrac{1}{2}\int_0^\phi (x\,dy - y\,dx) = \tfrac{1}{2}ab\int_0^\phi \sec\phi\,d\phi = \tfrac{1}{2}ab\log(\sec\phi + \tan\phi);$$

$$i.e. = \tfrac{1}{2}ab\operatorname{gd}^{-1}\phi = \tfrac{1}{2}abu, \text{ as before.}$$

Cor.—If $b = a$, the curve becomes a rectangular hyperbola. Hence the area of a sector of a rectangular hyperbola bounded by Ox and a radius vector to the point $(a\cosh u, a\sinh u)$ is $\frac{1}{2}a^2u$.

EXAMPLES LXVIII.

1. Find the area bounded by the line $y = 2x + 3a$, the ordinates for which $x = a$, and $x = 2a$, and Ox. Verify by geometry.

2. Find the area bounded by $xy = c^2$, $x = c$, $x = 2c$, and Ox.

3. Find the area of one of the portions of the sine-curve $y = a\sin x/a$, cut off by Ox.

4. Find the area of the parabolic segment cut off from the curve $ay = (x - a)(2a - x)$ by the axis of x.

5. Show that the two portions of the curve $a^2y = x(x - a)(x - 2a)$ cut off by Ox have the same area, namely $\frac{1}{4}a^2$.

6. Find the area of the segment of the circle $x^2 + y^2 = a^2$ cut off by the line $x = a/2$.

7. Find the area bounded by the curve $x^2/a^2 - y^2/b^2 = 1$, $x = 2a$, $x = a$, and Ox.

8. Find the area of the segment of the parabola $y^2 + ax = a^2$, cut off by Oy.

9. Find the area of the segment of the curve $x(a^2 + y^2) = a(a^2 - y^2)$ cut off by Oy.

10. Find the area between the witch, $xy^2 = a^2(a - x)$ and Oy its asymptote.

11. Find the area cut off from the curve $27ay^2 = 4(x - 2a)^3$ by the line $x = 5a$.

12. Find the area of the triangle formed by the line $r\cos\theta = a$, and two radii, for which $\theta = \alpha$ and β respectively. Verify by geometry.

13. Find the area of the triangle formed by the lines $p = r\cos(\theta - \alpha)$, $\theta = 0$, and $\theta = 2\alpha$.

14. Find the area of the segment of the circle $r = 2a\sin\theta$, cut off by the line $\theta = \frac{1}{4}\pi$.

15. Find the area of the portion of the circle $r = 2a\cos\theta$, cut off by the lines $\theta = 0$ and $\theta = \alpha$.

16. Find the area of the smallest loop of the curve $r = a\theta$.

17. Find the areas of the portions of the cardioide $r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{1}{2}\theta$ cut off by Oy.

18. Find the area bounded by $r = a\sec^2\theta$, $\theta = \frac{1}{4}\pi$, and $\theta = -\frac{1}{4}\pi$.

19. Find the area of the portion of the curve $r^2(2 + \cos\theta) = a^2$ which is in the first quadrant.

20. Find the area of the sector of the parabola $y^2 = 4ax$ bounded by radii drawn to the points (h, k), (h', k'). Hence find the area of the segment cut off by the line $y/x = k/h$.

21. Find the area of the sector of the catenary $y = c\cosh x/c$ bounded by radii drawn to the points $(0, c)$ and $(c\log_e 2, 5c/4)$.

22. Find the area of the sector of the ellipse $4x^2 + y^2 = 1$ bounded by radii drawn to points in the first quadrant for which $x = 0$ and $x = \frac{1}{4}$.

23. Show that the area of a sector of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is given by $\frac{1}{2}ab\log\left(\frac{x}{a} + \frac{y}{b}\right)$ + a constant.

24. Find the area of the sector of the curve $(a^2 + x^2)y = a^3$, bounded by radii drawn to the points $(0, a)$, $(a, \frac{1}{2}a)$.

25. Find the area included between the curve $y^2 = 4ax$, and $y = 2x$.

26. Find the area of the segment of the circle $x^2 + y^2 = 25a^2$ cut off by $x + y = 7a$.

27. Find the area included between $4(x^2 + y^2) = 9a^2$, and $y^2 = 4ax$.

28. Find the area of one of the portions between $x^2 + 4y^2 = 68$, and $xy = 8$.

29. Find the area of the part of the curve $b^2x^2 + a^2y^2 = a^2b^2$ which is within the parabola $b^2x^2 = (a^2 - b^2)ay$.

30. Find the areas of the three parts into which the circle $x^2 + y^2 = 3a^2$ is divided by the rectangular hyperbola $x^2 - y^2 = a^2$.

31. Find the area between the curve $y = \operatorname{sech} x$ and Ox.

32. Find the area included between the cissoid $y^2(2a - x) = x^3$, and its asymptote.

33. Find the area of the loop of the curve $xy^2 = (x - a)^2(2a - x)$.

34. Find the same for the curve $9y^2 = (x + 7)(x + 4)^2$.

35. Find the same for the curve $a^2y^2 = x^2(a - x)(2a - x)$.

36. Find the area of the whole curve $x^4 - 3ax^3 + 2a^2x^2 + a^2y^2 = 0$.

37. Find the area between the tractory $x = a \log \cot \frac{1}{2}\theta - a \cos \theta$, $y = a \sin \theta$; and the axis of x.

38. Find the area of the four-cusped hypocycloid $(x/a)^{\frac{2}{3}} + (y/a)^{\frac{2}{3}} = 1$, by putting $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

39. Find the area of the whole curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$.
Hence deduce that the area of the evolute of the ellipse, the equation being $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$, is $\frac{3\pi}{8} \cdot \frac{(a^2 - b^2)^2}{ab}$.

40. Find the area of the whole curve $(x/a)^{\frac{2}{5}} + (y/a)^{\frac{2}{5}} = 1$.

41. Find the area of the whole curve $(x/a)^{2/(2n+1)} + (y/a)^{2/(2n+1)} = 1$.

42. The parabola $r(1 - \cos \theta) = 2a$ is cut at P and Q on the same side of the axis by SP, SQ, making angles of 60° and 120° with the axis. Find the area of the sector cut off.

43. Find the area of a loop of the curve $r = a \sin 3\theta$.
Show that if the curves $r = \pm a \sin 3\theta$ be inscribed in a circle, their total combined area is half that of the circle.

44. Find the area of a small loop of the curve $r = a \sin \frac{1}{2}\theta$; also the area occupied by the whole curve.

45. Find the area of the sector of the rectangular hyperbola $r^2 \sin 2\theta = 2a^2$,

bounded by the radii drawn to the points of intersection with the line $a\sqrt{6} = r(\cos\theta + \sin\theta)$.

46. Find the area of a loop of the curve $r = a\cos 5\theta + b\sin 5\theta$.

47. Find the area cut off from the loop of the curve $y^2(a^2 + x^2) = 3x^2(a^2 - x^2)$, by the line $y = x$.

48. Find the area between the curve $\dfrac{y^2}{c^2} = \dfrac{2cx}{2y^2 + 3c^2}$ and the line $11y - 8x + 9c = 0$.

49. Find the ratio of the two parts into which the parabola $2a = r(1 + \cos\theta)$ divides the cardioide $r = 2a(1 + \cos\theta)$.

50. Find the point of intersection of the curves

$$y/c = \tfrac{3}{4}e^{x/c}, \text{ and } y/c = \cosh x/c,$$

and prove that the area bounded by these curves and the axis of y is

$$\left(\frac{\sqrt{2} - 1}{2}\right)^2 c^2.$$

51. A weightless string of length l, attached to a fixed point O, passes through a small ring, and the lower portion hangs vertically, carrying a small weight P. Find the locus of P, when the ring takes different positions in a horizontal line AB which is vertically beneath O; and show that the area included between the curve and AB is

$$l\sqrt{l^2 - h^2} - h^2\log\frac{l + \sqrt{l^2 - h^2}}{h},$$

where h is the vertical height of O above AB.

52. If the equiangular spiral $r = ae^{\theta\cot\alpha}$ cut Ox in points given generally by $r = a_n$, when $\theta = 2n\pi$; show that

(1) The area swept out by r in one revolution, from $r = a_n$ to $r = a_{n+1}$,
$= \frac{1}{2}(a_{n+1}^2 - a_n^2)\tan\alpha$.

(2) The *whole* area actually swept out by r, from $r = 0$ to $r = a_n$,
$= \frac{1}{2}a_n^2\tan\alpha$.

(3) The area enclosed in a single convolution, from $r = a_n$ to $r = a_{n+1}$,
$= \frac{1}{2}(a_{n+1}^2 - 2a_n^2 + a_{n-1}^2)\tan\alpha$.

53. Prove that the area of a closed curve is given by the integral

$$\frac{1}{2}\int_0^{2\pi}\left\{p^2 - \left(\frac{dp}{d\psi}\right)^2\right\}d\psi. \quad \text{[See Ans.]}$$

Apply it to the case of the circle given by $p = a(1 + \cos\psi)$.

ANSWERS.

1. $6a^2$. **2.** $c^2 \log_e 2$. **3.** $2a^2$. **4.** $\frac{1}{6}a^2$. **6.** $\frac{1}{2}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)a^2$.

7. $\{\sqrt{3} - \frac{1}{2}\log(2+\sqrt{3})\}ab$. **8.** $\frac{4}{3}a^2$. **9.** $(\pi - 2)a^2$. **10.** πa^2.

11. $\frac{24}{5}a^2$. **12.** $\frac{1}{2}a^2(\tan\beta - \tan\alpha)$. **13.** $p^2 \tan\alpha$. **14.** $\frac{1}{4}a^2(\pi - 2)$.

15. $a^2(\alpha + \sin\alpha\cos\alpha)$. **16.** $\frac{1}{24}\pi^3 a^2$. **17.** $\frac{a^2}{4}\left(\frac{3\pi}{4} \pm 2\right)$. **18.** $\frac{4}{3}a^2$.

19. $\frac{\pi a^2}{6\sqrt{3}}$. **20.** $\frac{1}{6}(h'k' - hk)$; $\frac{1}{6}hk$. **21.** $\frac{c^2}{8}(6 - 5\log 2)$. **22.** $\pi/24$.

24. $\frac{1}{4}a^2(\pi - 1)$. [Put $x = a\tan\theta$; $\therefore y = a\cos^2\theta$.] **25.** $\frac{1}{3}a^2$.

26. $(\frac{25}{2}\sin^{-1}\frac{3}{5} - \frac{7}{2})a^2$. **27.** $\left(\frac{1}{3\sqrt{2}} + \frac{9}{4}\cos^{-1}\frac{1}{3}\right)a^2$.

28. $17\sin^{-1}\frac{15}{17} - 16\log 2$. **29.** $\frac{1}{3}b^2c + ab\sin^{-1}e$.

30. Middle part $= 2[3\sin^{-1}\sqrt{\frac{2}{3}} + \log(\sqrt{2}+1)]a^2$. **31.** 2π. **32.** $3\pi a^2$.

33. $\frac{1}{2}(4 - \pi)a^2$. **34.** $\frac{8}{3}\sqrt{3}$. **35.** $[\frac{11}{12}\sqrt{2} - \frac{3}{4}\log_e(\sqrt{2}+1)]a^2$.

36. $\frac{3}{8}\pi a^2$. **37.** $\frac{1}{2}\pi a^2$. **38.** $\frac{3}{8}\pi a^2$. **39.** $\frac{3}{8}\pi ab$. **40.** $\frac{15}{128}\pi a^2$.

41. $\frac{1}{2^{2n}} \cdot \frac{1.3.5\ldots(2n+1)}{2.4.6\ldots2n} \cdot \pi a^2$. **42.** $\frac{44}{27}\sqrt{3}\,a^2$. **43.** $\frac{1}{12}\pi a^2$.

44. $\frac{1}{4}a^2(\pi - 2)$; $\frac{1}{2}a^2(\pi + 2)$. **45.** $a^2\log_e(2+\sqrt{3})$. **46.** $(a^2+b^2)\frac{1}{20}\pi$.

47. $\left\{\left(\frac{\pi}{6} - 1\right)\frac{\sqrt{3}}{2} + \frac{1}{2}\right\}a^2$, between $y = x$ and Oy.

48. $\frac{459}{320}c^2$. **49.** $9\pi - 16 : 9\pi + 16$.

53. See Fig. 47, Art. 220. Since $\sum_{\psi=0}^{\psi=2\pi}(\triangle OQZ - OPY) = 0$; and $\triangle OQZ - OPY = OPQ - OVY + VTZ = OPQ - \frac{1}{2}p^2 d\psi + \frac{1}{2}(dp/d\psi)^2 d\psi$; $\therefore$ etc.

440. Rectification of Curves.

Def.—We are said to *rectify* a curve when we find the length of a part, or the whole, of its arc.

441. Cartesians.—If ds be an element of the arc of a curve, then $ds^2 = dx^2 + dy^2$ (Art. 215), which may be written

$$= \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} dx^2, \text{ or } \left\{1 + \left(\frac{dx}{dy}\right)^2\right\} dy^2.$$

$$\therefore s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \ldots (1), \text{ or } \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \ldots (2).$$

Or, if $x = f(t)$, $y = \phi(t)$, then

$$ds^2 = [\{f'(t)\}^2 + \{\phi'(t)\}^2]\, dt^2, \text{ or briefly, } (f'^2 + \phi'^2)\, dt^2.$$

$$\therefore s = \int \sqrt{f'^2 + \phi'^2}\, dt \ldots (3).$$

Ex. Find the length of the arc of the parabola $y^2 = 4ax$, cut off by the latus-rectum.

Here $2y\,dy = 4a\,dx$, or $y\,dy = 2a\,dx$.

In this case we shall use formula (2) as being preferable to (1).

Thus, $$s = \int_{-2a}^{2a} \sqrt{1 + \frac{y^2}{4a^2}}\, dy$$

$$= 2\int_0^{2a} \frac{\sqrt{4a^2 + y^2}}{2a} dy \text{ [Art. 421]}$$

$$= \frac{1}{a}\left[\frac{y\sqrt{4a^2 + y^2}}{2} + 2a^2 \log\{y + \sqrt{4a^2 + y^2}\}\right]_0^{2a}$$

$$= a\left[2\sqrt{2} + 2\log\frac{2 + 2\sqrt{2}}{2}\right] = 2\{\sqrt{2} + \log_e(\sqrt{2} + 1)\}a.$$

Otherwise, we may put $x = am^2$, $y = 2am$;

$$\therefore dx = 2am\,dm,\ dy = 2a\,dm.$$

$$\therefore dx^2 + dy^2 = 4a^2(1 + m^2)dm^2.$$

$$\therefore s = \int \sqrt{dx^2 + dy^2} = 2a\int_{-1}^{1} \sqrt{1 + m^2}\, dm$$

$$= 2a\left[\frac{m\sqrt{1 + m^2}}{2} + \tfrac{1}{2}\log\{m + \sqrt{1 + m^2}\}\right]_{-1}^{1}$$

$$= 2\{\sqrt{2} + \log_e(\sqrt{2} + 1)\}a, \text{ as before.}$$

442. Polars.—We have, from Art. 216, the formula

$$ds^2 = r^2 d\theta^2 + dr^2,$$

which may be written

$$= \left\{r^2\left(\frac{d\theta}{dr}\right)^2 + 1\right\} dr^2, \text{ or } = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\} d\theta^2.$$

$$\therefore s = \int \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}\, dr, \text{ or } = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Ex. Find the length of the arc of the parabola $\frac{2a}{r} = 1 + \cos\theta$, cut off by the latus rectum.

Since $r = \frac{2a}{1+c}$, where $c \equiv \cos\theta$,

$$\therefore \frac{dr}{d\theta} = \frac{2as}{(1+c)^2};$$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{4a^2}{(1+c)^4}\{(1+c)^2 + s^2\} = \frac{4a^2(2+2c)}{(1+c)^4}$$

$$= \frac{8a^2}{(1+c)^3} = \frac{a^2}{\cos^6 \frac{1}{2}\theta}.$$

$$\therefore s = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\cos^3\frac{1}{2}\theta}$$

$$= 2a\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos^3\frac{1}{2}\theta}, \text{ since } \frac{1}{\cos^3\frac{1}{2}\theta} \text{ is an } \textit{even} \text{ function of } \theta \text{ [Art. 421]},$$

$$= 4a\int_0^{\frac{\pi}{4}} \frac{d\phi}{\cos^3\phi}, \text{ if } \phi = \tfrac{1}{2}\theta,$$

$$= 4a\int_0^{\frac{\pi}{4}} \sec^3\phi\, d\phi = 4a\int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2\phi}\, \sec^2\phi\, d\phi$$

$$= 4a\int_0^1 \sqrt{1+x^2}\, dx, \text{ if } x = \tan\phi,$$

$$= 4a\left[\frac{x\sqrt{1+x^2}}{2} + \tfrac{1}{2}\log(x + \sqrt{1+x^2})\right]_0^1$$

$$= 2a[\sqrt{2} + \log_e(\sqrt{2}+1)], \text{ as in the preceding article}$$

443. The Catenary or Chainette — Intrinsic and Cartesian Equations.

This curve is the curve assumed by a heavy, uniform, flexible and inextensible string when held at two points and allowed to hang freely.

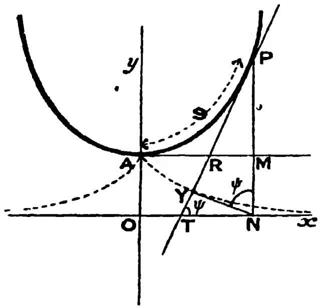

FIG. 113.

Its intrinsic equation [Art. 241], or the equation between s and ψ, can be readily found by the principles of statics.

Thus, consider the equilibrium of the portion AP of the string, A being the lowest point, or *vertex*, and Ay the *axis*; and let the arc $AP = s$.

The forces acting on AP are

(1) its weight, parallel to PM;
(2) the tension at P, along RP;
(3) the tension at A, along MR.

Hence PRM is the triangle of forces.

The tension at A is constant, $= T_0$ say; the weight of $AP \propto s = ks$, say; so that k is the weight per unit length of the string.

$$\therefore \tan\psi = \frac{PM}{RM} = \frac{ks}{T_0}; \quad \text{or } s = \frac{T_0}{k}\tan\psi,$$

or putting c for the constant T_0/k, the intrinsic equation is

$$s = c\tan\psi.$$

444. To express this in cartesians we have, taking OAy for the axis of y:—

$$\tan\psi = \frac{dy}{dx}, \quad \therefore s = c\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore s^2 + c^2 = c^2[(dy/dx)^2 + 1] = c^2(ds/dx)^2;$$

$$\therefore dx = \frac{c\,ds}{\sqrt{s^2+c^2}}.$$

$$\therefore x = c\int_0^s \frac{ds}{\sqrt{s^2+c^2}} = c\sinh^{-1}\frac{s}{c}, \text{ since } \sinh^{-1}0 = 0 \text{ [see Art. 64]},$$

$$= c\sinh^{-1}\frac{dy}{dx} \text{ from (1).}$$

$$\therefore \frac{dy}{dx} = \sinh\frac{x}{c}; \quad \therefore y = \left[c\cosh\frac{x}{c}\right]_0^x = c\cosh\frac{x}{c} - c, \text{ since } \cosh 0 = 1.$$

We have evidently taken the origin at A in the last line, since $\left[c\cosh\frac{x}{c}\right]_0^x$ is the difference of the ordinates for the points $P(x = x)$ and $A(x = 0)$.

If we choose for origin the point O, such that $OA = c$, the equation becomes

$$y = c\cosh\frac{x}{c} = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right).$$

Otherwise:—Since $s = c\tan\psi$,

$$\therefore\ \frac{ds}{dx} = c\sec^2\psi.\frac{d\psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But, *for any curve*,

$$\frac{ds}{dx} = \sec\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$\therefore$ from (2) and (3), $1 = c\sec\psi\frac{d\psi}{dx}$, or $dx = c\sec\psi\,d\psi$.

$$\therefore\ x = c\log(\sec\psi + \tan\psi) + C.$$

But when $\psi = 0$, $x = 0$; $\therefore\ 0 = 0 + C$.

$$\therefore\ x = c\log(\sec\psi + \tan\psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore\ \sec\psi + \tan\psi = e^{\frac{x}{c}}$$

and

$$\sec\psi - \tan\psi = \frac{1}{\sec\psi + \tan\psi} = e^{-\frac{x}{c}};$$

$\therefore$ subtracting, $\tan\psi$, or dy/dx, $= \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$.

Hence $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + C$, and C vanishes if we take $y = c$ when $x = 0$.

Otherwise:—From (4) above we have $x = c\,\mathrm{gd}^{-1}\psi$.

$$\therefore\ \tan\psi = \sinh\frac{x}{c};\ \text{i.e. } \frac{dy}{dx} = \sinh\frac{x}{c};\quad \therefore\ y = c\cosh\frac{x}{c} + C;\ \text{etc.}$$

445. Length of Arc—Properties of Curve.

The length of arc is readily found, thus

$$s = c\tan\psi = c\frac{dy}{dx} = c\sinh\frac{x}{c}.$$

We now give a few properties of the curve.

Since, as we have just shown,

$$\tan\psi = \sinh\frac{x}{c}, \quad \left(\text{or } \psi = \text{gd}\frac{x}{c}\right) \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore \sec\psi = \cosh\frac{x}{c}.$$

$$\therefore y = c\sec\psi, \text{ or } y\cos\psi = c.$$

Draw NY perpendicular to PT; then $\angle YNP = \psi$, whence

$$NY = PN\cos\psi = y\cos\psi = c = OA.$$

Again $PY = NY\tan\psi = c\tan\psi = s$; or $PY = \text{arc } AP$.

Now, suppose a string to be wrapped round the right half of the curve ending at A; then, if we take hold of the end at A, and unwrap the string gradually, keeping it tight, this end will describe a curve called the *involute* [see Ex. XXXVI., 13] of the catenary; the locus will evidently pass through Y, since $PY = \text{arc } PA$. This curve is called the *tractory*, or *tractrix*, and its equation can be shown to be given by

$$x = c\log(\sec\psi + \tan\psi) - c\sin\psi; \; y = c\cos\psi. \dagger$$

Evidently YN is the tangent to this curve, and is of constant length [See Ex. XXXIV., 18 (4)].

446. Since $T_0/k = c$, $\therefore T_0 = kc =$ the weight of a length OA of string.

Again, if T be the tension at P, then from the $\triangle PRM$,

$$\frac{T}{T_0} = \frac{PR}{RM} = \sec\psi = \frac{y}{c};$$

or $T = T_0\dfrac{y}{c} = ky =$ the weight of a length PN of the string.

The line Ox is called the directrix of the catenary.

EXAMPLES LXIX.

In the following examples, $\psi = \tan^{-1}(dy/dx)$ in every case.

1. Find the lengths of the following curves:—

(1) $y = \log\sec x$, between $x = 0$ and $x = \pi/3$.

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† For, let (α, β) be the coordinates of Y.

Then $\beta = YN\cos\psi = c\cos\psi$.

$$\alpha = ON - YN\sin\psi = x - c\sin\psi = c\,\text{gd}^{-1}\psi - c\sin\psi, \text{ by (5)},$$
$$= c\log(\sec\psi + \tan\psi) - c\sin\psi.$$

(2) The catenary, $y = \frac{c}{2}(e^{x/c} + e^{-x/c})$, from the vertex to any point.

(3) $4ax = y^2 - 2a^2 \log y/a - a^2$, from $(0, a)$ to any point.

(4) $y = \log(1 - x^2)$, from the origin.

(5) The cycloid, $x = a(\phi - \sin\phi)$; $y = a(1 - \cos\phi)$, from cusp to cusp.

(6) The tractory $x = a\sin\phi$; $y = a(\log\tan\frac{\phi}{2} + \cos\phi)$.

(7) $x = a\log\sec\psi$; $y = a(\tan\psi - \psi)$, from the origin.

2. Find the lengths of the following curves:—

(1) The circle, $r = 2a\cos\theta$.

(2) The cardioide, $r = a(1 - \cos\theta)$.

(3) The cardioide, $r = a(1 + \sin\theta)$.

(4) The spiral of Archimedes, $r = a\theta$, from the origin.

(5) The hyperbolic spiral, $r\theta = a$, between $\theta = \frac{5}{12}$ and $\theta = \frac{3}{4}$.

(6) The equiangular spiral $r = ae^{\theta\cot\alpha}$ between $\theta = 0$ and $\theta = 2\pi$.

3. Find, in the terms of ψ, the length of the parabola $x = a\cot^2\psi$; $y = 2a\cot\psi$, from the origin to any point. Also find the length cut off by the latus rectum.

4. Find the whole length of the loop of the curve $9ay^2 = x(x - 3a)^2$.

5. Find the length of the loop of the curve $9y^2 = (x + 7)(x + 4)^2$.

6. Find the length of the parabola $ay = x(x + a)$, from the origin to $x = a$.

7. Find the length of the arc of the semi-cubical parabola $y^3 = ax^2$, cut off by $x = a/8$.

8. Find the whole length of the four-cusped hypocycloid $x = -a\cos^3\psi$; $y = a\sin^3\psi$. Show that the *algebraical* value of the whole length is zero, and give an explanation.

9. Find the whole length of the curve $r^{\frac{1}{3}} = a^{\frac{1}{3}}\cos\frac{1}{3}\theta$.

10. The extremities LL' of the latus rectum of a parabola are joined by the arc of a catenary, whose axis is perpendicular to LL', and whose equation is $y = c\cosh\frac{x}{c}$. If $LL' = 2c$, find the difference in the lengths of the arcs of the two curves cut off by LL'.

11. If $s = a\tan\psi$, prove that $x = a\,\mathrm{gd}^{-1}\psi$; $y = a\sec\psi$.

12. If $s = 4a\sin\psi$, prove that $x = a(\phi + \sin\phi)$; $y = a(1 - \cos\phi)$; if $\phi = 2\psi$.

13. If $s = a\sin\frac{1}{3}\psi$, prove that $x = \frac{1}{3}a\sin\frac{\psi}{2}(3 - 2\sin^2\frac{\psi}{2})$; $y = -\frac{2}{3}a\cos^3\frac{\psi}{2}$.

14. If $s = a\log\tan\frac{1}{2}\psi$, prove that $x = a\log\sin\psi$; $y = a\psi$.

15. In the curve $x = a\cos\phi - \cos a\phi$; $y = a\sin\phi - \sin a\phi$, show that $\psi = \frac{a+1}{2}\phi$, and that the intrinsic equation is $s = \frac{4a}{a-1}\cos\frac{a-1}{a+1}\psi$.

16. In the epicycloid

$$x = (a+b)\cos\phi - b\cos\frac{a+b}{b}\phi; \quad y = (a+b)\sin\phi - b\sin\frac{a+b}{b}\phi$$

show that $\psi = \frac{a+2b}{2b}\phi$, and that the intrinsic equation is

$$s = \frac{4(a+b)b}{a}\cos\frac{a}{a+2b}\psi.$$

17. In the circle $r^2 + 2ar\cos\theta = 3a^2$, show that

$$\frac{ds}{d\theta} = \frac{\sqrt{3+c^2} - c}{\sqrt{3+c^2}}\,.\,a, \text{ where } c = \cos\theta.$$

Hence find the length of arc cut off (1) by the lines $\theta = 0$, and $\theta = \pi/4$; (2) by the axis of y.

18. Find the length of the curve $y^2/c^2 = 2cx/(2y^2 + 3c^2)$, cut off by the line $11y - 8x + 9c = 0$.

19. Find the whole length of the curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$.

20. Show that $dq = ds - p\,d\psi$ [see Art. 218]. Hence, show that the length of a closed curve is given by the integral $\int_0^{2\pi} p\,d\psi$.

Apply it to the case of the circle $p = a(1 + \cos\psi)$.

21. Show that the length of the arc of the hyperbola is given by the formula $s = \int(a^2\sin^2\phi + b^2)^{\frac{1}{2}}\,d\tan\phi$; and deduce that when a is small compared with b, the difference between the lengths of the curve and its asymptotes is approximately $\pi a\alpha$, where 2α is the acute angle between the asymptotes.

22. Show that the perimeter of an ellipse, of small eccentricity e, is approximately $2\pi a\left(1 - \frac{e^2}{4}\right)$. Show, also, that the perimeter exceeds that of a circle of the same area, by $e^4/16$ times either perimeter, nearly.

23. In the curve $p = r - a$, prove that $\dfrac{ds}{dr} = \dfrac{r}{a\sqrt{2r - a}}$, and that $s = \dfrac{1}{6a}(r + a)\sqrt{2r - a}$. [See Ex. XXXV., 5.]

24. Prove that the intrinsic equation of the parabola $\dfrac{2a}{r} = 1 + \cos\theta$, is $s = a(\log\tan\dfrac{\psi}{2} - \cot\psi\operatorname{cosec}\psi)$.

25. Prove that the intrinsic equation of the cardioide $r^{\frac{1}{3}} = a^{\frac{1}{3}}\sin\frac{1}{3}\theta$, is $s = 4a\sin^2\dfrac{\psi}{6}$.

ANSWERS.

1. (1) $\log(2 + \sqrt{3})$. (2) $\dfrac{c}{2}(e^{x/c} - e^{-x/c})$. (3) $\dfrac{y^2}{2a} - \dfrac{a}{2} - x$.

(4) $\log\dfrac{1 + x}{1 - x} - x$. (5) $8a$. (6) $a\log\sin\phi + C$. (7) $a(\sec\psi - 1)$.

2. (2) $8a$. (3) $8a$. (4) $a\int\sqrt{1 + \theta^2}\,d\theta$. (5) $\frac{11}{15} + \log\frac{4}{3}$.
(6) $ad\sec\alpha$, where d = the increase of r for the one revolution.

3. See Ex. 24. **4.** $4\sqrt{3}a$. **5.** $4\sqrt{3}$.

6. $\dfrac{a}{4}\{3\sqrt{10} - \sqrt{2} + \log\overline{(\sqrt{10} + 3)(\sqrt{2} - 1)}\}$. **7.** $\dfrac{61}{216}a$.

8. $6a$; $ds/d\psi$ changes sign at each cusp; hence, while ψ increases, s is diminishing for two of the branches.

9. $\dfrac{3\pi a}{2}$. **10.** $\{2\sinh 1 - \sinh^{-1}1 - \sqrt{2}\}c$. **17.** (1) $\left(\dfrac{\pi}{4} - \sin^{-1}\dfrac{1}{2\sqrt{2}}\right)l$.

(2) $\dfrac{2\pi a}{3}$. **18.** $(2\sqrt{2} - \frac{5}{16}\sqrt{5})c$. **19.** $\dfrac{4(a^2 + ab + b^2)}{a + b}$.

CHAPTER XXVIII.

VOLUMES AND SURFACES OF REVOLUTION—CENTROIDS—MOMENTS OF INERTIA.

447. Volumes.

Let CD be part of the curve $y = f(x)$, which revolves about the axis of x. Required the volume of the solid of revolution cut off by two planes through C and D, each perpendicular to Ox.

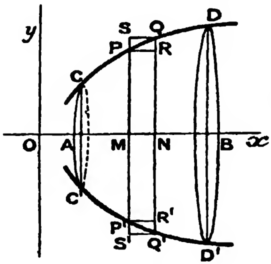

FIG. 114.

Let $OA = a$; $OB = b$; $OM = x$; $MN = dx$.

Then the volume of the element cut off by two planes through M and N, and parallel to the bounding planes, lies between that of the cylinders $PP'R'R$ and $SS'Q'Q$;

i.e. between $\pi y^2 dx$ and $\pi(y + dy)^2 dx$,

i.e. between $\pi y^2 dx$ and $\pi y^2 dx + \pi(2y + dy)\,dy\,dx$.

Hence the error in taking $\pi y^2 dx$ for the element of volume is less than $\pi(2y + dy)\,dy\,dx$, which is of the second order of infinitesimals, and therefore negligible compared with $\pi y^2 dx$.

Hence the required volume $= \int_a^b \pi y^2\,dx = \pi \int_a^b y^2\,dx$.

448. Volume of Sphere.—Regarding the sphere as generated by the revolution of the circle $x^2 + y^2 = a^2$ about the

axis of x, the volume of the portion contained between $x = x_1$, and $x = x_2$,

$$= \pi\int_{x_1}^{x_2} y^2\,dx = \pi\int_{x_1}^{x_2}(a^2 - x^2)\,dx = \pi\{a^2(x_2 - x_1) - \tfrac{1}{3}(x_2^3 - x_1^3)\}.$$

The volume of the whole sphere $= \pi\int_{-a}^{a}(a^2 - x^2)\,dx = \frac{4}{3}\pi a^3$, as may be seen by putting $x_2 = a$, and $x_1 = -a$, in the preceding result.

449. Volumes of Prolate Spheroid and Oblate Spheroid.

The *prolate spheroid* is generated by the revolution of an ellipse about its major axis. Its volume is evidently

$$\pi\int_{-a}^{a} y^2\,dx = \frac{\pi b^2}{a^2}\int_{-a}^{a}(a^2 - x^2)\,dx = \frac{b^2}{a^2} \times \frac{4}{3}\pi a^3 \quad \text{[Art. 448]}$$
$$= \tfrac{4}{3}\pi a b^2.$$

The *oblate spheroid* is generated by the revolution of an ellipse about its minor axis. Its volume is therefore

$$\pi\int_{-b}^{b} x^2\,dy = \pi\frac{a^2}{b^2}\int_{-b}^{b}(b^2 - y^2)\,dy = \pi\frac{a^2}{b^2}\Big[b^2y - \tfrac{1}{3}y^3\Big]_{-b}^{b} = \tfrac{4}{3}\pi a^2 b.$$

450. Paraboloid of Revolution.—This is the solid generated by the revolution of the parabola about its axis.

Taking $y^2 = 4ax$ for the equation to the parabola, the volume of a segment cut off by a plane through the point $(h, 0)$ perpendicular to Ox

$$= \pi\int_0^h y^2\,dx = 4\pi a\int_0^h x\,dx = 2\pi a h^2.$$

If k be the ordinate corresponding to $x = h$, then $k^2 = 4ah$, and the volume $= \frac{1}{2}\pi h k^2 =$ *half that of the circumscribed cylinder whose base and length are the same as those of the segment*, viz. πk^2 and h respectively.

451. Revolution of Cycloid about its Base.

The equations to the cycloid, referred to a cusp as origin, and base as axis of x, have been shown to be [Art. 308]

$$x = a(\theta - \sin\theta)\ ;\ y = a(1 - \cos\theta).$$

Hence the volume between two cusps

$$= \pi\int_0^{2\pi} y^2\,dx = \pi a^3\int_0^{2\pi}(1 - \cos\theta)^3\,d\theta$$

$$= \pi a^3\int_0^{2\pi}\left\{1 - 3\cos\theta + \tfrac{3}{2}(1 + \cos\theta) - \tfrac{1}{4}(\cos 3\theta + 3\cos\theta)\right\}d\theta$$

$$= \tfrac{5}{2}\pi a^3\int_0^{2\pi} d\theta + \text{terms which vanish [Art. 426, footnote]}$$

$$= 5\pi^2 a^3.$$

Otherwise :—From the symmetry of the cycloid, we may write

$$\text{Volume} = 2\pi\int_0^{\pi} y^2\,dx = 2\pi a^3\int_0^{\pi}(1 - \cos\theta)^3\,d\theta = 16\pi a^3\int_0^{\pi}\sin^6\tfrac{1}{2}\theta\,d\theta.$$

$$= 32\pi a^3\int_0^{\frac{\pi}{2}}\sin^6\phi\,d\phi,\ \text{if } \phi = \tfrac{1}{2}\theta,$$

$$= 32\pi a^3\,\frac{5.3.1}{6.4.2}\,\frac{\pi}{2} = 5\pi^2 a^3, \text{ as before.}$$

452. Surfaces—Curved Surface of Right Circular Cone.

Let OAB be a right circular cone.

It is evident that such a cone can be made practically by

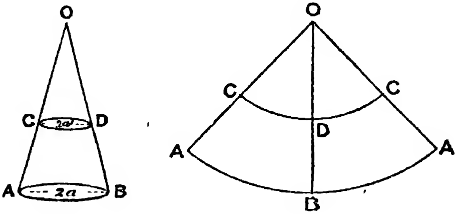

FIG. 115.

cutting out a piece of paper in the form of a sector of a circle, $OABA$, and bending it round until the edges OCA, OCA meet.

Hence the area of the curved surface of cone

$$= \text{area of sector } OABA = \tfrac{1}{2}\, OA \times \text{arc } ABA.\dagger$$

Now let $OB = l$, diameter $AB = 2a$; whence $ABA = 2\pi a$.
Then area $= \frac{1}{2}l \times 2\pi a = \pi la$.

453. Curved Surface of Frustum of Cone.

Let $CABD$ be the frustum, and let $CD = 2a'$, $DB = m$.
Then area of surface of frustum = area of $CABAC$

$$= \tfrac{1}{2}(ABA + CDC)m\,\ddagger = \pi(a + a')m = 2\pi a_1 m,$$

if a_1 be the *mean* radius between AB and CD, viz. $\frac{1}{2}(a + a')$.

The area is, in fact, equal to the *product of the slant side into the circumference of the middle section.*

454. Area of Surface of Revolution.

Let PQ ($= ds$) be an element of arc of the plane curve $y = f(x)$, which revolves round Ox, forming the surface whose area we require. Then the arc PQ, which we may regard as a straight line, will generate the frustum of a cone, $PP'Q'Q$.

Let $PM = y$, $QN = y + dy$; then the area of the surface $PP'Q'Q$

$$= 2\pi . \frac{y + (y + dy)}{2} . ds \;[\text{by Art. } 453]$$

$= 2\pi y ds +$ an infinitesimal of the second order.

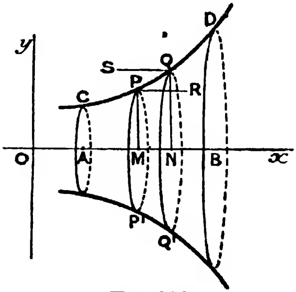

FIG. 116.

† This will be seen if we divide the sector into an infinite number of isosceles triangles, vertex O. Since the altitudes are each equal to OA, and the area of each is $\frac{1}{2}$ altitude $\times$ base; the whole area

$$= \tfrac{1}{2} \text{ altitude} \times (\text{sum of bases}) = \tfrac{1}{2}OA \times \text{arc } ABA.$$

‡ This will be seen if we divide the figure $CABAC$ into an infinite number trapeziums by lines radiating from O. The area of each

$$= \tfrac{1}{2}(\text{sum of parallel sides}) \times (\text{altitude } m).$$

Otherwise:—Draw PR, QS horizontally, and equal to ds. Then the surface generated by PQ will be intermediate between those generated by PR and QS, since every point of PQ is further from the axis than PR, and nearer than QS.

Hence, area lies between $2\pi y ds$ and $2\pi(y + dy)ds$; etc., as before.

The area between two planes through A and B perpendicular to Ox is therefore given by $2\pi \int_{x=a}^{x=b} y ds$, where $b = OB$, and $a = OA$

This may be written

$$2\pi \int_a^b f(x)\frac{ds}{dx}.dx, \text{ or } 2\pi \int_a^b f(x)\sqrt{1 + \{f'(x)\}^2}dx.$$

455. Area of Surface of Sphere.

Taking $x^2 + y^2 = a^2$ for the generating circle, we have

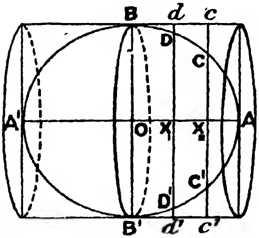

FIG 117.

$$y = \sqrt{a^2 - x^2},$$

$$\therefore y_1 = -\frac{x}{\sqrt{a^2 - x^2}};$$

and $$\left(\frac{ds}{dx}\right)^2 = 1 + y_1^2$$

$$= 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2};$$

$$\therefore ds = \frac{a\,dx}{\sqrt{a^2 - x^2}} = \frac{a\,dx}{y};$$

$$\therefore y ds = a\,dx.$$

$$\therefore 2\pi \int y ds = 2\pi a \int dx = 2\pi a x + C.$$

Hence, if $OX_1 = x_1$, $OX_2 = x_2$, the area of the surface $DCC'D'$

$= 2\pi a(x_2 - x_1)$

$= 2\pi a . X_1X_2$

$=$ *area of the corresponding zone*, $dcc'd'$, *of the circumscribed cylinder*

The whole area of the sphere $= 2\pi a\int_{-a}^{a} dx = 4\pi a^2$.

Otherwise :—The coordinates of any point on the circle are evidently

$$x = a\cos\phi,\quad y = a\sin\phi.$$

Hence $ds^2 = dx^2 + dy^2 = (a^2\sin^2\phi + a^2\cos^2\phi)d\phi^2 = a^2 d\phi^2$ (or at once from a figure);

$$\therefore\ 2\pi\int y\,ds = 2\pi a^2\int\sin\phi\,d\phi = -2\pi a^2\cos\phi + C.$$

The whole area $= \Big[-2\pi a^2\cos\phi\Big]_0^{\pi} = 4\pi a^2$.

456. Paraboloid of Revolution.

Taking $y^2 = 4ax$ as the generating parabola, we have

$$y = 2\sqrt{ax};\quad \therefore\ y_1 = \sqrt{\frac{a}{x}},\ \text{and}\ 1 + y_1^2 = \frac{a+x}{x}.$$

$$\therefore\ y\,ds = 2\sqrt{ax}.\sqrt{\frac{a+x}{x}}\,dx = 2\sqrt{a}\,\sqrt{a+x}\,dx.$$

Hence the area of the surface of a segment cut off by a plane through a point (h, k) of the parabola, perpendicular to Ox,

$$= 2\pi\int_{x=0}^{x=h} y\,ds = 4\pi\sqrt{a}\int_0^h\sqrt{a+x}\,dx = \frac{8\pi\sqrt{a}}{3}\Big[(a+x)^{\frac{3}{2}}\Big]_0^h$$

$$= \frac{8\pi\sqrt{a}}{3}\left\{(a+h)^{\frac{3}{2}} - a^{\frac{3}{2}}\right\}.$$

Now, since $k^2 = 4ah$, we have

$$a + h = \frac{4a^2 + 4ah}{4a} = \frac{4a^2 + y^2}{4a} = \frac{PG^2}{4a},$$

(where $PG =$ the normal), since the subnormal $NG = 2a$.

Hence area $= \dfrac{8\pi\sqrt{a}}{3}\left\{\dfrac{PG^3}{8a^{\frac{3}{2}}} - a^{\frac{3}{2}}\right\} = \dfrac{\pi}{3a}\left[PG^3 - 8a^3\right]$.

457. Prolate Spheroid and Oblate Spheroid.

For the definition, see Art. 449.

Taking the equation $y = \frac{b}{a}\sqrt{a^2 - x^2}$ for the generating ellipse, we have

$$y_1 = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}};$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + y_1^2 = \frac{a^2(a^2 - x^2) + b^2x^2}{a^2(a^2 - x^2)} = \frac{a^4 - a^2e^2x^2}{a^2(a^2 - x^2)} = \frac{b^2(a^2 - e^2x^2)}{a^2y^2};$$

$$\therefore \int y\,ds = \frac{b}{a}\int\sqrt{a^2 - e^2x^2}\,dx = \frac{b}{ae}\int\sqrt{a^2 - e^2x^2}\,d(ex)$$

$$= \frac{b}{ae}\left\{\frac{ex\sqrt{a^2 - e^2x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{ex}{a}\right\} + C$$

$\therefore$ whole surface of prolate spheroid

$$= 2\pi\int_{x=-a}^{x=a} y\,ds = 2\pi\cdot\frac{b}{ae}\{a^2e\sqrt{1 - e^2} + a^2\sin^{-1}e\}$$

$$= 2\pi b\left(b + \frac{a}{e}\sin^{-1}e\right), \text{ since } a\sqrt{1 - e^2} = b.$$

Similarly we may obtain the surface of the oblate spheroid.

458. Area of Surface generated by the Cycloid.

If the cycloid revolve about its axis, the equations to the curve referred to this as axis of x are

$$x = a(\theta - \sin\theta);\quad y = a(1 - \cos\theta)$$

$$\therefore dx = a(1 - \cos\theta)d\theta;\quad dy = a\sin\theta\,d\theta;$$

whence $$ds^2 = a^2(2 - 2\cos\theta)d\theta^2 = 4a^2\sin^2\frac{\theta}{2}\,d\theta^2.$$

$$\therefore 2\pi\int_{\theta=0}^{\theta=2\pi} y\,ds = 2\pi\cdot 2a^2\int_0^{2\pi}(1 - \cos\theta)\sin\frac{\theta}{2}\,d\theta = 8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}\,d\theta,$$

which will reduce to $\frac{64}{3}\pi a^2$.

If it revolve about the tangent at the mid-point of its arc, we must use the equations

$$x = a(\theta - \sin\theta);\quad y = a(1 + \cos\theta),$$

and the area of the surface generated will be found to be $\frac{32}{3}\pi a^2$, or half that of the previous surface.

459. Surface and Volume between two Surfaces of Revolution.

There is no special difficulty in finding the surface and volume included between two surfaces of revolution.

In either case we must find the points of intersection of the two generating curves.

To find the surface, we integrate $2\pi y ds$ for each curve between the limits given by the points of intersection, and *add* the results. To find the volume, we integrate $\pi y^2 dx$ for each curve, and take the *difference* of the results.

Ex. *The two parabolas* $y^2 = 4ax$, $x^2 = 4ay$, *revolve about* Ox. *Find* (1) *the surface*, (2) *the volume, of the included portion.*

The curves meet at the origin, and at $(4a, 4a)$.

(1) (a) For the parabola $y^2 = 4ax$, we have (Art. 456)

$$2\pi \int y ds = \left[\frac{8\pi\sqrt{a}}{3}(a+x)^{\frac{3}{2}}\right]_0^{4a} = \frac{8\pi}{3}(5\sqrt{5}-1)a^2.$$

(b) For the parabola $x^2 = 4ay$, we have

$$y_1 = \frac{x}{2a}; \quad \therefore 1 + y_1{}^2 = \frac{4a^2 + x^2}{4a^2}.$$

Hence $2\pi \int y ds = 2\pi \int_0^{4a} \frac{x^2}{4a} \cdot \frac{\sqrt{x^2 + 4a^2}}{2a} dx;$

and, putting $x = 2a \tan\theta$, this becomes $4\pi a^2 \int \frac{\sin^2\theta}{\cos^5\theta} d\theta$, the limits being $\tan^{-1} 2$ and 0.

Adopting successive reduction as in Art. 393, let $I = \int s^2 c^{-5} d\theta$.

To connect this with $\int c^{-3} d\theta$, we have

$$\frac{d}{d\theta} sc^{-4} = c^{-3} + 4s^2c^{-5};$$

$$\therefore I = \tfrac{1}{4} sc^{-4} - \tfrac{1}{4}\int c^{-3} d\theta.$$

Again, to connect $\int c^{-3} d\theta$ ($= I_1$ say) with $\int c^{-1} d\theta$, we have

$$\frac{d}{d\theta} sc^{-2} = c^{-1} + 2c^{-3}s^2 = -c^{-1} + 2c^{-3};$$

$$\therefore I_1 = \tfrac{1}{2} sc^{-2} + \tfrac{1}{2}\int c^{-1} d\theta = \tfrac{1}{2} sc^{-2} + \tfrac{1}{2} \operatorname{gd}^{-1}\theta.$$

Hence $I = \frac{1}{4}\left\{\frac{s}{c^4} - \frac{1}{2}\frac{s}{c^2} - \frac{1}{2}\operatorname{gd}^{-1}\theta\right\}.$

Therefore, noting that, when $\tan\theta = 2$, $\sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$,

the surface $= 4\pi a^2 I = \pi a^2\left[\frac{s}{c^4} - \frac{1}{2}\frac{s}{c^2} - \frac{1}{2}\log(\sec\theta + \tan\theta)\right] + C$,

which, taken between the limits, becomes

$$\pi a^2\left\{\frac{2}{\sqrt{5}}.25 - \frac{1}{2}.\frac{2}{\sqrt{5}}.5 - \tfrac{1}{2}\log_e(\sqrt{5}+2)\right\} = \pi a^2\{9\sqrt{5} - \tfrac{1}{2}\log_e(\sqrt{5}+2)\}.$$

The whole surface therefore

$$= \frac{8\pi}{3}(5\sqrt{5} - 1)a^2 + \pi a^2\{9\sqrt{5} - \tfrac{1}{2}\log_e(\sqrt{5}+2)\}$$

$$= \frac{\pi}{3}\{67\sqrt{5} - 8 - \tfrac{3}{2}\log_e(\sqrt{5}+2)\}a^2.$$

(2) For the volumes we have

$$(a)\ \pi\int_0^{4a} y^2 dx = 4\pi a\int_0^{4a} x\,dx = 2\pi a[x^2]_0^{4a} = 32\pi a^3.$$

$$(b)\ \pi\int_0^{4a} y^2 dx = \frac{\pi}{16a^2}\int_0^{4a} x^4\,dx = \frac{\pi}{16a^2}\left[\frac{x^5}{5}\right]_0^{4a} = \frac{64\pi a^3}{5}.$$

The difference $= \frac{96}{5}\pi a^3$.

EXAMPLES LXX.

1. Find the volumes of the solids generated by revolution, round Ox, of the following curves :—

(1) The straight line $y = mx$, from the origin to $x = h$.

(2) $y = mx + c$, from $x = -h$, to $x = +h$.

(3) $y = e^x$, from $x = 0$ to $x = -\infty$.

(4) $y = a\sin\frac{x}{a}$, for one of the portions cut off by Ox.

(5) $y = \sqrt{a(\sqrt{a+x} + \sqrt{a-x})}$, between $x = a$ and $x = -a$, taking the $+^{ve}$ sign with the roots.

(6) $9y^2 = (x+7)(x+4)^2$, for the whole loop, between $x = -7$ and $x = -4$.

2. Find the surfaces generated by the revolution, round Ox, of the following curves :—

(1) $y = mx$, from the origin to $x = h$.

(2) $y = mx + c$, from $x = -h$ to $x = +h$.

(3) $y = e^x$, from $x = 0$ to $x = -\infty$.

(4) The parabola $y^2 = x + 1$, from $x = 1$ to $x = 5$.

(5) $9y^2 = (x+7)(x+4)^2$, for the whole loop.

(6) The rectangular hyperbola $y^2 = x^2 + a^2$, from $x = 0$ to $x = a/\sqrt{2}$.

(7) The rectangular hyperbola $y^2 = x^2 - a^2$, from $x = a$ to $x = a\sqrt{5}/\sqrt{2}$.

(8) $y = a \sin \dfrac{x}{a}$, for one of the portions cut off by Ox.

3. Find the volume generated by the revolution of the cissoid, $y^2(2a - x) = x^3$, about the axis of x, between $x = 0$ and $x = a$.

4. Find the surface generated by the revolution of the tractory

$$x = a\left(\log\tan\frac{\phi}{2} + \cos\phi\right),\ y = a(1 - \sin\phi)$$

about Ox, between the points for which $\phi = \frac{1}{2}\pi$ and $\phi = \frac{1}{6}\pi$.

5. If the tractory, given by $x = a\left(\log\cot\dfrac{\phi}{2} - \cos\phi\right)$, $y = a\sin\phi$, revolve about its asymptote, the axis of x; show that whole area of the surface generated is equal to that of a sphere of radius a, and that the volume is half the volume of the sphere.

6. If S is the curved surface, and V the volume, generated by the revolution of the catenary $y = c\cosh(x/c)$ about Ox, between the origin and (h, k); prove that $S = \pi(ks + ch)$, where s is the length of the arc between these points; and that $V = \frac{1}{2}cS$.

7. Find the volume generated by the revolution of the curve $y = c\sin^{-1}\dfrac{x}{c}$ round Ox, between the points $(0, 0)$ and $\left(c, \dfrac{\pi}{2}c\right)$.

8. Find the surface generated by the revolution of the catenary $y = c\cosh^{-1}(x/c)$ about Ox, from $x = c$ to $y = c$.

9. Find the volume and surface generated by the revolution round Ox of the parabola $x^2 = 4ay$, from the vertex to the extremity of the latus rectum.

10. Find the surface generated by the revolution of the cycloid

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

round Oy, from $\theta = 0$ to $\theta = \pi$.

11. Find the volume and surface generated by the revolution of the curve $9ay^2 = 4x^3$ about Ox, between $x = 0$ and $x = a$.

12. Find the volume and surface generated by the revolution of the curve $9ay^2 = x(x - 3a)^2$ about Ox, for the whole loop.

13. Find the volume generated by the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 + \cos\phi)$ about Ox, from cusp to cusp.

14. Find the volume and surface generated by the revolution of the four-cusped hypocycloid $x = a\cos^3\phi$, $y = a\sin^3\phi$ about Ox.

15. A curve revolves about the axis of x. Show that for portions of the curve below the axis, the value of the surface generated is negative, while that of the volume is positive.

Show that the algebraical value of the surface of the cone generated by the line $y = mx$, between $x = h$ and $x = -h$, is zero; while the volume $= \frac{2}{3}\pi m^2 h^3$.

Explain, also, why the second result of Ex. 12 is negative if we take the upper sign for the value of y, i.e. $3\sqrt{a}\,.\,y = +\sqrt{x}(x - 3a)$.

16. Find the surface generated by the revolution of $4y = x^2 - 2\log x$ about Ox, from $x = 1$ to $x = 2$.

17. If t be the thickness of a zone of a sphere; b_1, b_2, the radii of the circular ends: prove that its volume $= \frac{1}{2}\pi b_1^2 t + \frac{1}{2}\pi b_2^2 t + \frac{1}{6}\pi t^3$, that is, half the sum of the cylinders described on the two ends, each of length t, together with the sphere inscribed in the zone.

18. Show that the above volume is equal to a cylinder of length t and base equal to the middle section of the zone (*i.e.* the section cut by a plane parallel to the ends and midway between them) minus half the sphere inscribed in the zone.

19. A parabola $y^2 = 4ax$, and its tangent at (h, k), revolve about its axis. Find the volume between the resulting paraboloid and its circumscribing cone.

20. The straight line $y + x = 2$ cuts off a segment from the parabola $y = 1 - x + x^2$, and the figure revolves round Ox. Find the volume of the solid generated.

21. The parabola $y^2 = 12ax$ intersects the circle $x^2 + y^2 = 64a^2$, and the whole figure revolves about Ox. Find the volume generated by the *larger* area intercepted between the two curves.

22. If the figure in the preceding example revolve about Oy, find the volume generated by the *smaller* area intercepted.

23. Find the volume generated by the revolution round Ox of the curve $y = \log x$, between $x = 0$ and $x = 1$.

24. The ellipse $x^2 + xy + y^2 = 1$ revolves about Ox. Find the volume generated by the portion of the curve between the points $(0, 1)$ and $\left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

25. A square having been inscribed in a circle, the area made up of the square and the segment of the circle cut off by one of the sides of the square revolves round the opposite side of the square. Find the volume and the surface of the solid thus generated.

26. Find the volume of the solid generated by the revolution of the catenary $x = c\cosh(y/c)$ about Ox, between $x = c$ and $y = c$.

27. The semi-cubical parabola $27ay^2 = 4(x - 2a)^3$ is cut by the line $x = 5a$, and the whole figure revolves about Ox. Find the surface and volume generated.

28. The equation of the cardioide being given by

$$x = 2a\cos\phi - a\cos 2\phi, \quad y = 2a\sin\phi - a\sin 2\phi;$$

find the volume and surface generated by the revolution of the whole curve round Ox, the limits of ϕ being O and π.

Answers.

1. (1) $\frac{1}{3}\pi m^2h^3$. (2) $\frac{2}{3}h(m^2h^2 + 3c^2)$. (3) $\frac{1}{2}\pi$. (4) $\frac{1}{2}\pi^2a^3$.
(5) $\pi a^3(4 + \pi)$. (6) $3\pi/4$.

2. (1) $\pi m\sqrt{1+m^2}\,h^2$. (2) $4\pi\sqrt{1+m^2}\,hc$. (3) $\pi\{\sqrt{2}+\log_e(\sqrt{2}+1)\}$.
(4) $49\pi/3$. (5) 3π. (6) $\pi\{1 + \frac{1}{\sqrt{2}}\log_e(\sqrt{2} + 1)\}a^2$.
(7) $\pi[\sqrt{10}-1-\frac{1}{\sqrt{2}}\log\{(\sqrt{5} + 2)/(\sqrt{2} + 1)\}]a^2$.
(8) $2\pi\{\sqrt{2} + \log_e(\sqrt{2} + 1)\}a^2$.

3. $\pi a^3\{8\log_e 2 - 16/3\}$. **4.** $\pi a^2\{2\log_e 2 - 1\}$. **7.** $\frac{\pi c^3}{4}\{\pi^2 - 8\}$.

8. $2\pi c^2\left(1 - \frac{1}{e}\right)$. **9.** $\frac{2\pi a^3}{5}$; $\frac{\pi a^2}{2}\{3\sqrt{2} - \log_e(\sqrt{2} + 1)\}$. **10.** $\frac{32\pi a^2}{3}$.

11. $\frac{\pi a^3}{9}$; $\{7\sqrt{2} + 3\log_e(\sqrt{2} + 1)\}\frac{\pi a^2}{18}$. **12.** $(3\pi a^3)/4$; $3\pi a^2$.

13. $2\pi a^3\int_0^\pi (1+c)^2(1-c)\,d\theta = 32\pi a^3\int_0^{\frac{\pi}{2}}\cos^4\phi\sin^2\phi\,d\phi$ (if $\phi=\theta/2$) $=\pi^2 a^3$.

14. $\dfrac{32\pi a^3}{105}$; $\dfrac{12\pi a^2}{5}$. **16.** $\dfrac{\pi}{16}\{27 - 16\log_e 2 - 4(\log_e 2)^2\}$.

19. $\frac{1}{6}\pi h k^2$. **20.** $\dfrac{64\pi}{15}$. **21.** $480\pi a^3$. **22.** $\dfrac{1792\sqrt{3}}{5}\pi a^3$. **23.** 2π.

24. $\dfrac{2\pi}{9\sqrt{3}}(2\sqrt{3}+7)$. **25.** $\dfrac{\pi a^3}{3}(3\pi - 2)$; $\sqrt{2}(\pi + 4 + 4\sqrt{2})\pi a^2$.

26. $\dfrac{\pi c^3}{2}\{c + \dfrac{5}{c} - 4 - 2\log c\}$. **27.** $\pi a^2\left\{\dfrac{7\sqrt{2}}{2} + \dfrac{3}{2}\log_e(\sqrt{2}+1)\right\}$.

28. $\dfrac{64}{3}\pi a^3$; $\dfrac{128\pi a^2}{5}$.

460. Centroids or Centres of Mass.

We shall now give a few examples on the application of the Integral Calculus to the finding of the *centroid*, or *centre of mass*, of an arc, an area, and a volume. We shall consider only the case in which the density is uniform. The two following propositions, the second of which is an extension of the first, are proved in works on Statics.

(1) If a series of particles of masses $m_1, m_2, \ldots$ be situated in a plane at the points $(x_1, y_1), (x_2, y_2), \ldots$ respectively, and if $(\bar{x}, \bar{y})$ be their centre of mass, then

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma(mx)}{\Sigma(m)};$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma(my)}{\Sigma(m)}.$$

(2) If $(x_1, y_1), (x_2, y_2), \ldots$ be the centres of mass of a series of bodies whose masses are $m_1, m_2, \ldots$ respectively, and if $(\bar{x}, \bar{y})$ be the centre of mass of the whole system, the equations in (1) will still hold.

461. Arc of Circle.

Let ACB be a uniform wire bent into the shape of an arc of a circle, radius a, and which subtends an angle 2α at the centre.

Let Ox bisect the arc at C, so that the c. of mass lies on OC.

Consider an element of arc PQ; its mass $= mds$, if $m =$ mass per unit arc, m being a constant.

Let the coordinates of P and Q be (x, y) and $(x + dx, y + dy)$ respectively; the c. of mass of PQ, regarded as a straight line, is at the mid-point

$$\left(x + \frac{dx}{2}, y + \frac{dy}{2}\right).$$

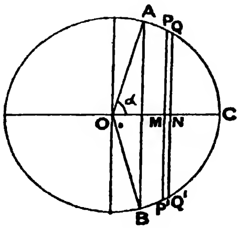

FIG. 118

Hence $\bar{x} = \Sigma[(x + \frac{1}{2}dx) . mds] \div \Sigma(mds)$
$= \Sigma(xds)/\Sigma(ds)$,

neglecting second orders, and noting that m is constant. This may be written $\int xds/\int ds$.

Now $\int ds =$ arc $ACB = 2a\alpha$.

To find $\int xds$, let $x = a\cos\phi$, $y = a\sin\phi$; then $ds = ad\phi$;

$$\therefore \int xds = a^2\int_{-\alpha}^{\alpha}\cos\phi\, d\phi = 2a^2\sin\alpha.$$

$$\therefore \bar{x} = \frac{2a^2\sin\alpha}{2a\alpha} = \frac{a\sin\alpha}{\alpha}.$$

If $\alpha = \frac{1}{2}\pi$, the arc becomes semicircular, in which case $\bar{x} = 2a/\pi$.

462. Segment of Circle

Let ACB in the above figure be the given segment; then its c. of mass lies on OC.

Let $m =$ mass per unit area; then the mass of a strip $PP'Q'Q$, of width dx, is $2my\,dx$ to the first order, and its c. of mass is evidently at the mid-point of MN.

Hence, neglecting infinitesimals beyond the first order,

$$\bar{x} = \frac{\int x . 2my\,dx}{\int 2my\,dx} = \frac{\int xy\,dx}{\int y\,dx} = \frac{-a^3\int_{-\alpha}^{\alpha}\sin^2\phi\cos\phi\, d\phi}{-a^2\int_{-\alpha}^{\alpha}\sin^2\phi\, d\phi}$$

$$= a\left[\tfrac{1}{3}\sin^3\phi\right]_{-\alpha}^{\alpha} \div \tfrac{1}{2}\left[\phi - \sin\phi\cos\phi\right]_{-\alpha}^{\alpha} = \frac{2a}{3} . \frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}.$$

If $a = \frac{1}{2}\pi$, the segment becomes a semicircle, in which case $\bar{x} = 4a/3\pi$.

463. Oblique Coordinates.

Ex. *Find the centre of mass of the segment of a parabola cut off by an oblique double ordinate* KL [see Fig. 99, Art. 427].

Let the segment be cut into narrow strips parallel to the bounding ordinate KL; then since Ox bisects every strip, the c. of mass of each will lie on Ox: hence the c. of mass of the whole segment will lie on Ox.

And since the *perpendicular* distance of any point M, on Ox, from Oy is $x \sin \omega$, we have, by Arts. 427 and 460,

$$\bar{x} \sin \omega = \frac{\int_0^h x \sin \omega \,.\, 2y\, dx \sin \omega}{\int_0^h 2y\, dx \sin \omega} = \frac{\int_0^h xy\, dx}{\int_0^h y\, dx} \cdot \sin \omega.$$

Hence
$$\bar{x} = \frac{2\sqrt{a'}\int_0^h x^{\frac{3}{2}}\, dx}{2\sqrt{a'}\int_0^h x^{\frac{1}{2}}\, dx} = \frac{\frac{2}{5}h^{\frac{5}{2}}}{\frac{2}{3}h^{\frac{3}{2}}} = \frac{3}{5}h.$$

That is to say, *the c. of mass lies on the diameter* OH, *and divides it in the ratio* 3 : 2.

464. Polar Coordinates.

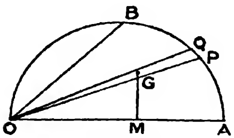

FIG. 119.

Ex. *Find the c. of mass of the sector* OAB *of the circle* $r = 2a \cos \theta$, *in which* $\angle$ BOA $= \frac{1}{4}\pi$.

Dividing the area into an infinite number of triangles such as OPQ, the c. of mass of which is at G, where $OG = \frac{2}{3}OP$ (ultimately), we have *for any curve*

$$\bar{x} = \frac{\int \frac{2}{3}r \cos\theta \,.\, \frac{1}{2}r^2 d\theta}{\int \frac{1}{2}r^2 d\theta} = \frac{2}{3}\frac{\int r^3 \cos\theta\, d\theta}{\int r^2 d\theta}; \quad \text{and similarly } \bar{y} = \frac{2}{3}\frac{\int r^3 \sin\theta\, d\theta}{\int r^2 d\theta}.$$

For the circle, $\int_0^{\frac{\pi}{4}} r^3 \cos\theta = 8a^3\int_0^{\frac{\pi}{4}} \cos^4\theta\, d\theta = 2a^3\int_0^{\frac{\pi}{4}}(1 + \cos 2\theta)^2\, d\theta$

$$= 2a^3\int_0^{\frac{\pi}{4}}\{1 + 2\cos 2\theta + \tfrac{1}{2}(1 + \cos 4\theta)\}d\theta = 2a^3\left\{\frac{3}{2}\cdot\frac{\pi}{4} + 1\right\} = \frac{a^3}{4}(3\pi + 8).$$

$$\int_0^{\frac{\pi}{4}} r^2 d\theta = 4a^2\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta = 2a^2\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)d\theta = 2a^2\left(\frac{\pi}{4}+\frac{1}{2}\right) = \frac{a^2}{2}(\pi+2).$$

$$\therefore\ \bar{x} = \frac{2}{3}\cdot\frac{a^3}{4}(3\pi+8)\Big/\frac{a^2}{2}(\pi+2) = \frac{a}{3}\cdot\frac{3\pi+8}{\pi+2}.$$

465. Segment of Sphere.

Referring to the figure of Art. 461, we have

(1) For the surface, if m = mass per unit area,

$$\bar{x} = \frac{2\pi\int x\,.\,my\,ds}{2\pi\int my\,ds} = \frac{\int xy\,ds}{\int y\,ds} = \frac{a^3\int\cos\phi\sin\phi\,d\phi}{a^2\int\sin\phi\,d\phi}$$
$$= a\left[\tfrac{1}{2}\sin^2\phi + C_1\right] \div \left[-\cos\phi + C_2\right].$$

If the limits of ϕ be α and β ($\alpha > \beta$), then, *remembering that the numerator and denominator are separate integrals,*

$$\bar{x} = \frac{a}{2}[\sin^2\alpha - \sin^2\beta] \div [\cos\beta - \cos\alpha]$$
$$= \frac{a}{2}(\cos\beta + \cos\alpha).$$

If h_1, h_2 be the values of x corresponding to α and β, then

$$h_1 = a\cos\alpha,\ h_2 = a\cos\beta,$$

$\therefore\ \bar{x} = \dfrac{h_1+h_2}{2}$, which is midway between $x = h_1$ and $x = h_2$.

Hence the c. of mass of the surface of a zone of a sphere is at the mid-point of the axis of the zone.

If $\alpha = \pi/2$, $\beta = 0$, we have a hemispherical surface, in which case $\bar{x} = a/2$.†

(2) For the volume, if m = mass per unit volume,

$$\bar{x} = \frac{\pi\int x\,.\,my^2dx}{\pi\int my^2dx} = \frac{\int xy^2dx}{\int y^2dx}$$

Now $\int xy^2dx = -a^4\int\cos\phi\sin^3\phi\,d\phi = -\dfrac{a^4}{4}\sin^4\phi + C_1$;

† These results follow from Art. 455; for the c. of mass of a zone of the sphere may be shown to be the same as that of the circumscribed cylinder, if we imagine the zone of either divided into an infinite number of zones.

$$\int y^2 dx = -a^3\int \sin^3\phi\, d\phi = -a^3\int(1-\cos^2\phi)\sin\phi\, d\phi$$
$$= a^3(\cos\phi - \tfrac{1}{3}\cos^3\phi) + C_2.$$

Hence $\bar{x}$ may be obtained.

For the hemisphere the limits of ϕ are 0 and $\pi/2$, giving $\bar{x} = \dfrac{a^4}{4} \div \dfrac{2}{3}a^3 = \dfrac{3}{8}a.$

466. General Example.

Ex. *Find the centroid of the segment of the parabola* $y^2 = 4ax$ *cut off by the line* $3y - 2x = 4a$.

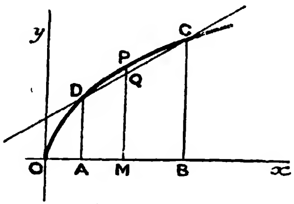

FIG. 120.

The points of intersection are $(a, 2a)$ and $(4a, 4a)$.

Let (x, y_1) denote a point on the straight line, and (x, y_2) a point on the parabola.

Then, the area of an element $PQ = (y_2 - y_1)dx$; the coordinates of its c. of mass are $\left(x, \dfrac{y_2 + y_1}{2}\right)$.

Hence $\bar{x} = \displaystyle\int_a^{4a} x(y_2 - y_1)\,dx \div \int_a^{4a}(y_2 - y_1)\,dx.$

$$\text{Now} \int_a^{4a} x(y_2 - y_1)\,dx = 2\sqrt{a}\int_a^{4a} x\sqrt{x}\,dx - \tfrac{2}{3}\int_a^{4a}(2ax + x^2)dx$$
$$= [\tfrac{4}{5}\sqrt{a}\,.\,x^{\frac{5}{2}} - \tfrac{2}{3}ax^2 - \tfrac{2}{9}x^3]_a^{4a}$$
$$= [\tfrac{4}{5}(32-1) - \tfrac{2}{3}(16-1) - \tfrac{2}{9}(64-1)]a^3 = \tfrac{4}{5}a^3$$

$$\int_a^{4a}(y_2 - y_1)dx = 2\sqrt{a}\int_a^{4a}\sqrt{x}\,dx - \tfrac{2}{3}\int_a^{4a}(2a + x)\,dx$$
$$= \left[\tfrac{4}{3}\sqrt{a}x^{\frac{3}{2}} - \tfrac{4}{3}ax - \tfrac{1}{3}x^2\right]_a^{4a}$$
$$= [\tfrac{4}{3}(8-1) - \tfrac{4}{3}(4-1) - \tfrac{1}{3}(16-1)]a^2 = \frac{a^2}{3}.$$

Hence $\bar{x} = \frac{4}{5}a^3 \div \frac{1}{3}a^2 = \frac{12}{5}a$

Again, $\bar{y} = \displaystyle\int_a^{4a}\frac{y_2 + y_1}{2}(y_2 - y_1)dx \div \int_a^{4a}(y_2 - y_1)\,dx.$

Now, $$\int_a^{4a} \frac{y_2+y_1}{2}\cdot(y_2-y_1)\,dx = \tfrac{1}{2}\int_a^{4a}(y_2^2-y_1^2)\,dx$$

$$= \tfrac{1}{2}\cdot 4a\int_a^{4a} x\,dx - \tfrac{1}{2}\cdot\tfrac{4}{9}\int_a^{4a}(2a+x)^2\,dx$$

$$= \left[ax^2 - \tfrac{2}{9}(4a^2x + 2ax^2 + \tfrac{1}{3}x^3)\right]_a^{4a}$$

$$= [(16-1) - \tfrac{2}{9}\{4(4-1) + 2(16-1) + \tfrac{1}{3}(64-1)\}]a^3 = a^3.$$

And, as above, $\int_a^{4a}(y_2-y_1)\,dx = \frac{a^2}{3}$;

$$\therefore\ \bar{y} = a^3 \div \frac{a^2}{3} = 3a.$$

The coordinates are therefore $(\frac{12}{5}a,\ 3a)$.

NOTE.—We might have found that $\bar{y} = 3a$, by noting that the mid-points of the chords parallel to DQC lie on the line $y = 3a$.

467. Theorems of Pappus.

I. **Prop.**—*If a plane curve revolve about the axis of* x, *the surface of the solid generated is equal to the length of the whole arc of the curve multiplied by that of the path described by the centre of mass of the arc.*

Let PQM be an ordinate meeting the closed area AB in P and Q; and let $PM = y_2$; $QM = y_1$.

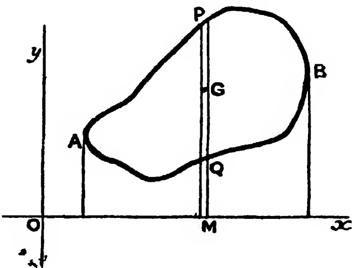

FIG. 121.

Then the surface of the solid of revolution is given by

$$2\pi\int y_2\,ds_2 + 2\pi\int y_1\,ds_1,$$

integrated from A to B, ds_2 and ds_1 being elements of arc at P and Q respectively.

But if $(\bar{x}, \bar{y})$ be the c. of mass of the whole arc ($=s$, say), then

$$\int y_2\,ds_2 + \int y_1\,ds_1 = \bar{y}s.$$

Hence the surface $= 2\pi\bar{y}.s$, which proves the proposition.

NOTE.—The theorem is true whether the curve be closed or not.

II. **Prop.**—*If a plane closed curve revolve about the axis of* x, *the volume of the solid generated is equal to the area of the given curve multiplied by the length of the path described by the centre of mass of the area.*

Using the above notation, the volume of the solid of revolution is given by $\pi\int y_2^2 dx - \pi\int y_1^2 dx$, integrated from A to B; *i.e.* $\pi\int(y_2^2 - y_1^2)dx$.

But if $(\bar{x}, \bar{y})$ be the c. of mass of the whole area ($= A$, say), then

$$\bar{y}A = \int\frac{y_2 + y_1}{2}.(y_2 - y_1)\,dx = \tfrac{1}{2}\int(y_2^2 - y_1^2)\,dx;$$

$$\therefore\ \pi\int(y_2^2 - y_1^2)\,dx = 2\pi\bar{y}.A, \text{ which proves the proposition.}$$

NOTE.—The theorem is also true in the case of an area bounded by an unclosed curve, two ordinates, and the axis of x.

468. Example. *A tore being the surface generated by the revolution of a circle about a line in its plane: find its area, and the volume included by it.*

Let a = radius of circle, b = the distance of its centre from Ox.

Then, since the c. of mass of both the arc and area are at the centre of the circle, we have :—

(1) Surface $= 2\pi b\,.\,2\pi a = 4\pi^2 ab$.

(2) Volume $= 2\pi b\,.\,\pi a^2 = 2\pi^2 a^2 b$.

469. Moments of Inertia.

Def.—Let the mass of each element of a body be multiplied by the square of its distance from a given point, line, or plane; then the sum of all these products is called the *Moment of Inertia* of the body about the point, line, or plane, as the case may be. If M be the mass of the body, and the sum of the above products be expressed in the form Mk^2; then k is called the *radius of gyration* about the point, line, or plane. We shall denote "moment of inertia" by M.I.

Hence the radius of gyration $= \sqrt{(\text{M.I.}) \div (\text{mass of body})}$.

470. In the case of a plane area, if we denote by dS an element of area surrounding the point (x, y), then the M.I. about $Oy = \Sigma x^2 m dS$, where m = mass per unit area.

Similarly, the M.I. about $Ox = \Sigma y^2 m dS$.

If r be the radius vector of the point (x, y); then, since $x^2 + y^2 = r^2$, we have

$$x^2 m dS + y^2 m dS = r^2 m dS;$$

and, as this is true for every element, it follows that

$$\Sigma x^2 m dS + \Sigma y^2 m dS = \Sigma r^2 m dS,$$

or, *the sum of the moments of inertia of any plane area about any two rectangular axes in its plane, is equal to the moment of inertia about the origin.*

A similar rule applies in the case of three dimensions.

471. Examples.

Ex. 1. *A uniform rod, of length* a, *about one end.*

Let $OA = a$, $OP = x$, m = mass per unit length of rod, and O the point about which the moment is taken.

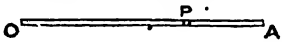

FIG. 122.

Then $M = ma$, and the mass of an element at $P = m\,dx$.

Hence, M.I. about O

$$= \int_0^a x^2 . m\,dx = \tfrac{1}{3}ma^3 = M . \frac{a^2}{3}.$$

Ex. 2. *A uniform rod, of length* 2a, *about its centre.*

The M.I. can easily be shown to be

$$\int_{-a}^{a} x^2 . m\,dx = \tfrac{2}{3}ma^3 = M . \frac{a^2}{3},$$

since in this case $M = 2am$.

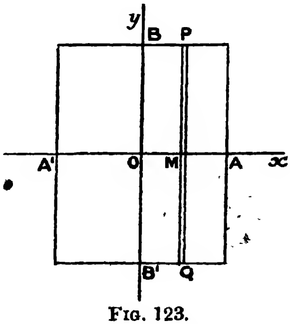

FIG. 123.

Ex. 3. *A uniform rectangular lamina, about the line joining the mid-points of opposite sides.*

Take, for axes of coordinates, lines through the centre parallel to the sides of the rectangle. Let $OA = a$, $OB = b$.

To find the M.I. about BOB', let PQ be a strip parallel to Oy, and of

width dx. Then if m be the mass per unit *width*, the mass of $PQ = m\,dx$; and, since every point of PQ is at the same distance from Oy, we have

$$\text{M.I.} = \int_{-a}^{a} x^2 . m\,dx = \tfrac{2}{3}ma^3 = M\frac{a^2}{3}, \text{ since } M = 2am.$$

NOTE.—We shall get the same result for the M.I. of a rectangular block (or a prism with ends perpendicular to its axis) *about a plane through its centre, perpendicular to its axis;* for the above figure may be taken as a section through the axis, PQ denoting a slice, every element of which is at a distance x from the plane through BOB'. Also m will be the mass per unit width along OA, and this will lead to the same expression as before, viz. $M.\frac{1}{3}a^2$. This result should be remembered.

Ex. 4. *An elliptical lamina, about the axes and centre.*

(1) To find the M.I. about the minor axis, we have, using the equation, $b^2x^2 + a^2y^2 = a^2b^2$,

mass of strip $PQ = m.2y\,dx$, if m = mass per unit area.

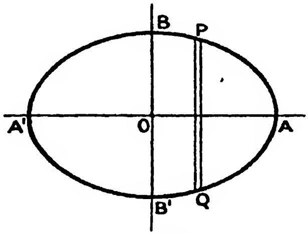

FIG. 124.

$$\therefore \text{ M.I.} = 2m\int_{-a}^{a} x^2 . y\,dx = \frac{2mb}{a}\int_{-a}^{a} x^2\sqrt{a^2 - x^2}\,dx$$

$$= \frac{2mb.a^4}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta, \text{ if } x = a\sin\theta,$$

$$= \frac{mba^3}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\theta\,d\theta = \frac{mba^3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1 - \cos 4\theta)\,d\theta$$

$$= \frac{mba^3}{4}\Big[\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{ (Art. 426, footnote)}$$

$$= \frac{mba^3\pi}{4} = M.\frac{a^2}{4}, \text{ since } M = m.\pi ab$$

(2) Similarly, the M.I. about the major axis $= M.\frac{b^2}{4}$.

(3) It follows from Art. 470, that the M.I. about $O = M.\frac{a^2+b^2}{4}$.

Cor.—For the circle $b = a$; *hence the M.I. of a circular lamina about a diameter* $= \mathrm{M}.\frac{1}{4}\mathrm{a}^2$; *and the M.I. about the centre* $= \mathrm{M}.\frac{1}{2}\mathrm{a}^2$.

Ex. 5. *A prolate spheroid, about the middle plane* BOB′, *perpendicular to the major axis.*

If m = mass per unit volume; then, since every element of the volume $\pi y^2 dx$, which is parallel to the plane through BOB', is at a distance x from this plane, we have

$$\text{M.I.} = \int_{-a}^{a} x^2 . m . \pi y^2 dx = \frac{m\pi b^2}{a^2}\int_{-a}^{a} x^2(a^2 - x^2)\,dx = \frac{m\pi b^2}{a^2}\left[\frac{a^2x^3}{3} - \frac{x^5}{5}\right]_{-a}^{a}$$

$$= \frac{4m\pi b^2 a^3}{15} = M.\frac{a^2}{5}, \text{ since } M = m.\tfrac{4}{3}\pi b^2 a \text{ [Art 449].}$$

Similarly, for an oblate spheroid.

Cor.—In the case of a sphere, $b = a$; hence the M.I. of the volume of a sphere about a plane through the centre $= M.\frac{1}{5}a^2$.

472. Theorem of Parallel Axes.—We shall now prove that the M.I. of a body about any axis is equal to the M.I. about a parallel axis through the c. of mass, together with the M.I. of the whole mass (collected at the c. of mass) about the former axis.

In other words, if Mk^2 be the M.I. about the c. of mass, and c the distance between the axes, then we shall prove that the M.I. about the given axis $= M(k^2 + c^2)$.

Take for axis of z the axis through the c. of mass;† and let MN be the given axis, where the coordinates of M are $(a, b, 0)$.

Then if $P(x, y, z)$ be any point in the body, and m

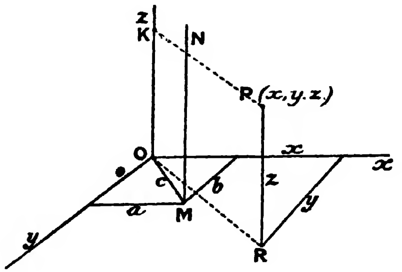

Fig. 125.

† The c. of mass need not be at the origin.

an element of mass surrounding that point, we have, by hypothesis and Art. 460,†

$$\Sigma mx = 0,\ \Sigma my = 0.$$

Also, if Mk^2 be the M.I. about Oz, then $Mk^2 = \Sigma m(x^2 + y^2)$, since $PK^2 = RO^2 = x^2 + y^2$.

Hence M.I. about

$$\begin{aligned} MN &= \Sigma(m.MR^2) = \Sigma[m\{(x-a)^2 + (y-b)^2\}] \\ &= \Sigma[m(x^2+y^2) - 2amx \quad - 2bmy + m(a^2+b^2)] \\ &= \Sigma m(x^2+y^2) \quad - 2a\Sigma mx - 2b\Sigma my + (a^2+b^2)\Sigma m \\ &= \quad Mk^2 \qquad - 0 \qquad - 0 \qquad + c^2M \\ &= M(k^2+c^2). \end{aligned}$$

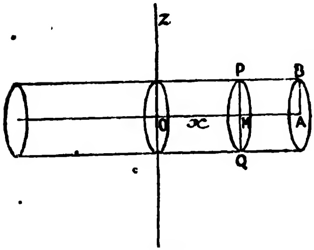

FIG. 126.

Ex. Find the M.I. of a right circular cylinder about an axis through its centre perpendicular to the axis of the cylinder.

Let PQ be an element of thickness dx, parallel to the ends. Then, if $OM = x$, $OA = a$, $AB = b$, and m = mass per unit length of the cylinder; M.I. of PQ about Oz

= M.I. about PQ + M.I. of PQ (collected at M) about Oz

$$= m\,dx.\frac{b^2}{4} + m\,dx.x^2 \text{ [Ex. 4, Cor., Art. 471].}$$

$$\therefore \text{ whole M.I.} = m\frac{b^2}{4}\int_{-a}^{a} dx + m\int_{-a}^{a} x^2\,dx = \frac{mb^2}{4}.2a + \frac{m}{3}.2a^3,$$

or, putting $2am = M$,
$$= M\left(\frac{b^2}{4} + \frac{a^2}{3}\right).$$

EXAMPLES LXXI.

1. Find the centres of mass of the areas included between the axis of x, two ordinates corresponding to given values of x, and the following curves:—

(1) $y = mx$, from $x = 0$ to $x = h$, and from $x = a$ to $x = b$.

† Extended, however, to the case of three dimensions.

(2) $4ay = x^2$, from $x = 0$ to $x = 2a$.

(3) $y^2 = 4ax$, from $x = 0$ to $x = h$.

(4) $y = a\sin\dfrac{x}{a}$, from $x = 0$ to $x = a\pi$.

(5) $y = e^x$, from $x = 0$ to $x = -\infty$.

(6) $y^2 = a^2 - ax$, from $x = 0$ to $x = a$.

(7) $a^2y^2 + b^2x^2 = a^2b^2$, from $x = 0$ to $x = a$.

(8) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, from cusp to cusp.

2. Find the centres of mass of the volumes generated by the following curves, and included between two given values of x:—

(1) $y = mx$, from $x = 0$ to $x = h$.

(2) $y = x + 1$, from $x = 0$ to $x = 1$.

(3) $y^2 = 4ax$, from $x = 0$ to $x = h$.

(4) $y = x^n$, from $x = 0$ to $x = h$.

(5) $a^2y^2 + b^2x^2 = a^2b^2$, from $x = 0$ to $x = a$

(6) $y = e^x$, from $x = 0$ to $x = \frac{1}{2}$.

3. Find the centre of mass of the arc of the catenary $y = a\cosh(x/a)$, cut off by the lines $x = \pm a$.

4. Find the centre of mass of the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, between two cusps.

5. Find the centre of mass of the arc of the parabola $y^2 = 4ax$, above Ox, between the origin and latus rectum.

6. Find the centre of mass of the surface of the cone formed by the revolution of the line $y = mx$ about Ox, and bounded by $x = h$.

7. Find the centre of mass of the surface of a frustum of the cone in the preceding example, between $x = b$ and $x = a$ $(a > b)$. Show that it divides the axis of the frustum in the ratio $2a + b : 2b + a$.

8. Find the centre of mass of the volume of a frustum of the same cone; and show that if α and β are the radii of its circular ends, corresponding to a and b respectively, it divides the axis of the frustum in the ratio $3\alpha^2 + 2\alpha\beta + \beta^2 : 3\beta^2 + 2\beta\alpha + \alpha^2$.

9. Find the centre of mass of the surface generated by the revolution of the parabola $y^2 = 4ax$ round Ox, between $x = 0$ and $x = a$; also between $x = 0$ and $x = 3a$.

10. Find the cartesian coordinates of the centre of mass of the triangle formed by the line $p = r\cos(\theta - \alpha)$, and the lines $\theta = \alpha$, $\theta = 2\alpha$.

11. Find the cartesian coordinates of the centre of mass of the sector of the circle $r = a$, between $\theta = 0$ and $\theta = \alpha$.

12. Find the moments of inertia of:—

(1) A rod of length $2a$, about its centre.

(2) A circle, radius a, about a diameter.

(3) A solid sphere, about a plane through the centre.

(4) A right circular cylinder of length $2a$, about a plane through its centre parallel to its ends.

(5) A rod of length $a + b$, about a point at distances a and b from its ends.

(6) A right circular cylinder, radius a, about a plane through its axis.

(7) The same, about its axis.

(8) A rectangular block, length a, breadth b, thickness c, about a plane through its centre at right angles to its length.

(9) A complete circular arc, about a diameter.

(10) The area of a parabolic segment, about a tangent at its vertex; the segment being bounded by a double ordinate corresponding to $x = h$.

(11) The surface of a sphere about a plane through the centre.

13. Find the M.I. of a rectangle, 8 in. long by 6 in. broad: (1) about a long side; (2) about a short side; and hence (3) about a corner.

14. Given a triangle OAB right angled at A, in which $OA = a$, $AB = b$; find its M.I.: (1) about OA; (2) about a line through O at right angles to OA; and hence (3) about O.

15. Show that the M.I. of a triangle ABC about BC is $Md^2/6$, where d is the distance of A from BC.

16. Find the M.I. of a circle about a tangent, by the principle of parallel axes.

17. Find the M.I. of a rod of length $2a$ about a point at a distance b from one end, by the same method.

18. Show that the M.I. of a system of two or more bodies is equal to the sum of the M.I.'s of the bodies severally. Apply this to Ex. 12 (5)

19. Apply this method to find the moment of inertia of a figure made up of a semicircle, radius a, and a square (the side of the latter being the diameter of the former), about the line of division of the two figures. Find the radius of gyration.

20. A girder has a section in the form of the letter **I**, which is made up of three equal rectangles of sides a and b ($a>b$), placed so as to form that letter. Find the M.I. of the area of the section about a line through the centre, parallel to the longer sides of the two end rectangles.

21. Find the M.I. of a circular annulus, of external and internal radii a and b: (1) about a diameter; (2) about an axis through the centre perpendicular to its plane.

22. Show that (a) the sum of the M.I.'s of a body about two perpendicular planes is equal to the M.I. about their common section, and (b) the sum about three mutually perpendicular planes is equal to the M.I. about their point of intersection.

23. Apply this to find the M.I. of a sphere: (1) about a diameter; (2) about the centre.

24. Apply this to find the M.I. of a spherical shell: (1) about a plane through the centre; (2) about a diameter.

25. Apply this to find the M.I. of a right circular cylinder about an axis through the centre, perpendicular to the axis of the cylinder.

26. Apply the principle of parallel axes to find the M.I. of a sphere about a diameter.

27. A circular arc AOA' subtends an angle 2α at the centre of the circle. Find, by polars, the M.I. about O, the centre of the arc. What does this become when the arc is (1) semicircular, (2) circular?

28. A square of side a revolves about a line in its plane, drawn through an angular point, perpendicular to the diagonal through that point. Apply the theorems of Pappus to find the total surface generated, and also the volume.

29. An equilateral triangle is inscribed in a circle, and the figure revolves about any line in its plane. Show that the ratio of the volume described by the triangle to that described by the circle is constant, and find its value.

30. Given that the volume of a sphere is $\frac{4}{3}\pi a^3$, employ Pappus's Theorem to find the centre of mass of a semicircle.

31. Two equal circles of radius a, in the same plane, intersect, their common chord being equal to a. Find the volume and surface generated by the revolution of the included area about a diameter of either circle, parallel to the common chord.

32. Having given an ellipse whose eccentricity is $\frac{1}{3}$, and a circle of equal area which touches the ellipse at one extremity of the major axis; compare the volumes of the solids generated by the revolution of these curves about the tangent at their common point.

33. A focal chord of a parabola is drawn inclined 45° to the axis of the curve. If the arc thus cut off revolve about the chord, find the volume bounded by the surface generated.

ANSWERS.

1. (1) $\left(\frac{2}{3}h,\ \frac{1}{3}mh\right)$; $\left(\frac{2}{3}\cdot\frac{a^2+ab+b^2}{a+b},\ \frac{m}{3}\cdot\frac{a^2+ab+b^2}{a+b}\right)$

(2) $(3a/2,\ 3a/10)$. (3) $(\frac{3}{5}h,\ \frac{3}{8}k)$ if $k^2 = 4ah$. (4) $\bar{y} = a\pi/8$.

(5) $(-1, \frac{1}{2})$. (6) $(\frac{2}{5}a, \frac{3}{8}a)$. (7) $(4a/3\pi, 4b/3\pi)$. (8) $(a\pi, \frac{5}{6}a)$ [Art. 451]

2. (1) $\bar{x} = \frac{3}{4}h$. (2) $\frac{17}{28}$. (3) $\frac{2}{3}h$. (4) $\frac{2n+1}{2n+2}h$. (5) $\frac{3}{8}a$. (6) $\frac{1}{2(e-1)}$

3. $\bar{y} = \frac{e^4+4e^2-1}{4e(e^2-1)}$. **4.** $(a\pi,\ 4a/3)$.

5. $x = \frac{a}{4}\cdot\frac{3\sqrt{2}-\log_e(\sqrt{2}+1)}{\sqrt{2}+\log_e(\sqrt{2}+1)}$; $\bar{y} = \frac{4a}{3}\cdot\frac{2\sqrt{2}-1}{\sqrt{2}+\log_e(\sqrt{2}+1)}$.

6. $\frac{2}{3}h$. **7.** See 1 (1). **8.** $\frac{3}{4}\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2}$. **9.** $\frac{2}{35}(5+3\sqrt{2})a$; $\frac{58}{35}a$.

10. $\bar{x} = \frac{1}{3}(3\cos\alpha - \sec\alpha)p$; $\bar{y} = p\sin\alpha$.

11. $\bar{x} = \frac{2a}{3}\cdot\frac{\sin\alpha}{\alpha}$; $y = \frac{2a}{3}\cdot\frac{1-\cos\alpha}{\alpha}$.

12. (1) $Ma^2/3$. (2) $Ma^2/4$. (3) $Ma^2/5$. (4) $Ma^2/3$.
(5) $M(a^2-ab+b^2)/3$. (6) $Ma^2/4$. (7) $Ma^2/2$.
(8) $Ma^2/3$. (9) $Ma^2/2$. (10) $M.3h^2/7$. (11) $Ma^2/3$.

13. $12M$; $64M/3$; $100M/3$. **14.** $Mb^2/6$; $Ma^2/2$; $M(3a^2+b^2)/6$.

16. $M.5a^2/4$. **17.** $M\{a^2/3+(a-b)^2\}$ **19.** $M(3\pi+128)a^2/12(\pi+8)$.

20. $\dfrac{M}{36}(7a^2 + 12ab + 8b^2)$, M being the mass of the area regarded as a thin lamina.

21. $M(a^2 + b^2)/4$; $M(a^2 + b^2)/2$. **23.** $M.2a^2/5$; $M.3a^2/5$.

24. $Ma^2/3$; $M.2a^2/3$. **25.** $M\left(\dfrac{b^2}{4} + \dfrac{a^2}{3}\right)$. **26.** $M.2a^2/5$. See Ex., Art. 472.

27. $M.2a^2\left(1 - \dfrac{\sin\alpha}{\alpha}\right)$; $2a^2M\left(1 - \dfrac{2}{\pi}\right)$; $2a^2M$. **28.** $4\sqrt{2}\pi a^2$; $\sqrt{2}\pi a^3$.

29. $3\sqrt{3} : 4\pi$. **30.** $4a/3\pi$ from the diameter.

31. $(2\pi - 3\sqrt{3})\pi a^3/2\sqrt{3}$; $2\pi^2 a^2/\sqrt{3}$. **32.** $\sqrt{3} : \sqrt[4]{8}$. **33.** $128\pi a^3/15$.

DIFFERENTIAL EQUATIONS

CHAPTER XXIX.

INTRODUCTORY.

473. A *differential equation* is an equation which involves two or more variables together with differential coefficients; or it may involve differential coefficients only.

There are two kinds of differential equations :—

(1) An *ordinary differential equation* involves only *one* independent variable; as $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = m^2y$. [See Art. 108.]

(2) A *partial differential equation* involves two or more independent variables and partial differential coefficients with respect to them; as $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

We shall, however, only deal with ordinary differential equations. Our treatment of the subject will be necessarily brief, though sufficient, it is hoped, to enable the reader to apply it to some of the practical problems in physics, in which it plays an important part.

In Art. 108 we have shown how to form a differential equation from an equation not involving differential coefficients. The problem now before us is to obtain, where possible, the original equation from the differential equation. The former is called a *solution*, or *integral*, of the latter.

474. Order and Degree.—The order of a differential equation is that of the highest differential coefficient, and similarly for

the degree. Thus the equation $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^3 + my^2 = 0$ is of the *second* order, since it contains d^2y/dx^2; and of the *first* degree, since d^2y/dx^2 is of that degree. An equation which, when rationalized and freed from fractions, involves the *dependent* variable and its derivatives in the first power only, is said to be *linear*; as in the equation

$$x^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + x^3y = x - 1.$$

475. General Integral and Complete Primitive—Arbitrary Constants.

Let $f(x, y, a)$ be a function of x containing a constant a; then a differential equation of the first order can be obtained from the equation $f(x, y, a) = 0$, either by differentiation merely, or by differentiation followed by the elimination of the constant a.

Thus, from the equation

$$y^2 = 4ax \quad \ldots \quad (1)$$

we have, by differentiation,

$$2y\frac{dy}{dx} = 4a \quad \ldots \quad (2)$$

and (2) is a differential equation formed by the first method.

If now we eliminate a between (1) and (2), we have

$$2xy\frac{dy}{dx} = 4ax = y^2,$$

or

$$2x\frac{dy}{dx} = y \quad \ldots \quad (3)$$

which is a differential equation formed by the second method.

When the *second* method is adopted, the original equation $f(x, y, a) = 0$ is called the *general integral* if we are *solving* the differential equation; but is called the *complete primitive* if we are *forming* the differential equation.

Thus (1) is the *general integral*, or the *complete primitive* of (3), according as we are passing from (3) to (1), or from (1) to (3). It is neither of these with respect to (2).

Since the constant a does not appear in (3), it follows that whatever value be attached to a, so long as it *is* constant, we shall always obtain the same equation (3). Hence, if we are given the equation (3) to solve, the constant a which appears in the integral (1) is perfectly arbitrary, and is called an *arbitrary constant.*

Again, from the equation $f(x, y, a, b) = 0$, involving two arbitrary constants, we can, by differentiating twice, obtain *three* equations, from which the *two* constants may be eliminated. The resulting differential equation will be of the *second* order.

Thus, for the circle $\quad (x - a)^2 + y^2 = b^2 \quad . \quad . \quad . \quad (4)$
we obtain by differentiation,

$$x - a + yy_1 = 0 \quad . \quad . \quad . \quad . \quad (5)$$

and again $\quad 1 + y_1^2 + yy_2 = 0 \quad . \quad . \quad . \quad . \quad (6)$
and in (6) we have a differential equation of the second order, in which the two constants a and b are eliminated.

Generally from the equation $f(x, y, a_1, a_2, \dots a_n) = 0$, . (A) containing n arbitrary constants, we can, by differentiating n times, obtain in all $n + 1$ equations, from which we can eliminate the n constants, the resulting equation being of the nth order.

And (A) is called the *general integral* or *complete primitive* of this differential equation, according to the rule given above.

476. Justification of the Term "General Integral"—Particular Integral.

We have seen that, if all of the constants are to be eliminated, an equation containing n arbitrary constants is bound to give rise to a differential equation whose order is *not lower* than the nth, since we must (in general) have not less than $n + 1$ equations from which to eliminate n constants, and must therefore differentiate n times. Conversely, *a differential equation of the* n*th order cannot give rise to more than* n *arbitrary constants in the general integral;* for, suppose there were $n + 1$ (say) in the latter,

this would give rise to an equation of order $n + 1$, as otherwise we could not eliminate the $n + 1$ constants. But this contradicts the hypothesis that the equation is of the nth order; hence the above statement must hold. Since the general integral contains the greatest possible number of arbitrary constants, it is (as its name implies) the most general solution possible.

If we omit some of the constants, or give particular values to them, we have a *particular integral.*

Ex. 1. The equation

$$y = a\cos x + b\sin x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

leads to the differential equation

$$y_2 + y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And (1) is the general integral, or complete primitive, of (2), since there are *two* constants eliminated, and (2) is of the *second* order. But

$$y = \sin x, \;\; y = \cos x, \;\; \text{and } y = \cos x - \sin x,$$

are all particular integrals, and may be shown to lead, each of them, to (2).

Ex. 2. In Art. 475, equation (1) is a particular integral of (2), for if we integrate (2) we obtain $y^2 = 4ax + C$, where C is an arbitrary constant. This is, of course, the general integral of (2), the constant a in this case *not* being arbitrary.

Ex. 3. In the same article, if we differentiate (3), we obtain

$$2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{dy}{dx},$$

of which (1) is a particular integral; for, on integrating back this last equation, we obtain $2x\dfrac{dy}{dx} = y + C$ instead of (3), thus giving a second arbitrary constant.

477. Geometrical Interpretation.

Consider the differential equation of the first order and degree,

$$f(x, y, dy/dx) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the general integral of this contains an *arbitrary* constant c, we can, by making c assume all possible values from $-\infty$ to $+\infty$, obtain the equations to an infinite number of curves, whose differential equation

is (1). If we give x and y any values we please, we can always find a corresponding value of dy/dx from (1); hence, for every point in the (xy plane there is a definite value for dy/dx. Again, if

$$\phi(x, y, c) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

be the general integral, then, by giving x and y any values we please, we can always obtain a value of c such that these values will satisfy (2). Hence, *through every point in the* xy *plane a curve of the system* (2) *will pass, and will have a definite direction at that point.*

[**Ex.** The general integral of $2xy_1 = y$ is $y^2 = 4ax$.

Let (2, 1) be a given point; then, if the parabola passes through this point, $1 = 8a$, or $a = \frac{1}{8}$.

Also substituting (2, 1) for (x, y) in the differential equation, we have

$$4y_1 = 1, \text{ or } y_1 = \tfrac{1}{4}.$$

Hence, the parabola $2y^2 = x$ passes through (2, 1) and has the direction given by $\tan\psi = \frac{1}{4}$.]

A differential equation of the first order is said to represent a *single infinity* of curves in a plane.

Again, a differential equation of the second order, if reduced to one of the first order, will furnish an arbitrary constant. If $f(x, y, dy/dx, c) = 0$ be the reduced equation, then by giving c all possible values between $-\infty$ and $+\infty$ we obtain an infinite number of values of dy/dx for a given point. Hence, there is an infinite number of curves passing through every point of the plane.

Moreover, if we fix on one value of c and solve the equation

$$f(x, y, dy/dx, c) = 0,$$

we obtain a second arbitrary constant, giving rise to a single infinity of curves for that one value of c. But there are an infinite number of these sets. Hence, there is a *double infinity* of curves for a differential equation of the second order. Similarly for the general case.

CHAPTER XXX.

EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE.

478. Let M and N be two functions of x and y; then the equation $M + N\dfrac{dy}{dx} = 0$ is of the first order and first degree. It may be also written $Mdx + Ndy = 0$. It must be understood that there is no method of solving the equation in its general form; and, in fact, all we shall do here is to consider the very special forms which are capable of solution, and which fall under the following heads:—

(1) Variables Separable;
(2) Homogeneous Equations;
(3) Exact Differential Equations;
(4) The Linear Equation.

479. (I) Variables Separable.

When the equation can be written in the form $f(x)dx + \phi(y)dy = 0$, the variables are said to be separable. The equation is then immediately integrable, the solution being

$$\int f(x)dx + \int \phi(y)dy = C.$$

Ex. 1. *Solve* $2x\dfrac{dy}{dx} = y$.

This may be written $\dfrac{2dy}{y} = \dfrac{dx}{x}$, whence

$$2\log y = \log x + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the general integral.

If we put $C = \log 4a$, (1) becomes

$$\log y^2 = \log 4ax, \quad \text{or } y^2 = 4ax.$$

Ex. 2. *Solve the equation* $xy + (b^2 - x^2)\dfrac{dy}{dx} = 0.$

We have $\dfrac{x\,dx}{b^2 - x^2} + \dfrac{dy}{y} = 0$; and, integrating, we obtain

$$-\tfrac{1}{2}\log(b^2 - x^2) + \log y = C,$$

or, writing $C = \log c$ for convenience,

$$\log y = \log c + \tfrac{1}{2}\log(b^2 - x^2) = \log c\sqrt{b^2 - x^2}.$$

$$\therefore\ y = c\sqrt{b^2 - x^2},\quad \text{or } y^2 = c^2(b^2 - x^2),\ c^2 \text{ being } \textit{arbitrary}.$$

This represents a system of conics whose centre is the origin, and which, for all values of c^2, pass through the two points $(\pm b, 0)$.

NOTE.—c^2 may be $+^{\text{ve}}$ or $-^{\text{ve}}$.

***Ex. 3.** $(1 + y)dx + x(x + y)dy = 0$ . . . . (1)

The variables in this case are not immediately separable.

But putting $x + y = z$, and therefore $dx + dy = dz$, (1) becomes

$$(1 + z - x)dx + xz(dz - dx) = 0,$$

$$i.e.\ (1 + z - x - xz)dx + xzdz = 0,$$

$$i.e.\ (1 - x)(1 + z)dx + xzdz = 0;$$

$$\therefore\ \frac{1 - x}{x}dx + \frac{zdz}{1 + z} = 0.$$

$$\therefore\ \log x - x + z - \log(1 + z) = C,$$

or

$$\log x + y - \log(1 + x + y) = C = \log c, \text{ say.}$$

$$\therefore\ \frac{e^y . x}{1 + x + y} = c,\ \text{ or } xe^y = c(1 + x + y).$$

480. (2) Homogeneous Equations.

In the equation $Mdx + Ndy = 0$, let M and N be *homogeneous expressions in* x *and* y *of the same degree*, say the nth. We may then write

$$M = x^n f(y/x),\ N = x^n\phi(y/x).\quad [\text{Art. } 175.]$$

The equation can therefore be put in the form

$$f(y/x)dx + \phi(y/x)dy = 0, \text{ if we divide out by } x^n \quad . \quad (1)$$

The method adopted is to put $y = vx$.

Then $dy = vdx + xdv$, and (1) becomes

$$f(v)dx + \phi(v)\{vdx + xdv\} = 0.$$

The variables are now separable; thus we have

$$\{f(v) + v.\phi(v)\}\, dx + x.\phi(v)dv + 0,$$

or
$$\frac{dx}{x} + \frac{\phi(v)dv}{f(v) + v.\phi(v)} = 0$$

which may be integrated at once.

Ex. 1. *Solve* $(x^2 + y^2)dx + xydy = 0.$

Put $y = vx$, and we obtain

$$x^2(1 + v^2)dx + vx^2(vdx + xdv) = 0;$$

or
$$(1 + 2v^2)dx + xvdv = 0;$$

$$\frac{dx}{x} + \frac{vdv}{1 + 2v^2} = 0.$$

Integrating,
$$\log x + \tfrac{1}{4}\log(1 + 2v^2) = C;$$

or
$$x\left(1 + \frac{2y^2}{x^2}\right)^{\frac{1}{4}} = c \text{ say, where } c = e^C,$$

$$\therefore\ x^2(x^2 + 2y^2) = c^4.$$

Ex. 2. *Solve* $y(ydx - xdy) + x\sqrt{x^2 + y^2}\,dy = 0.$ . . . . . . (1)

Arranging terms, $y^2dx + x(\sqrt{x^2 + y^2} - y)dy = 0.$
Put $y = vx$, and divide down by x^2;

$$\therefore\ v^2dx + (\sqrt{1 + v^2} - v)(vdx + xdv) = 0,$$

or
$$v\sqrt{1 + v^2}\,dx + x(\sqrt{1 + v^2} - v)dv = 0.$$

$$\therefore\ \frac{dx}{x} + \left(\frac{1}{v} - \frac{1}{\sqrt{1 + v^2}}\right)dv = 0.$$

Integrating, $\log x + \log v - \log(v + \sqrt{1 + v^2}) = C;$

or
$$\frac{xv}{v + \sqrt{1 + v^2}} = e^C = c, \text{ say}.$$

Replacing v by y/x and reducing, we obtain

$$xy = c(y + \sqrt{x^2 + y^2}) \text{ as the general integral.}$$

Rationalizing, this becomes

$$(x - c)^2y^2 = c^2(x^2 + y^2);$$

that is
$$(x^2 - 2cx)y^2 = c^2x^2.$$

Rejecting the factor x, this becomes finally

$$(x - 2c)y^2 = c^2x.$$

481. The following alternative methods of solving the preceding equation will be instructive :—

I. Noting that $d\left(\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$, we have from (1)

$$-y\,.\frac{xdy - ydx}{x^2} + \sqrt{1 + \frac{y^2}{x^2}}\,.\,dy = 0,$$

or
$$-yd\left(\frac{y}{x}\right) + \sqrt{1 + \frac{y^2}{x^2}}\,dy = 0.$$

Putting v for $\dfrac{y}{x}$, $-ydv + \sqrt{1 + v^2}\,dy = 0$, and the variables are separable; etc., as before.

II. Turn to polars. Then [Art. 435, note]

$xdy - ydx = r^2d\theta$; also $x^2 + y^2 = r^2$, and $dy = \sin\theta\, dr + r\cos\theta\, d\theta$.

$$\therefore\ r\sin\theta(-r^2d\theta) + r\cos\theta\,.\,r(\sin\theta\, dr + r\cos\theta\, d\theta) = 0,$$

or
$$\cos\theta\sin\theta\, dr + r(\cos^2\theta - \sin\theta)d\theta = 0.$$

$$\therefore\ \frac{dr}{r} + \left(\frac{\cos\theta}{\sin\theta} - \frac{1}{\cos\theta}\right)d\theta = 0.$$

$$\therefore\ \log r + \log\sin\theta - \log(\sec\theta + \tan\theta) = C;$$

$$\therefore\ r\sin\theta = c(\sec\theta + \tan\theta),$$

$$\text{i.e. } y = c.\frac{\sqrt{x^2 + y^2} + y}{x};\ \text{etc., as before.}$$

482. Form $a_1x + b_1y + c_1 = (a_2x + b_2y + c_2)\dfrac{dy}{dx}$.

This equation, though not homogeneous, may be made so as follows :—

Let $x = x' + h$, $y = y' + k$; then $dx = dx'$ and $dy = dy'$.

The equation becomes

$$a_1x' + b_1y' + c_1 + a_1h + b_1k = (a_2x' + b_2y' + c_2 + a_2h + b_2k)\frac{dy'}{dx'} \quad (1)$$

Now choose h and k so that

$$a_1h + b_1k + c_1 = 0\ ;\ a_2h + b_2k + c_2 = 0 \quad . \quad . \quad . \quad (2)$$

Then (1) reduces to $a_1x' + b_1y' = (a_2x' + b_2y')\dfrac{dy'}{dx'}$, which is homogeneous. After solving, we may replace x' and y' by $x - h$ and $y - k$ respectively; h and k being known from (2).

Ex. *Solve* $2(x + y - 1)dx = (3x + y - 5)dy$.

Putting $x = x' + h$, $y = y' + k$, we have

$$h + k - 1 = 0,\ 3h + k - 5 = 0; \text{ whence } h = 2,\ k = -1.$$

The transformed equation is $2(x' + y')dx' = (3x' + y')dy'$.

Putting $y' = vx'$, and dividing out by x', we have

$$2(1 + v)dx' = (3 + v)(vdx' + x'dv);$$

or

$$(v^2 + v - 2)dx' + x'(3 + v)dv = 0.$$

$$\therefore \frac{dx'}{x'} + \frac{(v + 3)dv}{(v - 1)(v + 2)} = 0.$$

Resolving the second fraction into partial fractions, we obtain

$$\log x' + \tfrac{4}{3}\log(v - 1) - \tfrac{1}{3}\log(v + 2) = C,$$

that is

$$x'\left(\frac{y' - x'}{x'}\right)^{\frac{4}{3}} \div \left(\frac{y' + 2x'}{x'}\right)^{\frac{1}{3}} = \text{constant, } c \text{ say,}$$

or

$$(y' - x')^4 = c^3(y' + 2x').$$

Replacing x' and y' by $x - 2$ and $y + 1$ respectively, the general integral is $(y - x + 3)^4 = c^3(y + 2x - 3)$.

483. Exceptional Case.

Suppose $a_1/a_2 = b_1/b_2$ ($= m$ say), then h and k become infinite, and we cannot apply the method.

Writing the equation in the form

$$\{m(ax + by) + c\}dx = (ax + by + c')dy, \quad . \quad . \quad . \quad . \quad (1)$$

put $ax + by = v$; then $adx + bdy = dv$, and therefore in (1), multiplying first by b,

$$b(mv + c)dx = (v + c')(dv - adx),$$

and the variables are separable. Thus we obtain

$$\{(a + mb)v + bc + ac'\}dx = (v + c')dv; \text{ etc.}$$

Ex. *Solve* $(2x - 4y + 1)dx = (x - 2y)dy$ . . . . (a)

Put $x - 2y = v$, $\therefore dx - 2dy = dv$.

$$\therefore \text{ in } (a) \ (2v + 1)dx = \frac{v}{2}(dx - dv),$$

or

$$(3v + 2)dx + vdv = 0.$$

$$dx + \frac{vdv}{3v + 2} = 0.$$

$$\therefore x + \frac{v}{3} - \frac{2}{9}\log(3v + 2) = C.$$

$$\therefore 9x + 3(x - 2y) - 2\log(3x - 6y + 2) = 9C,$$

or

$$3(2x - y) - \log(3x - 6y + 2) = c.$$

EXAMPLES LXXII.

1. Solve the equations :—

(1) $(y + 1)dx = xdy$. (2) $(1 + y^2)\frac{dy}{dx} = 2xy^2$.

(3) $\sqrt{1 + y}\,dx = \sqrt{1 + x}\,dy$. (4) $xdx + ydy = xydy$.

(5) $\cos x \cos y dx + \sin x \sin y dy = 0$.

(6) $(1 - x^2)(1 - y)\,dx = xy(1 + y)dy$.

(7) $x\sqrt{y^2 - 1}\,dx = y\sqrt{x^2 - 1}\,dy$. (8) $y\left(2x + y\frac{dy}{dx}\right) = x(2y^2 + 5)$.

2. Solve :—

(1) $xdy = (x + y)dx$. (2) $(x + y)\frac{dy}{dx} = y - x$.

(3) $3xy\frac{dy}{dx} = 4x^2 + y^2$. (4) $(2y - x)dy = (y + 2x)dx$.

(5) $x(x^2 - y^2)dy + y^3\,dx = 0$. (6) $\phi(\theta + 2\phi)d\theta = \theta(\phi + 2\theta)d\phi$.

(7) $\frac{dy}{dx} = (y - \sqrt{x^2 + y^2})/x$.

(8) $(x^2 + 2xy)\frac{dy}{dx} + 2xy + y^2 + 3x^2 = 0$.

(9) $(x\tan\frac{y}{x} - y)dx + xdy = 0$.

(10) $\frac{dx}{\sqrt{x^2 + y^2}} + \left(\frac{1}{y} - \frac{x}{y\sqrt{x^2 + y^2}}\right)dy = 0$.

3. Solve :—

(1) $\dfrac{dy}{dx} = \dfrac{2y - 3x + 5}{2x - 3y - 5}$.

(2) $(x - 2y)dy = (2x + 5y - 9)dx$.

(3) $(x - 3y - 5)dy = (3x + y - 5)dx$.

(4) $\dfrac{dy}{dx} = \dfrac{3y + 5x + 4}{7x - y - 10}$.

(5) $(2x - 3y + 1)dx = 2(2x - 3y + 3)dy$.

(6) $(x + y)(2\dfrac{dy}{dx} + 3) = 1$.

(7) $(2x - 3y + 1)\left(\dfrac{dy}{dx} - 1\right) = 2\dfrac{dy}{dx} + 1$.

4. Solve :—

(1) $(x - y)dy = dx$, [put $x - y = z$].

(2) $xy(xdy + ydx) = (1 + y)dy$, [put $xy = z$].

(3) $\cos(x + y)dy = dx$, [put $x + y = z$].

(4) $\dfrac{dy}{dx} + y^2 + \dfrac{1}{4x^2} = 0$, [put $x = 1/z$].

(5) $(xy - 2)\dfrac{dy}{dx} = xy^3$, [put $x = 1/z$].

ANSWERS.

1. (1) $y + 1 = cx$. (2) $y^2 - x^2y = cy + 1$. (3) $\sqrt{1 + x} = \sqrt{1 + y} + c$.
(4) $x + \log(x - 1) = \frac{1}{2}y^2 + c$. (5) $\sin x = c\cos y$.
(6) $2\log x + 4\log(1 - y) = x^2 - 4y - y^2 + c$. (7) $\sqrt{x^2 - 1} = \sqrt{y^2 - 1} + c$.
(8) $y + \frac{1}{2}\log(2y^2 - 2y + 5) - \frac{4}{3}\tan^{-1}\dfrac{2y - 1}{3} = x^2 + c$.

2. (1) $y = x(\log x + c)$. (2) $\frac{1}{2}\log(x^2 + y^2) + \tan^{-1}(y/x) = c$.
(3) $(y^2 - 2x^2)^3 = c^4x^2$. (4) $xy + x^2 - y^2 = c^2$. (5) $2x^2\log y - y^2 = cx^2$.
(6) $\theta^2\phi^2 = c(\theta \sim \phi)^3$. (7) $x^2 + 2cy = c^2$. (8) $x(x^2 + xy + y^2) = c^3$.
(9) $x\sin(y/x) = c$. (10) $\log(x + \sqrt{x^2 + y^2}) = c_1$; whence $y^2 + 2cx = c^2$.

3. (1) $(y - x + 2)(y + x)^5 = c^6$. (2) $\log(x + y - 3) + \dfrac{3(x - 2)}{2(x + y - 3)} = c$.
(3) $3\log(x'^2 + y'^2) = 2\tan^{-1}(y'/x') + c$, where $x' = x - 2$, $y' = y + 1$.

(4) $\log(5x'^2 - 4x'y' + y'^2) = 10\tan^{-1}\dfrac{y'-2x'}{x'} + c$, where $x' = x - 1$, $y' = y + 3$.

(5) $x - 2y = 4\log(2x - 3y + 9)$. (6) $2(x+y) + 2\log(x+y-1) + x = c$.

(7) $x - y = 3\log(2x - 3y + 8) + c$.

4. (1) $y = \log(x - y - 1) + c$. (2) $x^2y^2 = 2y + y^2 + c$.

(3) $\tan\dfrac{x+y}{2} = y + c$. (4) $(2xy - 1)(\log x + c) = 2$.

(5) $y^3 = c(xy - 1)(xy + 2)^2$.

484. (3) Exact Differential Equations.

When the differential equation

$$Mdx + Ndy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

can be obtained from the complete primitive by simple differentiation, it is said to be *exact*.

Thus the equation $x^2 + 2xy + x - y = c$ becomes, after differentiation,

$$(2x + 2y + 1)dx + (2x - 1)dy = 0,$$

which is therefore an exact equation.

Suppose

$$u = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

to be the complete primitive of the exact equation (1), u being a function of x and y. Differentiating (2), we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

Since this must be identical with (1), we have, comparing coefficients of dx and dy,

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}.$$

But

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}; \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y};$$

hence, by Art. 172,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and this is the condition that the equation may be exact.

Ex. $(x^2 - 2xy + y)dx + (y^2 - x^2 + x)dy = 0.$

Compare with $\dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = 0.$

Then, from Art. 172, as we have seen,

$$\frac{\partial}{\partial y}(x^2 - 2xy + y) \text{ must be equal to } \frac{\partial}{\partial x}(y^2 - x^2 + x);$$

which is the case, since each $= -2x + 1$.

Hence the equation is exact.

485. Rule for solving an Exact Equation.

To solve an exact equation, we have the following rule :— Integrate Mdx on the supposition that y is constant, and Ndy on the supposition that x is constant; add the two results together, *with the exception* that terms common to both results are only to be written down once.

These terms will always contain *both* x and y, and may therefore easily be detected.

Ex. In the case of the equation above, namely

$$(x^2 - 2xy + y)dx + (y^2 - x^2 + x)dy = 0,$$

first treat y as a constant; then

$$\int(x^2 - 2xy + y)dx = \tfrac{1}{3}x^3 - x^2y + xy + \text{terms containing } y \text{ only} \quad . \quad (4)$$

Next treat x as a constant; then

$$\int(y^2 - x^2 + x)dy = \tfrac{1}{3}y^3 - x^2y + xy + \text{terms containing } x \text{ only} \quad . \quad (5)$$

Hence the integral is $\tfrac{1}{3}x^3 - x^2y + xy + \tfrac{1}{3}y^3 = c'$;

or $$x^3 + y^3 + 3xy(1 - x) = c, \text{ where } c = 3c'.$$

Note.—The terms $-x^2y + xy$, which contain both x and y, and which occur in both results (4) and (5), are written down once only.

486. Equations, not exact, solved by an Integrating Factor.

If the condition (3) of Art. 484 is not satisfied, the equation

is not exact. But assuming that (1) *has* a solution $u = c$; then, comparing the equations

$$Mdx + Ndy = 0; \quad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0,$$

we have $\dfrac{\partial u}{\partial x}\Big/M = \dfrac{\partial u}{\partial y}\Big/N = f$ say; and f may be assumed to be a function of x and y without altering the truth of the last statement. Hence, putting $f(x, y)$ for f,

$$\partial u/\partial x = M.f(x, y); \quad \partial u/\partial y = N.f(x, y).$$

Thus, if we multiply the equation $Mdx + Ndy = 0$ by the factor $f(x, y)$, which is called *an integrating factor*, we obtain an exact equation. The difficulty, however, is to find this integrating factor. In many cases it can be found by inspection, in other cases there are rules for finding it. For these rules the reader is referred to Boole's *Differential Equations*, or to Murray's *Introductory Course in Differential Equations*, pp. 23–25. We shall only consider a few simple cases in which it may be found by inspection.

Ex. 1. $ydx + 2xdy = 0$.

This is not an exact equation. Multiplying by $1/xy$, however, it becomes

$$\frac{dx}{x} + \frac{2dy}{y} = 0; \text{ whence } \log x + 2\log y = c',$$

or $$xy^2 = c, \text{ if } \log c = c'.$$

Or, multiplying by y, we have $y^2dx + 2xydy = 0$, an exact equation, giving $xy^2 = c$, as before.

Ex. 2. $ydx - (x + y^2)dy = 0$.

This is not an exact equation. An integrating factor is $1/y^2$, which gives

$$\frac{ydx - xdy}{y^2} - dy = 0.$$

$$\therefore \frac{x}{y} - y = c, \text{ or } x - y^2 = cy.$$

Ex. 3. $(1 + xy)dx + x^2dy = 0$.

This may be written $dx + x(ydx + xdy) = 0$; which suggests the factor $1/x$.

$$\therefore \frac{dx}{x} + ydx + xdy = 0, \text{ or } \log x + xy = c.$$

EXAMPLES LXXIII.

1. Solve :—

(1) $(2x - 5y)dy + (3x + 2y)dx = 0$.

(2) $2(x - 3y + 1)\dfrac{dy}{dx} = 4x - 2y + 1$.

(3) $xdy + ydx = 3x^2dx$.

(4) $\dfrac{dx}{x} - \dfrac{dy}{y} = xdx + ydy$.

(5) $\dfrac{xdy + ydx}{xy} = \dfrac{2xdx}{1 + x^2}$.

(6) $\dfrac{dy}{dx} = \dfrac{x^2 - ay}{ax - y^2}$.

(7) $y(4x - y - 1)dx + x(2x - 2y - 1)dy = 0$.

(8) $\{\sin y + (1 - y)\cos x\}dx + \{(1 + x)\cos y - \sin x\}dy = 0$.

(9) $\dfrac{xdx}{\sqrt{x^2 + y^2}} + \left\{1 + \dfrac{y}{\sqrt{x^2 + y^2}}\right\}dy = 0$.

2. Solve, using an integrating factor :—

(1) $xdy + 2ydx = 0$.

(2) $xdy + 3ydx = 0$.

(3) $xdy + (2y + 3x)dx = 0$.

(4) $xdy - ydx = x^2dx$.

(5) $xdy - ydx = y^2dx$.

(6) $xdy + ydx = 2x^3y^2dx$.

(7) $y^2(ydx + xdy) + ydx - xdy = 0$.

(8) $(x^2 \cos x - y)dx + xdy = 0$.

(9) $y\sqrt{1 - y^2}\,dx + (x\sqrt{1 - y^2} + y)dy = 0$.

(10) $\sqrt{y}(\sqrt{x} + \sqrt{y})dx + \sqrt{x}(\sqrt{x} - \sqrt{y})dy = 0$.

(11) $y(1 + y + x^2)dx = x(1 + x^2)dy$.

(12) $(x - x^2y)\dfrac{dy}{dx} + y + xy^2 = 0$.

ANSWERS.

NOTE.—The expressions in square brackets [ ] are integrating factors.

1. (1) $3x^2 + 4xy - 5y^2 = c^2$. (2) $2x^2 - 2xy + 3y^2 + x - 2y = c$.
(3) $xy = x^3 + c$. (4) $\log(x/y) = \frac{1}{2}(x^2 + y^2) + c$. (5) $xy = c(1 + x^2)$.
(6) $x^3 + y^3 - 3axy = c^3$. (7) $xy(2x - y - 1) = c$.
(8) $(1 + x)\sin y + (1 - y)\sin x = c$. (9) $y + \sqrt{x^2 + y^2} = c$.

2. (1) $x^2y = c$. (2) $x^3y = c$. (3) $x^2(x + y) = c^3$.
(4) $y = x(x + c)$, $[1/x^2]$. (5) $x + y(x - c) = 0$, $[1/y^2]$.
(6) $-\dfrac{1}{xy} = x^2 + c$, $\left[\dfrac{1}{x^2y^2}\right]$. (7) $x(y^2 + 1) = cy$, $\left[\dfrac{1}{y^2}\right]$.
(8) $x \sin x + y = cx$, $[1/x^2]$. (9) $xy = \sqrt{1 - y^2} + c$, $[1/\sqrt{1 - y^2}]$.
(10) $2\sqrt{xy} + x - y = c$, $[1/\sqrt{xy}]$. (11) $y \tan^{-1}x + x = cy$, $[1/y^2(1 + x^2)]$.
(12) $\dfrac{d(xy)}{x^2y^2} + \dfrac{dx}{x} - \dfrac{dy}{y} = 0$; whence $\log\dfrac{x}{y} = \dfrac{1}{xy} + c$.

487. (4) The Linear Equation of the First Order.—Referring to the definition of Art. 474, it will be seen that the most general form of the linear equation of the first order is

$$\frac{dy}{dx} + f(x) \cdot y = \phi(x).$$

To simplify the notation let $f(x) = P$, $\phi(x) = Q$; then the equation is

$$\frac{dy}{dx} + Py = Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

P and Q being functions of x.

The equation may be solved by means of the integrating factor $e^{\int P dx}$. Thus, multiplying (1) by this factor,

$$e^{\int P dx} \cdot \frac{dy}{dx} + P \cdot e^{\int P dx} \cdot y = Q \cdot e^{\int P dx}.$$

The left-hand side is an exact differential coefficient, and the right-hand side is a function of x purely.

Hence, by integration, we have

$$e^{\int P dx} y \dagger = \int Q e^{\int P dx} . dx + C,$$

which is the general form of the solution. This, of course, assumes that the integral on the right can be obtained; but in any case it is called the solution.

Ex. 1. *Solve* $x^2 \dfrac{dy}{dx} + xy = -1$.

We have $\dfrac{dy}{dx} + \dfrac{y}{x} = -\dfrac{1}{x^2}$.

The integrating factor is $e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

$$\therefore\ x\frac{dy}{dx} + y = -\frac{1}{x};\ \text{which gives, on integration,}$$

$$xy = -\log x + C,\quad \text{or } xy + \log x = C.$$

This example is the same as Ex. 3 of the last article.

Ex. 2. *Solve* $\dfrac{dy}{dx} + y \cot x = \sec^2 x$.

The integrating factor is $e^{\int \cot x dx} = e^{\log \sin x} = \sin x$.

The equation therefore becomes

$$\sin x \frac{dy}{dx} + y \cos x = \sin x \sec^2 x.$$

Integrating, $y \sin x = \int \sin x \sec^2 x dx = \sec x + C$, the required integral.

Ex. 3. $(a^2 - x^2)\dfrac{dy}{dx} + ay + x(a - x) = 0$.

We have $\dfrac{dy}{dx} + \dfrac{a}{a^2 - x^2} y = -\dfrac{x}{a + x}$.

Since $\displaystyle\int \frac{a dx}{a^2 - x^2} = \frac{1}{2} \log \frac{a + x}{a - x} = \log \sqrt{\frac{a + x}{a - x}}$,

$$\therefore\ \text{the integrating factor is } \sqrt{\frac{a+x}{a-x}};$$

$$\therefore\ \sqrt{\frac{a+x}{a-x}} \cdot \frac{dy}{dx} + \sqrt{\frac{a+x}{a-x}} \cdot \frac{ay}{a^2 - x^2} = -\frac{x}{\sqrt{a^2 - x^2}}.$$

---

† This can be verified by differentiating $e^{\int P dx} y$, and noting that $\dfrac{d}{dx}[\int P dx] = P$. Note that the left-hand side of the integral is equal to *y multiplied by the integrating factor*; this will often save labour in working examples. See Ex. 3, below.

Integrating, $y\sqrt{\dfrac{a+x}{a-x}} = \sqrt{a^2 - x^2} + C,$ [see footnote above]

or $$y\sqrt{a+x} = (a-x)\sqrt{a+x} + C\sqrt{a-x},$$

which reduces to $(x+y-a)^2(a+x) = C^2(a-x)$.

488. Kindred Form.

The equation $\dfrac{dy}{dx} + Py = Qy^n$, P and Q being functions of x alone, can be solved as follows :—

We have $$\frac{1}{y^n}\cdot\frac{dy}{dx} + P\cdot\frac{1}{y^{n-1}} = Q \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now let $\dfrac{1}{y^{n-1}} = z$; whence $-\dfrac{n-1}{y^n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

Using this in (1),

$$-\frac{1}{n-1}\frac{dz}{dx} + Pz = Q\ ; \text{ or } \frac{dz}{dx} + (1-n)Pz = (1-n)Q,$$

which is evidently reduced to the linear form.

Ex. *Solve* $\dfrac{dy}{dx} + \dfrac{y}{x} = y^3x^2$.

Dividing by y^3, $$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y^2} = x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put $\dfrac{1}{y^2} = z$, $\therefore\ -\dfrac{2}{y^3}\dfrac{dy}{dx} = \dfrac{dz}{dx}$;

$\therefore$ in (1) $$-\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}\cdot z = x^2,$$

or $$\frac{dz}{dx} - \frac{2}{x}z = -2x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since $\displaystyle\int -\frac{2}{x}dx = -2\log x = \log\frac{1}{x^2}$, the integrating factor is $\dfrac{1}{x^2}$. Hence in (2)

$$\frac{1}{x^2}\frac{dz}{dx} - \frac{2}{x^3}z = -2.$$

Integrating, $$z/x^2 = -2x + C,$$

that is, $$\frac{1}{x^2y^2} + 2x = C.$$

EXAMPLES LXXIV.

1. Solve :—

(1) $\dfrac{dy}{dx} + \dfrac{y}{x} = 4(1 + x^2)$.

(2) $x(1 + x)\dfrac{dy}{dx} + xy = a^2$.

(3) $(1 + x^2)\dfrac{dy}{dx} + 2xy = \cos x$.

(4) $x\dfrac{dy}{dx} - 2y = x^4\sqrt{1 + x^2}$.

(5) $(1 - x^2)\dfrac{dy}{dx} - xy = a + bx$.

(6) $\dfrac{dy}{dx} = x - y$.

(7) $\dfrac{dy}{dx} - ay = \sin x$.

(8) $\dfrac{dy}{dx} + y\tan x = 1$.

(9) $x(x + 1)\dfrac{dy}{dx} - (2x + 1)y = 2x^4(x + 1)$.

(10) $\sin 2x\dfrac{dy}{dx} + 2y = \cos 2x$.

2. Solve :—

(1) $x\dfrac{dy}{dx} - y = 3x^2y^2$.

(2) $\dfrac{dy}{dx} + y = xy^3$.

(3) $2\dfrac{dy}{dx} + y\tan x = y^3\cos 2x$.

(4) $2\dfrac{dy}{dx} - y\sec x = y^3\tan x$.

(5) $6(x + 1)dy = y(1 - y^3)dx$.

(6) $1 - x\dfrac{dy}{dx} = x^2\dfrac{dy}{dx}$.

(7) $dx = x\sin y(1 - x\cos y)dy$.

ANSWERS.

1. (1) $xy = 2x^2 + x^4 + c$. (2) $(1 + x)y = a^2\log x + c$.

(3) $y(1 + x^2) = \sin x + c$. (4) $3y = x^2(1 + x^2)^{\frac{3}{2}} + cx^2$.

(5) $(y + b)\sqrt{1 - x^2} = a\sin^{-1}x + c$. (6) $y - x + 1 = ce^{-x}$.

(7) $(a^2+1)y + a\sin x + \cos x = ce^{ax}$. (8) $y\sec x = \log(\sec x + \tan x) + c$.

(9) $y = x(x + 1)\{x^2 - 2x + 2\log(x + 1) + c\}$.

(10) $(2y + 1)\tan x = 2x + c$.

2. (1) $x + x^3y = cy$. (2) $\dfrac{1}{y^2} - x - \frac{1}{2} = ce^{2x}$.

(3) $3\cos x = y^2(2\sin^3 x - 3\sin x + c)$.

(4) $(\tan x + \sec x)(y^2 + 1) = (x + c)y^2$. (5) $\sqrt{x+1}\,(1 - y^3) = cy^3$.

(6) $(1 + x)e^y = cx$, [use dx/dy]. (7) $1 - x - x\cos y = cxe^{\cos y}$, [use dx/dy].

CHAPTER XXXI.

EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE.

489. We shall now consider the methods which we may adopt for solving the equation $f(x, y, dy/dx) = 0$, in which dy/dx, or p as we shall now write it, occurs in a degree higher than the first. There are three methods, according as we solve for p, y, or x.

490. I. Solve for p.

If the equation is of the nth degree in p, there will be n real or imaginary roots of the form

$$p = \phi(x, y).$$

To solve each of the latter equations we must adopt where possible the methods given in the last chapter.

Ex. 1. *Solve* $xy(p^2 + 1) - (x^2 + y^2)p = 0$.

This factorizes into $(xp - y)(yp - x) = 0$.

$$\therefore \text{(i) } x\frac{dy}{dx} - y = 0;\ \text{or } \frac{dy}{y} = \frac{dx}{x};\ \text{whence } y = cx.$$

$$\text{(ii) } y\frac{dy}{dx} - x = 0;\ \text{or } ydy = xdx;\ \text{whence } y^2 = x^2 + c^2.$$

Ex. 2. *Solve* $p^2y + 2px = y$.

Solving for p, $$p = \frac{-x \pm \sqrt{x^2 + y^2}}{y},$$

or $$ydy = (-x \pm \sqrt{x^2 + y^2})dx,$$

which is a pair of homogeneous equations.

Putting $y = vx$, and taking the two cases in one, we have, after dividing by x,

$$v(vdx + xdv) = (-1 \pm \sqrt{1 + v^2})dx,$$

or

$$\frac{dx}{x} + \frac{vdv}{(1 + v^2) \mp \sqrt{1 + v^2}} = 0.$$

Now $\dfrac{v}{1 + v^2 \mp \sqrt{1 + v^2}} = \dfrac{v}{\sqrt{1+v^2}(\sqrt{1+v^2} \mp 1)} = \dfrac{v(\sqrt{1 + v^2} \pm 1)}{\sqrt{1 + v^2}(1 + v^2 - 1)}$

$$= \frac{\sqrt{1 + v^2} \pm 1}{v\sqrt{1 + v^2}} = \frac{1}{v} \pm \frac{1}{v\sqrt{1 + v^2}}.$$

$$\therefore \frac{dx}{x} + \frac{dv}{v} \pm \frac{dv}{v\sqrt{1 + v^2}} = 0.$$

Let $1 + v^2 = z^2$, $\therefore vdv = zdz$.

$$\therefore \int \frac{dv}{v\sqrt{1 + v^2}} = \int \frac{zdz}{(z^2 - 1)z} = \int \frac{dz}{z^2 - 1}$$

$$= \tfrac{1}{2}\log\frac{z - 1}{z + 1} = \tfrac{1}{2}\log\frac{(z - 1)^2}{z^2 - 1} = \tfrac{1}{2}\log\frac{(\sqrt{1 + v^2} - 1)^2}{v^2} = \log\frac{\sqrt{1 + v^2} - 1}{v}.$$

$$\therefore \log x + \log v \pm \log\frac{\sqrt{1 + v^2} - 1}{v} = C.$$

Hence (i) $x \cdot \dfrac{y}{x} \cdot \dfrac{\sqrt{x^2 + y^2} - x}{y} = c,$

that is

$$\sqrt{x^2 + y^2} - x = c;$$

or (ii) $x \cdot \dfrac{y}{x} \cdot \dfrac{y}{\sqrt{x^2 + y^2} - x} = c.$

$$\therefore y^2 = c(\sqrt{x^2 + y^2} - x);$$

$$\therefore y^2(\sqrt{x^2 + y^2} + x) = c(x^2 + y^2 - x^2) = cy^2.$$

$$\therefore \sqrt{x^2 + y^2} + x = c.$$

The two solutions are, therefore, given by

$$\sqrt{x^2 + y^2} \pm x = c;$$

or

$$x^2 + y^2 = (c \pm x)^2 = c^2 + x^2 \pm 2cx;$$

or

$$y^2 = c^2 \pm 2cx.$$

Both solutions, however, are included in the equation

$$y^2 = c^2 + 2cx,$$

since c may be +ve or —ve. [See Ex. 1, Art. 492.]

491. II. Solve for y.

Let $y = \phi(x, p)$ be one of the solutions . . . . (1)

Differentiate with respect to x; then

$$\frac{dy}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial p}\cdot\frac{dp}{dx}, \text{ which may be written}$$

$$p = \phi_1(x, p) + \phi_2(x, p)\cdot\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad (2)$$

This equation is of the first order and degree in p and x, and may therefore be solved by means of the preceding chapter.

Let the solution be $\chi(x, p) = 0$ . . . . (3)

If we eliminate p between (1) and (3) we shall obtain the required integral.

NOTE.—When the elimination of p is troublesome, we may express x and y each in terms of p, and call this the solution.

Ex. *Solve* $y = x^2 - \frac{1}{2}p^2$.

Differentiate with respect to x; then

$$p = 2x - p\frac{dp}{dx},$$

or
$$(p - 2x)dx + pdp = 0, \text{ which is homogeneous.}$$

Put $p = vx$.

$$\therefore (v - 2)dx + v(vdx + xdv) = 0;$$

or
$$(v^2 + v - 2)dx + xvdv = 0,$$

$$\frac{dx}{x} + \frac{vdv}{(v + 2)(v - 1)} = 0;$$

$$\therefore \log x + \tfrac{1}{3}\int\left(\frac{1}{v - 1} + \frac{2}{v + 2}\right)dv = 0.$$

$$\therefore \log x + \tfrac{1}{3}\log(v - 1)(v + 2)^2 = c',$$

$$3\log x + \log\frac{(p - x)(p + 2x)^2}{x^3} = 3c';$$

or
$$(p - x)(p + 2x)^2 = c, \text{ say.}$$

From the given equation $p = \sqrt{2}\,.\sqrt{x^2 - y}$.

$\therefore$ the general integral is

$$(\sqrt{2}\sqrt{x^2 - y} - x)(\sqrt{2}\sqrt{x^2 - y} + 2x)^2 = c,$$

which reduces to $\quad x(x^2 - 3y) + \sqrt{2}(x^2 - y)^{\frac{3}{2}} = c;$

and, again, on rationalizing, to

$$(x^2 + y)^2(x^2 - 2y) + 2x(x^2 - 3y)c = c^2.$$

492. III. Solve for x.

Let $x = f(y, p)$ be one of the solutions.

Differentiate with respect to y, and note that $dx/dy = 1/p$; then

$$\frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dy}.$$

This equation is of the first order and degree in p and y, and may be solved by means of the preceding chapter.

Eliminate p between the latter solution and the original equation; the resulting equation will be the general integral.

Ex. 1. *Solve* $p^2y + 2px = y$ . . . . (1)

We have $x = \frac{(1-p^2)y}{2p}$; and, differentiating with respect to y,

$$\frac{1}{p} = \frac{1-p^2}{2p} + \frac{-2p^2-1+p^2}{2p^2}y\frac{dp}{dy};$$

$$\therefore 2p = p - p^3 - (1+p^2)y\frac{dp}{dy};$$

$$p(1+p^2) + (1+p^2)y\frac{dp}{dy} = 0;$$

$$\therefore pdy + ydp = 0,$$

whence $$py = c.$$

$\therefore$ in (1), eliminating p,

$$\frac{c^2}{y} + \frac{2cx}{y} = y,$$

or $$y^2 = c^2 + 2cx.$$ [See Ex. 2, Art. 490.]

Ex. 2. *Solve* $2x = 3p^2$.

Here y is absent, but the method is the same as above.

Thus $$\frac{2}{p} = 6p\frac{dp}{dy}; \quad \text{or } dy = 3p^2dp;$$

$$\therefore y = p^3 + c.$$

Eliminating p between this and $2x = 3p^2$, we have

$27(y-c)^2 = 8x^3$, the general integral.

493. Clairaut's Form.

The equation $$y = px + f(p) \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$
is called by this name.

To solve it, differentiate with respect to x; then

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx},$$

or
$$\frac{dp}{dx}\{x + f'(p)\} = 0.$$

$\therefore$ (i) $\dfrac{dp}{dx} = 0$, or $p = c$; and therefore in (1)

$$y = cx + f(c), \text{ which is the general integral.}$$

Or, (ii) $x + f'(p) = 0$;

and eliminating p between this and (1), we obtain an equation *without an arbitrary constant*, since in this case no integration has been performed. This equation is called the *singular solution.*

Ex. *Solve* $y = px + \dfrac{a}{p}$ . . . . (2)

We have
$$p = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx};$$

$$\therefore \frac{dp}{dx}\left(x - \frac{a}{p^2}\right) = 0.$$

$\therefore$ (i) $\dfrac{dp}{dx} = 0$, or $p = c$;

$$\therefore\ y = cx + \frac{a}{c} \text{ is the general integral.}$$

(ii) $x - \dfrac{a}{p^2} = 0$, or $p = \sqrt{\dfrac{a}{x}}$;

$$\therefore \text{ in (2) } y = \sqrt{ax} + \sqrt{ax} = 2\sqrt{ax}; \quad \text{or } y^2 = 4ax,$$

which is the singular solution.

494. Geometrical Interpretation—The Envelope.

The singular solution in Clairaut's equation, as we have seen, is the equation which is obtained by eliminating p between the equations

$$y = px + f(p), \text{ and } x + f'(p) = 0 \quad . \quad . \quad . \quad (1)$$

We shall now show that this equation in x and y is that of the

envelope of the general integral $y = cx + f(c)$, which represents a family of straight lines.

For, differentiating the latter with respect to c, we get

$$0 = x + f'(c).$$

The envelope is therefore obtained by eliminating c between the equations

$$y = cx + f(c), \text{ and } x + f'(c) = 0 \quad . \quad . \quad . \quad (2)$$

Comparing (1) and (2), it will be seen that we must obtain the same eliminant in either case; for it is immaterial whether we use the letter c or p, since in either case it has to be eliminated.

Ex. In the example of Art. 493, the envelope of the line

$$y = cx + \frac{a}{c}$$

is the parabola $y^2 = 4ax$. This is apparent from the well-known fact that the tangent to the parabola, $y^2 = 4ax$, at any point may be written in the form $y = mx + \dfrac{a}{m}$.

495. The Singular Solution satisfies the Differential Equation.

Consider the singular solution from a geometrical point of view. The envelope, which it represents, touches every member of the family of straight lines $y = cx + f(c)$; and therefore at every point of the envelope the value of dy/dx, or p, is the same as for the corresponding member which touches it at that point.

But if we solve algebraically the equation

$$y = px + f(p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and so obtain

$$p = \phi(x, y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

then, for any given point (x, y), p has a perfectly definite value.†

Hence, at every point of the envelope, the value of p as found

† For (2) contains no *arbitrary* constants. Of course p may have more than one value, according to the degree of the equation in p; but we should in that case consider one root at a time.

from the equation to the envelope (since it is the same as for the tangent line at that point) must be such as to satisfy (2), and therefore (1).

Ex. In the example of Art. 493 the singular solution is $y^2 = 4ax$. To show that we can obtain therefrom the differential equation $y = px + \dfrac{a}{p}$; we have, by differentiation,

$$2yp = 4a, \text{ or } y = 2a/p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\therefore y^2 = 2ay/p, \text{ or } 4ax = 2ay/p, \text{ or } y = 2px \quad . \quad . \quad \text{(ii)}$$

Adding (i) and (ii), and halving, we obtain $y = px + \dfrac{a}{p}$.

496. We may add that other differential equations besides Clairaut's form have singular solutions, but we cannot discuss them here.

497. Kindred Form.

The equation $y = xf(p) + \phi(p)$ may be solved by differentiation with respect to x, as in the case of Clairaut's equation, which is in reality a particular form of this more general equation.

Thus, $$p = f(p) + \{xf'(p) + \phi'(p)\}\frac{dp}{dx}.$$

$$\therefore \{p - f(p)\}\frac{dx}{dp} - xf'(p) = \phi'(p),$$

or $$\frac{dx}{dp} + \frac{f'(p)}{f(p) - p}\cdot x = \frac{\phi'(p)}{p - f(p)},$$

which is linear in x.

Ex. *Solve* $y = (2p + 1)x + \dfrac{1}{p+1}$ . . . . (1)

We have $$p = 2p + 1 + \left\{2x - \frac{1}{(p+1)^2}\right\}\frac{dp}{dx};$$

$$\therefore (p+1)\frac{dx}{dp} + 2x = \frac{1}{(p+1)^2};$$

or $$\frac{dx}{dp} + \frac{2}{p+1}x = \frac{1}{(p+1)^3}.$$

Since $\int \frac{2dp}{p+1} = 2\log(p+1)$, the integrating factor is $(p+1)^2$.

$$\therefore (p+1)^2 x = \int \frac{dp}{p+1} = \log(p+1) + c,$$

which gives x in terms of p.

From (1), y can be obtained in terms of p, and these two equations constitute the solution.

EXAMPLES LXXV.

1. Solve by method I.:—

(1) $(2x+1)p^2 = 1$.
(2) $p^2 + yp = 2y^2$.
(3) $p^2 + (y-x)p = xy$.
(4) $(1-p^2)xy = (x^2-y^2)p$.
(5) $xp^2 + py = x - y$.
(6) $yp^2 + 2xp = 4(x+y)$.
(7) $y(1+y^2)(p^2-2) = (1+y^4)p$.

2. Find, by method II., the general integrals of:—

(1) $y = p + p^2$.
(2) $y = x + p^2$.
(3) $y + p^2 - 5px + 5x^2 = 0$.
(4) $x + 2py = p^2x$.
(5) $2x^3y = px^4 + p^3$.
(6) $p^2x^3 + x^2py = 1$.
(7) $y^2(1+p^2)^3 = a^2p^6$.

3. Find, by method III., the general integrals of:—

(1) $x(p-1)^2 = a$.
(2) $x = 2y - 3p^2$.
(3) $p^2x = 2(1+p)y$.
(4) $y - 2px = 4yp^2$.
(5) $p^2(3y^2+2x) = 2(3py-1)$.

4. Find the general integral, and the singular solution of:—

(1) $y = px + p^2$.
(2) $y = px + 2\sqrt{p}$.
(3) $y = px - \sin p$.
(4) $1 + yp = xp^2$.
(5) $x^2p^2 - 2(xy+1)p + y^2 = 0$.
(6) $y = px + a\sqrt{1+p^2}$.
(7) $p^2(x^2-a^2) - 2pxy + y^2 + a^2 = 0$.

5. Solve by the method of Art. 497:—

(1) $y = 2px + p^2$.
(2) $y = p^2x + p^3$.
(3) $y = p^2x + \sqrt{1-p^2}$.
(4) $9(y + xp\log p) = (2 + 3\log p)p^3$.
(5) $y = \frac{x}{p} + p$.

6. Solve by any method :—

(1) $y = x\sqrt{p} + \sqrt{p}$.
(2) $x^2 - 2px + p^2y^2 = 0$.
(3) $x(1 + p^2) = 2a$.
(4) $(y - px)(p + 1) = 1$.
(5) $xp^2 + (y - x)p = y$.
(6) $y = xp(1 + p)$.
(7) $y = p^2x + p$.
(8) $(p^2 + 2yp)(1 - x^2) = x^2y^2$.
(9) $x^2p(p + 3) = (y + x)(y - 2x)$.
(10) $p^2x^2 + px(2y + x) + y^2 = 0$.

ANSWERS.

1. (1) $(y - c)^2 = 2x + 1$. (2) $\log y + 2x = c$; $\log y = x + c$.
(3) $\log y = -x + c$; $2y - x^2 = c$. (4) $y = cx$; $x^2 + y^2 = c^2$.
(5) $2xy - x^2 = c^2$; $y + x = c$.
(6) $\log(y^2 + 2xy + 2x^2) = 2\tan^{-1}\dfrac{y + x}{x} + c$; $y = 2x + c$.
(7) $4x + 2\log y + y^2 + c = 0$; $\log(1 + y^2) = 2x + c$.

2. (1) $x = \log p + 2p + c$. (2) $x = 2p + 2\log(p - 1) + c$.
(3) $y + cx + c^2 = x^2$. (4) $1 + 2cy = c^2x^2$. (5) $2y = cx^2 + c^3$.
(6) $x = cxy + c^2$.
(7) $x = -a/(1 + p^2)^{\frac{3}{2}} + c$, $y = ap^3/(1 + p^2)^{\frac{3}{2}}$; putting $p = \tan\psi$, we obtain $x - c = -a\cos^3\psi$; $y = a\sin^3\psi$, the four-cusped hypocycloid.

3. (1) $(y - x - c)^2 = 4ax$. (2) $y = \frac{3}{2}(p^2 + p) + \frac{3}{4}\log(2p - 1)$.
(3) $(x - c)^2 = 2cy$. (4) $y^2 - cx = c^2$. (5) $x + cy + c^2 = \frac{1}{2}y^2$.

4. (1) $y = cx + c^2$; $x^2 + 4y = 0$. (2) $xy + 1 = 0$.
(3) $y = x\cos^{-1}x - \sqrt{1 - x^2}$. (4) $y^2 + 4x = 0$.
(5) $2xy + 1 = 0$. (6) $x^2 + y^2 = a^2$. (7) $x^2 - y^2 = a^2$.

5. (1) $p^2x = -\frac{2}{3}p^3 + c$. (2) $(1 - p)^2x = \frac{3}{2}p^2 - p^3 + c$.
(3) $(1 - p)^2x + \sin^{-1}p + \sqrt{1 - p^2} = c$. (4) $3px = p^3 + c$.
(5) $\sqrt{p^2 - 1}\,x = p(\cosh^{-1}p + c)$.

6. (1) $\dfrac{1}{y} = \dfrac{1}{x+1} + c.$ (2) $x^2 = 2y \pm (y\sqrt{1-y^2} + \sin^{-1} y) + c.$

(3) $y = \sqrt{2ax - x^2} + a \sin^{-1}\dfrac{x-a}{x} + c$; or, $x = a(1 + \sin\phi)$,

$y = a(\phi + \cos\phi)$, the cycloid.

(4) $(y + x)^2 = 4x.$ (5) $xy = c^2$; $y = x + c.$

(6) $\log p^2x = (1 + pc)/p.$ (7) $(p-1)^2x + p - \log p = c.$

(8) $\log y' = -x \pm \sin^{-1} x + c.$ (9) $2xy + x^2 = c$; $y + 2x\log x = cx.$

(10) $xy = x^2 + c^2.$

CHAPTER XXXII.

EQUATIONS OF THE SECOND ORDER.

498. In this chapter we shall consider the various special forms of the equation $f(y_2, y_1, y, x) = 0$; we shall also show under what circumstances the general linear equation of the second order can be solved.

499. Form $f(y_2, x) = 0$.

Solving the equation algebraically for y_2 or d^2y/dx^2, suppose we obtain $d^2y/dx^2 = \phi(x)$.

We can then integrate twice, and so obtain the general integral.

Ex. *Solve* $\sqrt{1-x^2}\,y_2 + x = 0$.

We have $\dfrac{d^2y}{dx^2} = -\dfrac{x}{\sqrt{1-x^2}}$;

$$\therefore \frac{dy}{dx} = -\int\frac{xdx}{\sqrt{1-x^2}} = \sqrt{1-x^2} + c_1.$$

$$\therefore y = \int(\sqrt{1-x^2} + c_1)dx = \tfrac{1}{2}x\sqrt{1-x^2} + \tfrac{1}{2}\sin^{-1}x + c_1x + c_2$$

the general integral.

500. Form $f(y_2, y) = 0$.

Since $y_2 = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dy}\left(\dfrac{dy}{dx}\right).\dfrac{dy}{dx} = \dfrac{dp}{dy}.p$, the equation may be written $f\left(p\dfrac{dp}{dy}, y\right) = 0$; and solving algebraically we may obtain $p\dfrac{dp}{dy} = \phi(y)$.

Hence, integrating, $p^2 = 2\int\phi(y)dy + C.$

This gives p or dy/dx in terms of y, and the integral can then be easily found.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2} + a^2y = 0$ . . . . (1)

We have $p\dfrac{dp}{dy} + a^2y = 0$, or $pdp + a^2ydy = 0$.

$$\therefore\ p^2 + a^2y^2 = c_1 = a^2c^2, \text{ say,}\dagger$$

$$\therefore\ p^2 = a^2(c^2 - y^2);$$

$$\therefore\ \frac{dy}{dx} = a\sqrt{c^2 - y^2}, \text{ or } dx = \frac{dy}{a\sqrt{c^2 - y^2}};$$

$$\therefore\ x = \frac{1}{a}\left(\sin^{-1}\frac{y}{c} + \alpha\right), \ \frac{\alpha}{a} \text{ being an arbitrary constant;}$$

$$\therefore\ y = c\sin(ax - \alpha),$$

which is the general integral, c and α being arbitrary constants.

We may also write the equation in the form

$$y = c(\sin ax\cos\alpha - \cos ax\sin\alpha)$$
$$= A\sin ax + B\cos ax,$$

where $\qquad A = c\cos\alpha,\ B = -c\sin\alpha.$

This equation and the following are of great importance in Dynamics.

Ex. 2. *Solve* $\dfrac{d^2y}{dx^2} = a^2y$.

We shall obtain, by the above method, $dx = \dfrac{dy}{a\sqrt{y^2 + c^2}}$.

$$\therefore\ x = \frac{1}{a}\left(\sinh^{-1}\frac{y}{c} + \alpha\right);$$

$$\therefore\ y/c = \sinh(ax - \alpha) = \tfrac{1}{2}(e^{ax-\alpha} - e^{-ax+\alpha}),$$

or $$y = \frac{c}{2}e^{-\alpha}\cdot e^{ax} - \frac{c}{2}e^{\alpha}\cdot e^{-ax} = Ae^{ax} + Be^{-ax},$$

where $$A = \frac{c}{2}e^{-\alpha},\ B = -\frac{c}{2}e^{\alpha}.$$

---

† Or, multiplying (1) by $2\dfrac{dy}{dx}$, we have

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + a^2\cdot 2y\frac{dy}{dx} = 0.$$

$$\therefore\ \text{integrating, } \left(\frac{dy}{dx}\right)^2 + a^2y^2 = c_1;\ \text{etc.}$$

Since $\sinh(ax - \alpha) = \sinh ax \cosh\alpha - \cosh ax \sinh\alpha$, the general integral may be written

$$y = A \sinh ax + B \cosh ax.$$

501. Form $f(y_2, y_1) = 0$.

This is evidently expressible in the form $f(dp/dx, p) = 0$; and solving for dp/dx in terms of p, we can find; by integration, x in terms of p. We then solve algebraically for p in terms of x, and integrate.

Ex. *Solve* $(d^2y/dx^2)^2 - 1 = 1/(dy/dx)^2$.

We have $\left(\dfrac{dp}{dx}\right)^2 = 1 + \dfrac{1}{p^2} = \dfrac{1+p^2}{p^2}$.

$$\therefore dx = \frac{p\,dp}{\sqrt{1+p^2}};$$

$$\therefore x = \sqrt{1+p^2} + c_1;$$

$$\therefore 1 + p^2 = (x - c_1)^2,$$

whence

$$p = \sqrt{(x-c_1)^2 - 1};$$

$$\therefore y = \int \sqrt{(x-c_1)^2 - 1}\,dx$$

$$= \frac{(x-c_1)\sqrt{(x-c_1)^2 - 1}}{2} - \tfrac{1}{2}\cosh^{-1}(x - c_1) + c_2.$$

502. Form $f(y_2, y_1, x) = 0$, y being absent.

This may be written $f(dp/dx, p, x) = 0$, which is of the first order in p and x, and may therefore be solved by the methods of the preceding chapters; p will then be obtained in terms of x, and the rest of the work is as indicated above.

Ex. *Solve* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2x = 0$.

We have

$$x\frac{dp}{dx} + p - 2x = 0,$$

or

$$(xdp + pdx) - 2xdx = 0.$$

$$\therefore px - x^2 = c_1;$$

$$\therefore p \equiv \frac{dy}{dx} = \frac{c_1 + x^2}{x} = \frac{c_1}{x} + x;$$

$$\therefore y = c_1 \log x + \tfrac{1}{2}x^2 + c_2.$$

503. Form $f(y_2, y_1, y) = 0$, x being absent.

This may be written $f\left(p\dfrac{dp}{dy}, p, y\right) = 0$, which is of the first order in p and y.

Ex. *Solve* $y\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\left(\dfrac{dy}{dx} - 2y\right) = 0.$

We have
$$yp\frac{dp}{dy} + p(p - 2y) = 0.$$
or
$$y\frac{dp}{dy} + p - 2y = 0.$$
$$\therefore\ py - y^2 = c_1;$$
that is,
$$y\frac{dy}{dx} = y^2 + c_1;$$
$$\therefore\ dx = \frac{y\,dy}{y^2 + c_1};\quad \therefore\ x = \tfrac{1}{2}\log(y^2 + c_1) + c_2.$$

504. The Linear Equation of the Second Order.

The linear equation of the second order in its general form may be written
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
where P, Q, and R are functions of x.

Let $y = w$ be a solution of the equation
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$
found by inspection or otherwise.

Then we can obtain the general integral of (1) by putting $y = vw$, v and w being functions of x.

By differentiation, $\dfrac{dy}{dx} = v\dfrac{dw}{dx} + w\dfrac{dv}{dx}$,
$$\frac{d^2y}{dx^2} = v\frac{d^2w}{dx^2} + 2\frac{dv}{dx}\cdot\frac{dw}{dx} + w\frac{d^2v}{dx^2}.$$

Substituting these values in (1), we obtain
$$v\left\{\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Qw\right\} + \frac{dv}{dx}\left\{2\frac{dw}{dx} + Pw\right\} + w\frac{d^2v}{dx^2} = R.$$

But by hypothesis $\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Qw = 0$; hence

$$w\frac{d^2v}{dx^2} + \left\{2\frac{dw}{dx} + Pw\right\}\frac{dv}{dx} = R;$$

or, putting p for dv/dx, and remembering that w is a *known* function of x, so that dw/dx is known,

$$w\frac{dp}{dx} + \left\{2\frac{dw}{dx} + Pw\right\}p = R,$$

which is linear in p, and can therefore be solved by the ordinary rule.

Thus
$$\frac{dp}{dx} + \left\{\frac{2}{w}\frac{dw}{dx} + P\right\}p = \frac{R}{w}.$$

Since $\int\left(\frac{2}{w}\frac{dw}{dx} + P\right)dx = 2\log w + \int P dx$, the integrating factor is $w^2 e^{\int P dx}$; and the solution is

$$pw^2 e^{\int P dx} = \int Rwe^{\int P dx}\, dx + c.$$

Hence p or dv/dx is found in terms of x; and, on integration, v can also be found.

Ex. *Solve* $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{1}{2}x^4$ . . . . (i)

Since $y = x$ is a solution of the equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$, as may be seen by inspection, put $y = vx$,

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}, \text{ and } \frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2};$$

$$\therefore \text{ in (i), } x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + x\left(v + x\frac{dv}{dx}\right) - vx = \tfrac{1}{2}x^4;$$

or, putting p for $\frac{dv}{dx}$,

$$x\frac{dp}{dx} + (x^2 + 2)p = \tfrac{1}{2}x^4,$$

or
$$\frac{dp}{dx} + \left(x + \frac{2}{x}\right)p = \tfrac{1}{2}x^3.$$

Since $\int\left(x + \frac{2}{x}\right)dx = \frac{1}{2}x^2 + 2\log x$, the integrating factor is, $x^2e^{\frac{1}{2}x^2}$.

Hence $px^2e^{\frac{1}{2}x^2} = \frac{1}{2}\int x^5 e^{\frac{1}{2}x^2}\,dx = 2\int z^2 e^z dz$, if $z = \frac{1}{2}x^2$,

$$= 2z^2e^z - 4\int ze^z dz = (2z^2 - 4z + 4)e^z + c$$

$$= (\tfrac{1}{2}x^4 - 2x^2 + 4)e^{\frac{1}{2}x^2} + c.$$

$$\therefore \frac{dv}{dx} = \frac{1}{2}\left(x^2 - 4 + \frac{8}{x^2}\right) + \frac{c}{x^2e^{\frac{1}{2}x^2}},$$

whence $v \equiv \dfrac{y}{x} = \frac{1}{6}x^3 - 2x - \dfrac{4}{x} + c\displaystyle\int \frac{dx}{x^2e^{\frac{1}{2}x^2}} + c_1.$

∴ the general integral is

$$y = \tfrac{1}{6}x^4 - 2x^2 - 4 + cx\int \frac{dx}{x^2e^{\frac{1}{2}x^2}} + c_1x.$$

The integral on the right cannot be evaluated in finite terms, but in any case the above is called the solution.

The case in which P and Q are constants will be considered in the next chapter.

EXAMPLES LXXVI.

Solve the equations :—

1. $\dfrac{d^2s}{dt^2} = a.$

2. $\dfrac{d^2s}{dt^2} + a\sin\omega t = 0.$

3. $(a + x)^2\dfrac{d^2y}{dx^2} = 1.$

4. $\dfrac{d^2y}{dx^2} + y = 0.$

5. $\dfrac{d^2x}{dt^2} = x.$

6. $\dfrac{d^2\theta}{dt^2} + a^2\theta = 0.$

7. $\dfrac{d^2s}{dt^2} = -ks.$

8. $y^3\dfrac{d^2y}{dx^2} + 1 = 0.$

9. $x\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}.$

10. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x - 1.$

11. $(1 + x)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + x = 0.$

12. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 1 = 0.$

13. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 1 = 0.$

14. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$

15. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0.$

16. $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 0.$

17. $\dfrac{d^4y}{dx^4} = a^2\dfrac{d^2y}{dx^2}.$

18. $\dfrac{d^2T}{d\phi^2} - T + a = 0.$

19. $1 + \left(\frac{dy}{dx}\right)^2 - \left(\frac{d^2y}{dx^2}\right)^2 = 0.$

20. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x.$

21. $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 1.$

22. $2y^2\frac{d^2y}{dx^2} = 1.$

23. $\frac{d^2y}{dx^2} = e^{2y}.$

24. $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = a.$

25. $(1 + y)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1.$

26. $\left(y + \frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$

27. $(x + 1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = e^x.$ [L.H. side vanishes when $y = e^x$.]

28. $x^2\frac{d^2y}{dx^2} + x^2\cot x\frac{dy}{dx} - 2y(1 + x\cot x) = x.$ [L.H. side vanishes when $y = x^2$.]

Answers.

1. $s = \frac{1}{2}at^2 + c_1t + c_2.$ **2.** $s = \frac{a}{\omega^2}\sin\omega t + c_1t + c_2.$ **3.** $y = cx - \log(a + x) + c_1.$

4. $y = A\cos x + B\sin x$, or $= a\sin(x + \alpha).$

5. $y = Ae^t + Be^{-t}$, or $= a\cosh(t + \alpha).$

6. $\theta = A\cos at + B\sin at = m\sin(at + \alpha).$ **7.** $s = A\cos\sqrt{k}t + B\sin\sqrt{k}t.$

8. $(x - c_1)^2 = c(y^2 + c).$ **9.** $y = \frac{1}{2}cx^2 + c_1.$ **10.** $y = c_1\log x - x + \frac{1}{4}x^2 + c_2.$

11. $y = c_1\log(1 + x) + \frac{1}{2}x - \frac{1}{4}x^2 + c_2.$ **12.** $y = c\log x - \frac{1}{2}(\log x)^2 + c_1.$

13. $y = \log\cos(c - x) + c_1.$ **14.** $y = c_1 + c_2e^{-x}.$ **15.** $y = c_1 + c_2x + c_3e^{-x}.$

16. $y = A\cos x + B\sin x + C.$ **17.** $y = Ae^{ax} + Be^{-ax} + Cx + D.$

18. $T = a + Ae^{\phi} + Be^{-\phi}.$ **19.** $y = \cosh(x - c_1) + c_2.$ **20.** $y = \frac{1}{2}e^x + ce^{-x} + c_1.$

21. $y = \frac{1}{2}\log(1 + x^2) + c_1\tan^{-1}x + c_2.$

22. $x = \frac{1}{c}\sqrt{cy^2 - y} + \frac{1}{2c\sqrt{c}}\cosh^{-1}(2cy - 1) + c_1.$

23. $e^{2x} + 2cc_1e^{cx-y} = c_1^2.$ **24.** $c^2(x - c_1)^2 = ay^2 + c.$

25. $x = \sqrt{(y + 1)^2 + c_1} + c_2.$ **26.** $2c^2x = y\sqrt{y^2 + c^2} + c^2\sinh^{-1}\frac{y}{c} + y^2 + c_1.$

27. $y = c_2e^x + e^x\int\frac{e^x + c_1}{e^x(x + 1)}dx.$ **28.** $y = c_2x^2 - \frac{1}{3x} + c_1x^2\int\frac{dx}{x^4\sin x} - x^2\int\frac{\cot x\,dx}{x^3}.$

CHAPTER XXXIII.

LINEAR EQUATION WITH CONSTANT COEFFICIENTS.

505. The general form of the linear equation with constant coefficients is

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_1\frac{dy}{dx}+a_0y=X,$$

where a_n, a_{n-1}, ... are constants and X is a function of x. There are two cases to be considered: first, the case in which X is zero; and, secondly, the general case.

We shall here be chiefly concerned with equations of the second order only, although the principles underlying their treatment will be seen to apply in most cases to equations of higher orders.

506. I. Equation of the Second Order, in which X is Zero.

Let the equation be $y_2+ay_1+by=0$. . . . . (1)

Assume $y=e^{mx}$; that such an assumption can be made will be seen as we proceed.

Then $$y_1=me^{mx};\quad y_2=m^2e^{mx},$$

$$\therefore \text{ in (1), } (m^2+am+b)e^{mx}=0,$$

or $$m^2+am+b=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a quadratic from which m can be found.

Let m_1, m_2 be the two values of m; then we have

$$y=e^{m_1x} \text{ or } y=e^{m_2x}$$

which are two independent particular integrals.

The *general* solution is $y = Ae^{m_1 x} + Be^{m_2 x}$ . . . . . . (3)
as we can easily show; for

$$y_1 = m_1 A e^{m_1 x} + m_2 B e^{m_2 x},$$
$$y_2 = m_1^2 A e^{m_1 x} + m_2^2 B e^{m_2 x}.$$

Substituting these values of y, y_1, y_2 in (1), we obtain

$$Ae^{m_1 x}(m_1^2 + am_1 + b) + Be^{m_2 x}(m_2^2 + am_2 + b) = 0,$$

which is identically true since m_1 and m_2 are the roots of (2); and therefore (3) is evidently *a* solution.

Moreover, (3) contains *two* arbitrary constants; hence it is the *general* solution.

It will be seen that equation (2) can be obtained from (1) by writing m^2 for y_2, m for y_1, and 1 for y.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0.$

The *auxiliary* quadratic is $m^2 + m - 2 = 0$,

or $\qquad (m - 1)(m + 2) = 0$, whence $m = 1$ or -2.

The general solution is therefore $y = Ae^x + Be^{-2x}$.

Ex. 2. *Solve* $\dfrac{d^2y}{dx^2} - a^2y = 0.$

The equation $m^2 - a^2 = 0$, gives $m = \pm a$.
The general solution is therefore $y = Ae^{ax} + Be^{-ax}$.

507. Case of Equal Roots.

If $m_1 = m_2$, then the general solution takes the form

$$y = Ae^{m_1 x} + Be^{m_1 x} = (A + B)e^{m_1 x} \quad . \quad . \quad . \quad (1)$$

Since $A + B$ is equivalent to only a single arbitrary constant, the solution does not appear to be general enough.

Suppose, however, we put $m_2 = m_1 + h$, where h ultimately vanishes.

Then the general solution is

$$\begin{aligned} y &= Ae^{m_1x} + Be^{(m_1+h)x} \\ &= Ae^{m_1x} + Be^{m_1x}.e^{hx} \\ &= Ae^{m_1x} + Be^{m_1x}\left(1 + hx + \frac{h^2x^2}{2}\cdots\right) \\ &= (A+B)e^{m_1x} + Bhxe^{m_1x} + \text{higher powers of } h. \end{aligned}$$

Now, A and B are perfectly arbitrary; let us, then, choose A and B so that both $A + B$ and Bh are finite. The succeeding terms will be negligible in comparison with $Bhxe^{m_1x}$, and will ultimately vanish with h.

Write c_1 for $A + B$, and let c_2 be the limit of Bh; the solution is, therefore,

$$y = c_1e^{m_1x} + c_2xe^{m_1x} = (c_1 + c_2x)e^{m_1x}.$$

A second method will be given below.

Ex. *Solve* $y_2 + 4y_1 + 4y = 0$.

Putting $m^2 + 4m + 4 = 0$, $m = -2$; the roots being equal.

The general solution is $y = (c_1 + c_2x)e^{-2x}$.

508. Case of Imaginary Roots.

Suppose the roots of the quadratic $m^2 + am + b = 0$ to be imaginary; say $m = \alpha \pm \beta i$.

The solution is then

$$\begin{aligned} y &= Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \qquad \ldots\ldots\ldots\ldots \quad (1) \\ &= e^{\alpha x}\{Ae^{\beta xi} + Be^{-\beta xi}\} \\ &= e^{\alpha x}\{A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)\} \\ &= e^{\alpha x}\{(A+B)\cos\beta x + i(A-B)\sin\beta x\}. \end{aligned}$$

Choose A and B so that both $A + B$ and $i(A - B)$ are real; thus let $A = C + Di$, $B = C - Di$,† then

† It appears, from equation (1) above, that A and B must be imaginary coefficients, as otherwise y would be itself imaginary.

$$A + B = 2C; \quad i(A - B) = 2Di^2 = -2D,$$

$$\therefore y = e^{ax}(2C\cos\beta x - 2D\sin\beta x),$$

which may be written

$$y = e^{ax}(A\cos\beta x + B\sin\beta x).$$

If $a = 0$, the roots are equal and opposite, and in this case $y = A\cos\beta x + B\sin\beta x$, which is the solution of the equation

$$y_2 + \beta^2 y = 0.$$

Ex. *Solve* $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0.$

Here $m^2 - 2m + 2 = 0$, or $m = 1 \pm i$;

$$\begin{aligned} \therefore \text{ the solution is } y &= Ae^{(1+i)x} + Be^{(1-i)x} \\ &= e^x\{Ae^{ix} + Be^{-ix}\} \\ &= e^x(c_1\cos x + c_2\sin x), \text{ or } Ce^x\sin(x + a). \end{aligned}$$

509. The Operator D or d/dx.

We have already had occasion (see Art. 107) to treat the symbol D or d/dx as if it were an algebraical quantity. We shall now show how this can be applied to the solution of the class of differential equations under consideration. It must be clearly understood that we are only using a very convenient notation, and are taking advantage of the fact that some of the laws of differentiation are very analogous to the fundamental laws of algebra. This fact we proceed to show.

If u and v be functions of x, then

(1) $D(u + v) = Du + Dv$ by the fundamental laws of differentiation; hence D obeys the *Distributive Law*.

(2) $D(Du) = D^2u$; and generally $D^r(D^s u) = D^{r+s}u$; hence D obeys the *Index Law*.

(3) $D^r au = aD^r u$ if *a is constant*; hence, *in regard to constants*, D obeys the *Commutative Law*.

Hence we may treat D as if it were an algebraical quantity, provided that we make D commutative in respect to constants only.

Finally, by agreement, we have

(4) $D^2u + aDu + bu = (D^2 + aD + b)u$, and so in the general case.

510. To prove that $(D - m_2)(D - m_1) = (D - m_1)(D - m_2)$.

Consider the two operators $D - m_1$ and $D - m_2$, m_1 and m_2 being constants.

Let us operate with each of these in succession upon y.

Then
$$(D - m_1)y = Dy - m_1y$$
$$\begin{aligned}(D - m_2)(D - m_1)y &= (D - m_2)(Dy - m_1y)\\ &= D(Dy - m_1y) - m_2(Dy - m_1y)\\ &= D^2y - m_1Dy - m_2Dy + m_1m_2y\\ &= D^2y - (m_1 + m_2)Dy + m_1m_2y.\end{aligned}$$

This may be written
$$(D - m_2)(D - m_1)y = \{D^2 - (m_1 + m_2)D + m_1m_2\}y,$$
which shows that we may treat $(D - m_2)(D - m_1)$ as if it were an algebraical product, and then apply it to y.

We may similarly show that $(D - m_1)(D - m_2)y$ leads to the same result.

Hence $(D - m_2)(D - m_1) = (D - m_1)(D - m_2)$, as in the case of an algebraical product.

511. Alternative Method of Solving $y_2 + ay_1 + by = 0$.

Writing the equation in the form $(D^2 + aD + b)y = 0$, let the factors of $D^2 + aD + b$ be $D - m_1$ and $D - m_2$.

The equation is then equivalent to
$$(D - m_1)(D - m_2)y = 0.$$

Let $(D - m_2)y = v$; then $(D - m_1)v = 0$.
$$\text{i.e. } Dv - m_1v = 0, \text{ or } \frac{dv}{dx} - m_1v = 0,$$
a linear equation of the first order, the solution of which is
$$e^{-m_1x}v = c_1, \text{ or } v = c_1e^{m_1x}.$$

Hence $(D - m_2)y = c_1 e^{m_1 x}$,

or $$\frac{dy}{dx} - m_2 y = c_1 e^{m_1 x},$$

the integrating factor of which is $e^{-m_2 x}$.

Hence $$e^{-m_2 x} y = c_1 \int e^{(m_1 - m_2)x} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$= \frac{c_1}{m_1 - m_2} e^{(m_1 - m_2)x} + c_2,$$

$$\therefore y = \frac{c_1}{m_1 - m_2} e^{m_1 x} + c_2 e^{m_2 x},$$

$$= Ae^{m_1 x} + Be^{m_2 x}.$$

If $m_2 = m_1$, then (1) becomes

$$e^{-m_1 x} y = c_1 \int dx = c_1 x + c_2;$$

$$\therefore y = (c_1 x + c_2)e^{m_1 x}.$$

The case in which m_1 and m_2 are imaginary can be discussed in the same manner as before.

512. Complementary Function. — Two results in connection with the last article are of importance, and should, if possible, be committed to memory :—

(1) If $(D - m)y = 0$, then $y = ce^{mx}$.

(2) If $(D - m)y = v$, then $y = ce^{mx} + e^{mx}\int e^{-mx} v\,dx$.

In (2) the term $e^{mx}\int e^{-mx} v\,dx$ is called the *particular integral*, since it contains no arbitrary constant, and is in fact what the solution becomes on giving to c the particular value zero; while the term ce^{mx} is called the *complementary function*, for it is not of itself a solution of (2), since by (1) $(D - m)e^{mx} = 0$, but it contains the arbitrary constant and so makes the solution of (2) general. It is, in fact, the solution of $(D - m)y = 0$. See Art. 514.

Ex. *Solve* $y_2 - 4y_1 + 3y = 0$.

Here $(D^2 - 4D + 3)y = 0$, or $(D - 3)(D - 1)y = 0$;

$$\therefore (D - 3)v = 0, \text{ where } v = (D - 1)y.$$

$$\therefore v = ce^{3x}.$$

Hence $(D-1)y = ce^{3x}$;

$$\therefore\ y = e^x\int e^{-x}.ce^{3x}dx + \text{the complementary function}$$
$$= ce^x\int e^{2x}dx + c_1e^x$$
$$= \frac{c}{2}e^{3x} + c_1e^x \text{ or } Ae^{3x} + Be^x.$$

As an alternative we may write $y = e^x[\int e^{-x}.ce^{3x}dx + c_1] =$ etc.; that is to say, we may add a constant to the integral inside the bracket instead of adding the complementary function.

EXAMPLES LXXVII.

1. Solve by the method of Art. 506:—

(1) $\frac{d^2y}{dx^2} - y = 0.$ (2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0.$

(3) $\frac{d^2y}{dx^2} + (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y = 0.$ (4) $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0.$

(5) $\frac{d^2y}{dx^2} + 4y = 0.$ (6) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$

(7) $y_2 - 2y_1 + y = 0.$ (8) $2y_2 + 3y_1 + 4 = 0.$

(9) $a^2y_2 + 2aby_1 + (b^2+c^2)y = 0.$ (10) $ay_2 + by_1 = 0$

2. Solve by the methods of Arts. 511 and 512:—

(1) $\frac{dy}{dx} - 2y = 0.$ (2) $\frac{dy}{dx} - 2y = e^{2x}.$

(3) $a\frac{dy}{dx} = by.$ (4) $\frac{dy}{dx} + y = e^x + e^{2x}.$

(5) $2\frac{dy}{dx} - 3y = e^{2x} + 2.$ (6) $\frac{dy}{dx} + ay = x.$

(7) $\frac{d^2y}{dx^2} - 4y = 0.$ (8) $y_2 - 2y_1 - 3y = 0.$

(9) $(D-2)^2y = 0.$ (10) $(D-a)(D-b)y = 0.$

(11) $(D^2+a^2)y = 0.$ (12) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$

(13) $D(D-1)y = 0.$ (14) $4\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + y = 0.$

3. Solve :—

(1) $(D^2 + 2D + 5)y = 0.$ (2) $\{(D + a)^2 + b^2\}y = 0.$

(3) $\dfrac{d^2x}{dt^2} - 2\omega \cos\alpha \dfrac{dx}{dt} + \omega^2 x = 0.$

(4) $\dfrac{d^2x}{dt^2} - 2\omega \cosh\alpha \dfrac{dx}{dt} + \omega^2 x = 0.$

ANSWERS.

1. (1) $y = c_1e^x + c_2e^{-x}$. (2) $y = c_1e^x + c_2e^{-3x}$. (3) $y = c_1e^{-\alpha x} + c_2e^{-\beta x}$

(4) $y = c_1e^{\frac{1}{2}x} + c_2e^{2x}$. (5) $y = A\cos 2x + B\sin 2x$.

(6) $y = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$. (7) $y = e^x(c_1 + c_2x)$.

(8) $y = e^{-\frac{3}{4}x}\left(A\cos\frac{\sqrt{23}}{4}x + B\sin\frac{\sqrt{23}}{4}x\right)$.

(9) $y = e^{-(b/a)x}\left(A\cos\frac{cx}{a} + B\sin\frac{cx}{a}\right)$. (10) $y = c_1e^{-(b/a)x} + c_2$.

2. (1) $y = ce^{2x}$. (2) $y = e^{2x}(x + c)$. (3) $y = ce^{(b/a)x}$. (4) $y = \frac{1}{2}e^x + \frac{1}{3}e^{2x} + ce^{-x}$.

(5) $y = e^{2x} - \frac{2}{3} + ce^{\frac{3}{2}x}$. (6) $a^2y = ax - 1 + ce^{-ax}$. (7) $y = Ae^{2x} + Be^{-2x}$.

(8) $y = c_1e^{3x} + c_2e^{-x}$. (9) $y = (c_1 + c_2x)e^{2x}$. (10) $y = c_1e^{ax} + c_2e^{bx}$.

(11) $y = A\cos ax + B\sin ax$. (12) $y = (c_1 + c_2x)e^{-x}$. (13) $y = c_1 + c_2e^x$.

(14) $y = (c_1 + c_2x)e^{\frac{1}{2}x}$.

3. (1) $y = e^{-x}(A\cos 2x + B\sin 2x)$. (2) $y = e^{-ax}(A\cos bx + B\sin bx)$.

(3) $x = e^{\omega t\cos\alpha}\{A\cos(\omega t\sin\alpha) + B\sin(\omega t\sin\alpha)\}$.

(4) $x = e^{\omega t\cosh\alpha}\{A\cosh(\omega t\sinh\alpha) + B\sinh(\omega t\sinh\alpha)\}$.

513. II. Case in which X is not Zero—Complementary Function.

The equation under consideration is of the form

$$y_2 + ay_1 + by = X,$$

or

$$(D^2 + aD + b)y = X \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We shall, however, only consider the three following cases :—

(*a*) $X = me^{nx}$, (*b*) $X = m\cos nx$, or $m\sin nx$, (*c*) $X = mx^n$ (n being a $+^{ve}$ integer); and their combinations.

514. First, let $y = v$ be a particular integral of (1), *i.e.* one not containing any arbitrary constants.

Also, let $y = c_1u_1 + c_2u_2$ be the general solution of

$$(D^2 + aD + b)y = 0,$$

containing the two arbitrary constants c_1 and c_2.

Then we shall see that the general solution of (1) is

$$y = v + c_1u_1 + c_2u_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For $(D^2 + aD + b)y = (D^2 + aD + b)v + (D^2 + aD + b)(c_1u_1 + c_2u_2)$

$$= X + 0 = X,$$

showing that (2) is *a* solution. Moreover (2) contains *two* arbitrary constants; hence it must be the *general* solution.

The terms $c_1u_1 + c_2u_2$ constitute the *complementary function.*

Hence the method :—*Find the particular integral* (p.i.), *and then add the complementary function* (c.f.).

To find the particular integral, further use will be made of the properties of the symbol D. The method employed, here and previously, is known as that of the *Calculus of Operations*.

515. Case (*a*)—$X = me^{nx}$.

Def.—Let y be a function of x such that, when operated on with $f(D)$, it becomes v; that is, let $f(D)y = v$, $f(D)$ being regarded here and elsewhere as a rational integral algebraical function of D.

Then we may write $y = \dfrac{1}{f(D)}v$; the latter expression being defined as *that quantity which when operated on with* f(D) *becomes* v.

For example, $\dfrac{1}{D}\cdot f(x) = \int f(x)\,dx$.

516. To prove that $\frac{1}{f(D)} mv = m \cdot \frac{1}{f(D)} v$, m being constant.

Let $y = \frac{1}{f(D)} mv$; whence $f(D)y = mv$.

$\therefore\ v = \frac{1}{m} f(D)y = f(D)\frac{1}{m}y$, since D and its powers obey the commutative law with respect to constants.

$$\therefore\ \frac{1}{m}y = \frac{1}{f(D)}v, \text{ and } \therefore\ y = m \cdot \frac{1}{f(D)}v.$$

517. To prove that $\frac{1}{f(D)}(u+v) = \frac{1}{f(D)}u + \frac{1}{f(D)}v$.

Operate on both sides of the equation with $f(D)$.

The left-hand side becomes $u + v$.

The right-hand side becomes $f(D)\left[\frac{1}{f(D)}u + \frac{1}{f(D)}v\right]$

$$= f(D)\cdot\frac{1}{f(D)}u + f(D)\cdot\frac{1}{f(D)}v,$$

since D and its powers obey the distributive law,

$$= u + v.$$

The above statement is therefore true.

518. To prove that $f(D)e^{ax} = f(a)e^{ax}$, and that $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$.

Here $f(D)$ is a rational integral algebraical function of D.

Now
$$De^{ax} = ae^{ax},$$
$$D^2e^{ax} = a^2e^{ax};$$

$\therefore$ if p, q, r, be any constants, we have

$$(pD^2 + qD + r)e^{ax} = (pa^2 + qa + r)e^{ax},$$

and generally, $f(D)\,e^{ax} = f(a)\,e^{ax}$, which is the first statement.

Again, from the last result, we have

$$e^{ax} = \frac{1}{f(D)} \cdot f(a)\, e^{ax} = f(a) \cdot \frac{1}{f(D)} e^{ax}, \text{ [Art. 516];}$$

$$\therefore \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}, \text{ which is the second statement.}$$

519. (*a*) **Form** $(D^2 + aD + b)y = me^{nx}$.

We have $y = m \dfrac{1}{D^2 + aD + b} \cdot e^{nx} = \dfrac{me^{nx}}{n^2 + an + b}$ . . . (1)

The (c.f.) is $Ae^{m_1 x} + Be^{m_2 x}$ where m_1, m_2 are the roots of $D^2 + aD + b = 0$. [Art. 511.]

Hence the general solution is

$$y = \frac{me^{nx}}{n^2 + an + b} + Ae^{m_1 x} + Be^{m_2 x}.$$

Ex. 1. *Solve* $(D^2 - D - 2)y = 2e^{3x}$.

The (p.i.) is $y = 2 \cdot \dfrac{e^{3x}}{3^2 - 3 - 2} = \frac{1}{2}e^{3x}$.

Since $D^2 - D - 2 = (D + 1)(D - 2)$, the (c.f.) is $Ae^{-x} + Be^{2x}$.

The general solution is therefore, $y = \frac{1}{2}e^{3x} + Ae^{-x} + Be^{2x}$.

Ex. 2. *Solve* $(D - 1)^2 y = 3e^{-x}$.

The (p.i.) is $y = \dfrac{3e^{-x}}{(-1 - 1)^2} = \frac{3}{4}e^{-x}$; and the (c.f.) is $(c_1 + c_2 x)\, e^x$.

Hence the general solution is $y = \frac{3}{4}e^{-x} + (c_1 + c_2 x)e^x$.

Ex. 3. *Solve* $(D^2 - 6D + 18)y = e^x$.

The (p.i.) is $y = \dfrac{e^x}{1 - 6 + 18} = \frac{1}{13}e^x$.

Solving the equation, $m^2 - 6m + 18 = 0$, $m = 3 \pm 3i$.

Hence the (c.f.) is $e^{3x}(A \cos 3x + B \sin 3x)$.

520. Exceptional Cases.

First, suppose $n = m_1$, one of the roots of $D^2 + aD + b = 0$. Then, since $n^2 + an + b = 0$ by hypothesis, we have from (1) of the

last article, $y = \infty$. Hence the method fails. However, let $n = m_1 + h$, where h ultimately vanishes; then

$$y = \frac{1}{(D - m_1)(D - m_2)} m \,.\, e^{(m_1+h)x}$$

$$= \frac{1}{h(m_1 - m_2 + h)} m \,.\, e^{m_1 x} \,.\, e^{hx}, \text{ by the above rule,}$$

$$= \frac{me^{m_1 x}}{h(m_1 - m_2 + h)} [1 + hx + \tfrac{1}{2}h^2 x^2 + \dots],$$

and adding the (c.f.) $Ae^{m_1 x} + Be^{m_2 x}$, the general solution is

$$y = \left\{\frac{m}{h(m_1 - m_2 + h)} + A\right\} e^{m_1 x} + \frac{mxe^{m_1 x}}{m_1 - m_2 + h}$$

$$+ Be^{m_2 x} + \text{terms in } h;$$

and, choosing A so that $\dfrac{m}{h(m_1 - m_2 + h)} + A$ is finite when h vanishes, we have, in the limit,

$$y = \left(C + \frac{m}{m_1 - m_2} x\right) e^{m_1 x} + Be^{m_2 x}.$$

Secondly, suppose, in addition, that the roots of $D^2 + aD + b = 0$ are both equal to m_1.

Then $y = \dfrac{1}{(D - m_1)^2} m \,.\, e^{m_1 x} = \infty$, so that the method fails as before.

Putting $n = m_1 + h$, we have by the rule

$$y = \frac{1}{h^2} \,.\, me^{m_1 x} \,.\, e^{hx} = \frac{me^{m_1 x}}{h^2} [1 + hx + \tfrac{1}{2}h^2 x^2 + \dots];$$

and the general solution is

$$y = (c_1 + c_2 x)e^{m_1 x} + \frac{me^{m_1 x}}{h^2} [1 + hx + \tfrac{1}{2}h^2 x^2 + \dots]$$

$$= \left(c_1 + \frac{m}{h^2}\right) e^{m_1 x} + \left(c_2 + \frac{m}{h}\right) xe^{m_1 x} + \tfrac{1}{2}mx^2 e^{m_1 x} + \text{terms in } h;$$

and, choosing c_1 and c_2 so that both $c_1 + \frac{m}{h^2}$ and $c_2 + \frac{m}{h}$ are finite, we have, in the limit,

$$y = Ae^{m_1x} + Bxe^{m_1x} + \tfrac{1}{2}mx^2e^{m_1x}$$
$$= (A + Bx + \tfrac{1}{2}mx^2)e^{m_1x}.$$

Ex. 1. *Solve* $(D-1)(D-2)y = e^x$.

Since $D = 1$ gives an infinite value for y, let

$$(D-1)(D-2)y = e^{(1+h)x},$$

where h ultimately vanishes.

$$\therefore y = \frac{1}{(D-1)(D-2)}e^{(1+h)x} = \frac{1}{h(h-1)}e^{(1+h)x}$$

$$= \frac{1}{h(h-1)}e^x[1 + hx + \ldots] = \frac{e^x}{h(h-1)} + \frac{xe^x}{h-1} + \text{terms in } h,$$

the (c.f.) being $Ae^x + Be^{2x}$.

Hence in the limit, when $h = 0$, we have

$$y = (c_1 - x)e^x + Be^{2x}, \text{ where } c_1 = \underset{h=0}{lt}\left[A + \frac{1}{h(h-1)}\right].$$

Ex. 2. *Solve* $(D-1)^2y = e^x$.

If $(D-1)^2y = e^{(1+h)x}$, $\therefore y = \frac{e^{(1+h)x}}{h^2} = \frac{e^x}{h^2}[1 + hx + \frac{1}{2}h^2x^2 + \ldots]$,
the (c.f.) being $(c_1 + c_2x)e^x$.

Hence the solution will be $y = (A + Bx)e^x + \frac{1}{2}x^2e^x$.

Otherwise :—In Ex. 1, let $(D-1)y = v$; then $(D-2)v = e^x$.

$$\therefore v = \frac{1}{1-2}e^x = -e^x;$$

that is, $$(D-1)y = -e^x.$$

Hence, by Art. 512, for the (p.i.), we have

$$y = e^x\int e^{-x}(-e^x)dx = -xe^x,$$

and adding the (c.f.), $y = (A - x)e^x + Be^{2x}$.

We might also have put $(D-2)y = v$, but the present method is shorter.

In Ex. 2, putting $(D-1)y=v$, we have

$$(D-1)v=e^x; \quad \therefore\ v=e^x\int e^{-x}.e^x dx = xe^x.$$

$$\therefore\ (D-1)y=xe^x; \quad \therefore\ y=e^x\int e^{-x}.xe^x dx=\tfrac{1}{2}x^2e^x,$$

and adding the (c.f.), $y=(c_1+c_2x+\frac{1}{2}x^2)e^x$.

521. Case (b)—$X=m\cos nx$, **or** $m\sin nx$—**To prove that**

$f(D^2).\begin{Bmatrix}\cos nx\\ \sin nx\end{Bmatrix}=f(-n^2).\begin{Bmatrix}\cos nx\\ \sin nx\end{Bmatrix}$, **and** $\frac{1}{f(D^2)}\cdot\begin{Bmatrix}\cos nx\\ \sin nx\end{Bmatrix}=\frac{1}{f(-n^2)}\cdot\begin{Bmatrix}\cos nx\\ \sin nx\end{Bmatrix}$.

We have

$$D\cos nx=-n\sin nx,$$
$$D^2\cos nx=-n^2\cos nx,$$
$$D^3\cos nx=n^3\sin nx,$$
$$D^4\cos nx=n^4\cos nx=(-n^2)^2\cos nx,$$

and, generally, $(D^2)^m\cos nx=(-n^2)^m\cos nx$.

Hence, if p, q, r be any constants, we have

$$(pD^4+qD^2+r)\cos nx=(pn^4-qn^2+r)\cos nx,$$

and generally $\quad f(D^2)\cos nx=f(-n^2)\cos nx \quad . \quad . \qquad (1)$

Similarly, $\quad f(D^2)\sin nx=f(-n^2)\sin nx$;

which is the first statement.

Again, from (1), $\cos nx=\frac{1}{f(D^2)}.f(-n^2)\cos nx$

$$=f(-n^2)\cdot\frac{1}{f(D^2)}\cdot\cos nx.$$

$$\therefore\ \frac{1}{f(D^2)}\cos nx=\frac{1}{f(-n^2)}\cos nx.$$

Similarly, $\frac{1}{f(D^2)}\sin nx=\frac{1}{f(-n^2)}\sin nx$; which is the second statement.

522. To prove that $\frac{1}{f(D)}=\frac{1}{\phi(D).f(D)}\cdot\phi(D)$.

Let $f(D)y=v$, and operate on both sides with $\phi(D)$.

$$\therefore\ \phi(D).f(D)y=\phi(D)v.$$

$$\therefore\ y=\frac{1}{\phi(D).f(D)}\cdot\phi(D)v.$$

But $y = \dfrac{1}{f(D)} v$; whence the above statement follows.

We shall now apply these results.

523. (*b*) **Form** $(D^2 + aD + b)y = m \cos nx$.

We have $y = m \cdot \dfrac{1}{D^2 + aD + b} \cdot \cos nx$; and "multiplying" above and below by $D^2 - aD + b$, *in order that the denominator may contain only even powers of* D,

$$y = m \cdot \frac{1}{(D^2 + b)^2 - a^2D^2} \cdot (D^2 - aD + b) \cos nx \quad \text{[Art. 522]}$$

$$= m \cdot \frac{1}{(D^2 + b)^2 - a^2D^2} \cdot \{(-n^2 + b) \cos nx + an \sin nx\}$$

$$= m(b - n^2) \cdot \frac{1}{(D^2+b)^2 - a^2D^2} \cos nx + amn \cdot \frac{1}{(D^2 + b)^2 - a^2D^2} \cdot \sin nx$$

[Art. 517]

$$= \frac{m(b - n^2)}{(b - n^2)^2 + a^2n^2} \cos nx + \frac{amn}{(b - n^2)^2 + a^2n^2} \sin nx$$

$$= \frac{m(b-n^2) \cos nx + amn \sin nx}{(b-n^2)^2+a^2n^2}.$$

The (c.f.) must of course be added.

Note.—We may here observe that

$$f(D^2)(a \cos nx + b \sin nx) = f(-n^2)(a \cos nx + b \sin nx),$$

and that $\dfrac{1}{f(D^2)} (a \cos nx + b \sin nx) = \dfrac{1}{f(-n^2)} (a \cos nx + b \sin nx)$.

Also $$\frac{1}{f(D^2)} \cos (nx + \alpha) = \frac{1}{f(-n^2)} \cos (nx + \alpha);$$

this may be at once deduced from the preceding result.

The form $(D^2+aD+b)y=m \sin nx$ may be treated in the same manner.

524. Exceptional Case.

Suppose $a = 0$ and $b = n^2$; then y becomes infinite, and the method fails.

The equation is now $(D^2 + n^2)y = m \cos nx$. . . . . (1)

On the right-hand side of (1) write $n + h$ for n, where h ultimately vanishes.

Then $(D^2 + n^2)y = m \cos (n + h)x$.

The (c.f.) is $A \cos nx + B \sin nx$.

For the (p.i.) we have

$$y = \frac{m \cos (n+h)x}{-(n+h)^2 + n^2}$$

$$= -\frac{m}{h(2n+h)}[\cos nx \cos hx - \sin nx \sin hx]$$

$$= -\frac{m}{h(2n+h)}\left[\cos nx\left(1 - \frac{h^2x^2}{2}\cdots\right) - \sin nx\,(hx - \ldots)\right]$$

$$= -\frac{m}{h(2n+h)}\cos nx + \frac{mx}{2n+h}\sin nx + \text{higher powers of } h.$$

Adding the (c.f.), we have

$$y = \left\{A - \frac{m}{h(2n+h)}\right\}\cos nx + \left\{B + \frac{m}{2n+h}x\right\}\sin nx + \ldots.$$

Choosing A so that $A - \dfrac{m}{h(2n+h)}$ is finite in the limit, we have ultimately, writing A still for this limit,

$$y = A \cos nx + \left(B + \frac{m}{2n}x\right)\sin nx.$$

NOTE.—In practice we may write 1 for $\cos hx$, and hx for $\sin hx$, in the above work.

Ex. 1. *Solve* $(D-1)(D-2)y = 4\cos 2x$.

We have
$$y = 4\cdot\frac{(D+1)(D+2)}{(D^2-1)(D^2-4)}\cos 2x$$

$$= \frac{4(D^2+3D+2)}{(-4-1)(-4-4)}\cos 2x,$$

noting that for practical purposes it is immaterial whether we substitute

-2^2 for D^2 in the denominator *before* or *after* we operate with the numerator,

$$= \tfrac{1}{10}(-4 + 3D + 2)\cos 2x$$
$$= \tfrac{1}{10}(3D - 2)\cos 2x = -\tfrac{1}{5}(3\sin 2x + \cos 2x),$$

which is the (p.i.).

Otherwise :—
$$y = \frac{4}{D^2 - 3D + 2}\cos 2x$$
$$= \frac{4}{-4-3D+2}\cos 2x = -\frac{4}{3D+2}\cos 2x$$
$$= -\frac{4(3D-2)}{9D^2-4}\cos 2x = -\frac{4(3D-2)}{-40}\cos 2x = \text{etc.}$$

In the latter method we have substituted -2^2 for D^2 at the earliest opportunity, and a little thought will convince the reader that we are justified in doing so.

The (c.f.) is $Ae^x + Be^{2x}$; hence the general solution is
$$y = Ae^x + Be^{2x} - \tfrac{1}{5}(3\sin 2x + \cos 2x).$$

Ex. 2. *Solve* $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = \cos x + \sin x.$

We have
$$(D^2 - 2D + 3)y = \cos x + \sin x.$$

$\therefore$ putting $D^2 = -1$, $y = \dfrac{1}{-2(D-1)}(\cos x + \sin x)$
$$= \frac{D+1}{-2(D^2-1)}(\cos x + \sin x) = \tfrac{1}{4}(D+1)(\cos x + \sin x)$$
$$= \tfrac{1}{4}(-\sin x + \cos x + \cos x + \sin x) = \tfrac{1}{2}\cos x.$$

The roots of $D^2 - 2D + 3 = 0$ are $1 \pm \sqrt{2}\,i$; the (c.f.) is therefore $e^x(A\cos\sqrt{2}x + B\sin\sqrt{2}x)$. Hence the general solution is
$$y = e^x(A\cos\sqrt{2}x + B\sin\sqrt{2}x) + \tfrac{1}{2}\cos x.$$

Ex. 3. *Solve* $\dfrac{d^2y}{dx^2} + y = 2\cos x.$

Let $(D^2 + 1)y = 2\cos(1+h)x = 2(\cos x\cos hx - \sin x\sin hx)$
$$= 2(\cos x - hx\sin x) + \dots$$
$$\therefore\ y = \frac{2(\cos x - hx\sin x)}{-(1+h)^2+1} + \dots = -\frac{2}{2h+h^2}(\cos x - hx\sin x + \dots)$$
$$= -\frac{2}{2h+h^2}\cos x + \frac{2x\sin x}{2+h} + \dots$$

The (c.f.) is $A\cos x + B\sin x$.

Hence we have, ultimately, $y = A\cos x + (B + x)\sin x$.

525. Case (c)—$X = mx^n$, n being a Positive Integer—Expansion of $1/f(D)$ in Ascending Powers of D.

Let D be temporarily regarded as an algebraical quantity, and suppose $1/f(D)$ expanded in ascending powers as far as the term containing D^n, by ordinary long division. Let the remainder be $D^{n+1}\phi(D)$ †.

Then $$\frac{1}{f(D)} = a_0 + a_1 D + a_2 D^2 + \dots + a_n D^n + \frac{D^{n+1}\phi(D)}{f(D)}.$$

$$\therefore 1 - D^{n+1}\phi(D) = f(D).(a_0 + a_1 D + a_2 D^2 + \dots + a_n D^n).$$

Since the *operator* D obeys the laws of multiplication, this equation is true when D is no longer regarded as an algebraical quantity, but as an operator.

Now $[1 - D^{n+1}\phi(D)]x^n = x^n$; since $D^{n+1}x^n = 0$, and so for higher d.c.'s.

Hence $x^n = f(D).(a_0 + a_1 D + \dots + a_n D^n)x^n$;

$$\therefore \frac{1}{f(D)}x^n = (a_0 + a_1 D + \dots + a_n D^n)x^n$$

which may be briefly written $= (a_0 + a_1 D + \dots)x^n$, on the understanding that the terms on the right are continued until x disappears by repeated differentiation.

We may evidently, *in practice*, expand $1/f(D)$ as if it were an algebraical quantity; and we may adopt any method of expansion we please, such as, for instance, that of resolving into partial fractions.

526. (c) Form $(D^2 + aD + b)y = mx^n$.

The method to be adopted has been indicated above; we shall therefore give a numerical example.

Ex. 1. *Solve* $(D - 1)(D - 2)y = 8x^3$.

We have $$y = \frac{1}{(D-1)(D-2)}.8x^3 = 8\left(\frac{1}{D-2} - \frac{1}{D-1}\right)x^3$$
$$= 8\left[-\tfrac{1}{2}\left(1 - \frac{D}{2}\right)^{-1} + (1 - D)^{-1}\right]x^3$$

† The remainder must obviously contain terms of higher degree than D^n.

$$= 8\left\{-\frac{1}{2}\left(1 + \frac{D}{2} + \frac{D^2}{4} + \frac{D^3}{8}\right) + 1 + D + D^2 + D^3\right\}x^3$$

$$= 8[\tfrac{1}{2} + \tfrac{3}{4}D + \tfrac{7}{8}D^2 + \tfrac{15}{16}D^3]x^3$$

$$= 8[\tfrac{1}{2}x^3 + \tfrac{3}{4}.3x^2 + \tfrac{7}{8}.6x + \tfrac{15}{16}.6] = 4x^3 + 18x^2 + 42x + 45.$$

The (c.f.) is $Ae^x + Be^{2x}$.

$\therefore$ the general solution is $y = Ae^x + Be^{2x} + 4x^3 + 18x^2 + 42x + 45$.

Ex. 2. *Solve* $y_2 + y_1 = 3x^2$.

We have $D(D + 1)y = 3x^2$;

$$\therefore (D + 1)y = \frac{1}{D}.3x^2 = \int 3x^2dx = x^3 \text{ [Art. 515].}$$

$$\therefore y = \frac{1}{1 + D}x^3 = (1 - D + D^2 - D^3 \ldots)x^3 = x^3 - 3x^2 + 6x - 6.$$

The (c.f.) is $A + Be^{-x}$.

$\therefore$ the general solution is $y = A + Be^{-x} + x^3 - 3x^2 + 6x$,

noting that the term -6, being a constant, may be supposed to be included in the constant A.

527. Combinations of the Three Classes.

The statement of Art. 517 can be extended to three or more quantities.

Thus $\dfrac{1}{f(D)}(u + v + w + \ldots) = \dfrac{1}{f(D)}u + \dfrac{1}{f(D)}v + \dfrac{1}{f(D)}w + \ldots$

The following example will show how we can apply this to solve equations which are combinations of the three classes (a), (b), (c).

Ex. *Solve* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = e^x + x + \cos x + \cos 2x$.

We have $(D^2 + 2D + 2)y = e^x + x + \cos x + \cos 2x$.

$$\therefore y = \frac{1}{D^2 + 2D + 2}e^x + \frac{1}{D^2 + 2D + 2}x + \frac{1}{D^2 + 2D + 2}\cos x + \frac{1}{D^2 + 2D + 2}\cos 2x$$

$$= \tfrac{1}{5}e^x + \tfrac{1}{2}(1 - D + \ldots)x + \frac{1}{2D + 1}\cos x + \frac{1}{2D - 2}\cos 2x$$

$$= \tfrac{1}{5}e^x + \tfrac{1}{2}(x-1) + \frac{2D-1}{4D^2-1}\cos x + \tfrac{1}{2}\frac{D+1}{D^2-1}\cos 2x$$

$$= \tfrac{1}{5}e^x + \tfrac{1}{2}(x-1) + \frac{-2\sin x - \cos x}{-5} + \tfrac{1}{2}\,\frac{-2\sin 2x + \cos 2x}{-5},$$

and, adding the (c.f.), we have for the general solution,

$$y = \tfrac{1}{5}e^x + \tfrac{1}{2}(x-1) + \tfrac{1}{5}(2\sin x + \cos x) + \tfrac{1}{10}(2\sin 2x - \cos 2x)$$
$$+ e^{-x}(A\cos x + B\sin x).$$

528. The Homogeneous Linear Equation.

The equation

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1}x\frac{dy}{dx} + a_ny = X, \quad (1)$$

in which a_1, a_2 ... are constants, and X a function of x, is known by the above title.

To solve it, put $x = e^t$, when it can be shown to reduce to the ordinary linear equation with constant coefficients.

We shall only consider the second order equation, although the method applies in the general case.

We have $$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dy}{dt}\Big/e^t = \frac{dy}{dt}\Big/x;$$

$$\therefore\ x\frac{dy}{dx} = \frac{dy}{dt} = Dy, \text{ if } D \equiv \frac{d}{dt}.$$

Differentiating with respect to x, we have

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dt^2}\Big/\frac{dx}{dt} = \frac{D^2y}{x};$$

$$\therefore\ x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = D^2y.$$

$$\therefore\ x^2\frac{d^2y}{dx^2} = D^2y - x\frac{dy}{dx} = D^2y - Dy = D(D-1)y.$$

Similarly, we can show that

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y,$$

and $$x^r\frac{d^ry}{dx^r} = D(D-1)(D-2)\dots(D-r+1)y.$$

Substituting these values in (1), we shall obtain, after reduction, a linear equation with constant coefficients.

Thus, if the equation be $x^2\dfrac{d^2y}{dx^2} + ax\dfrac{dy}{dx} + by = f(x)$,

we obtain

$$D(D-1)y + aDy + by = f(e^t),$$

or $$\{D^2 + (a-1)D + b\}y = \phi(t), \text{ in which } D \equiv d/dt,$$

which can be solved by preceding methods.

Ex. 1. *Solve* $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x^3$.

We obtain, on putting $x = e^t$,

$$\{D(D-1) - D + 2\}y = e^{3t},$$

or $$(D^2 - 2D + 2)y = e^{3t};$$

the solution of which is $y = \frac{1}{5}e^{3t} + e^t(A\cos t + B\sin t)$

$$= \tfrac{1}{5}x^3 + x(A\cos\log x + B\sin\log x).$$

Ex. 2. *Solve* $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = x^2 + \cos\log x$.

We obtain $\{D(D-1) - 3D + 4\}y = e^{2t} + \cos t$,

or $$(D-2)^2y = e^{2t} + \cos t.$$

$$\therefore\ y = \frac{1}{(D-2)^2}e^{2t} + \frac{1}{(D-2)^2}\cos t$$

$$= \frac{1}{D-2}\cdot e^{2t}\int e^{-2t}.e^{2t}dt + \frac{1}{D^2 - 4D + 4}\cos t \text{ [Art. 512, or 520 (note)]}$$

$$= \frac{1}{D-2}\cdot te^{2t} + \frac{3+4D}{9-16D^2}\cos t, \text{ on putting } D^2 = -1 \text{ in the second term,}$$

$$= e^{2t}\int e^{-2t}.te^{2t}dt + \frac{3+4D}{25}\cos t$$

$$= \tfrac{1}{2}t^2e^{2t} + \tfrac{1}{25}(3\cos t - 4\sin t).$$

The (c.f.) is $(c_1 + c_2t)e^{2t}$.

Hence the general solution in terms of x is

$$y = \{c_1 + c_2\log x + \tfrac{1}{2}(\log x)^2\}x^2 + \tfrac{1}{25}(3\cos\log x - 4\sin\log x).$$

Ex. 3. *Solve* $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 2\log x.$

We obtain $\quad \{D(D-1)+2D\}y = D(D+1)y = 2t;$

$$\therefore\ (D+1)y = t^2;$$

$$y = \frac{1}{1+D}t^2 = (1 - D + D^2 \ldots)t^2 = t^2 - 2t + 2$$

$\therefore$ the general solution is

$$y = t^2 - 2t + 2 + A + Be^{-t} = A_1 + B\frac{1}{x} - 2\log x + (\log x)^2,$$

where $A_1 = A + 2$.

EXAMPLES LXXVIII.

1. Solve :—

(1) $(D-1)(D-2)y = e^{3x}$. (2) $(D^2-1)y = 3e^{2x}$.

(3) $y_2 + a^2y = a^2e^{ax}$. (4) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 5e^{-2x}$.

(5) $y_2 - 6y_1 + 10y = 5e^x$. (6) $(D+1)(D+2)y = 2e^{-x}$.

(7) $\dfrac{d^2y}{dx^2} - y = e^x$. (8) $y_2 - 4y_1 + 4y = 2e^{2x}$.

(9) $(2D+1)^2y = 4e^{-\frac{1}{2}x}$. (10) $(2D^2 + D - 3)y = e^x$.

(11) $9\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + y = 18e^{-\frac{1}{3}x}$.

2. Solve :—

(1) $(D^2-1)y = \cos x$. (2) $y_2 + 2y = 2\sin 3x$.

(3) $\dfrac{dy}{dx} + ay = \cos x$. (4) $(D-a)y = \cos ax + \sin ax$.

(5) $(D+1)(D-2)y = 3\cos 2x$. (6) $y_2 - 2y_1 + 3y = 9\sin\sqrt{2}\,x$.

(7) $(4D^2 - 5D + 1)y = 2\sin\frac{1}{2}x + \cos\frac{1}{2}x$.

(8) $\dfrac{d^2y}{dx^2} - a\dfrac{dy}{dx} = a\cos ax$.

(9) $(D^2 + a^2)y = (a^2 - b^2)\cos(bx + a)$.

(10) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \cos 2x$. (11) $(D^2+1)y = \cos x$.

(12) $y_2 + 4y = 4\cos 2x$. (13) $\dfrac{d^2y}{dx^2} + 2y = 2\sqrt{2}\sin\sqrt{2}x$.

(14) $(D^2 + k)y = 2\sqrt{k}\sin\sqrt{k}\,x$.

(15) $(D^2 + k)y = 2\sqrt{k}(\sin\sqrt{k}x + \cos\sqrt{k}x)$.

3. Solve :—

(1) $(D + 1)y = x^2$. (2) $\dfrac{dy}{dx} - y = -x^n$.

(3) $(D^2 + 2)y = 2x$. (4) $(D + 1)(D + 2)y = 8x^3$.

(5) $D(D - 2)y = -6x^2$. (6) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = x^2$.

(7) $6y_2 + 7y_1 - 3y = 27x^2$. (8) $(D - 1)^2y = x^3$.

4. Solve :—

(1) $(D^2 - 1)y = x + e^{2x}$.

(2) $(D - 1)(D - 2)y = 4x^2 + 10\cos x$.

(3) $\dfrac{d^2y}{dx^2} + 4y = 4\cos 3x\sin x$.

5. Solve :—

(1) $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$. (2) $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = x^2$.

(3) $x^2y_2 + xy_1 = a$. (4) $x^2y_2 + xy_1 = \log x$.

(5) $2x^2y_2 - 3xy_1 + 3y = 13\cos(\log x)$.

(6) $(2x^2D^2 + 3xD - 1)y = 2\sqrt{x}$.

(7) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 3y = x + \cos(\sqrt{3}\log x)$.

6. Show that

$f(D^2).m\cosh ax = m.f(a^2)\cosh ax$. and $\dfrac{1}{f(D^2)}.m\cosh ax = \dfrac{m}{f(a^2)}\cosh ax$.

Apply this to solve the equations :—

(1) $\dfrac{d^2y}{dx^2} + y = 2\cosh x$.

(2) $\dfrac{d^2x}{dt^2} + 2a\cos a\dfrac{dx}{dt} + a^2x = a\cosh at$.

ANSWERS.

1. (1) $y = \frac{1}{2}e^{3x} + Ae^{x} + Be^{2x}$. (2) $y = e^{2x} + Ae^{x} + Be^{-x}$.

(3) $y = \frac{1}{2}e^{ax} + A\cos ax + B\sin ax$. (4) $y = e^{-2x} + Ae^{-x} + Be^{3x}$.

(5) $y = e^{x} + e^{3x}(A\sin x + B\cos x)$. (6) $y = (A + 2x)e^{-x} + Be^{-2x}$.

(7) $y = (A + \frac{1}{2}x)e^{x} + Be^{-x}$. (8) $y = (c_1 + c_2x + x^2)e^{2x}$.

(9) $y = (A + Bx + \frac{1}{2}x^2)e^{-\frac{1}{2}x}$. (10) $y = Ae^{x} + Be^{-\frac{3}{2}x} + \frac{1}{5}xe^{x}$.

(11) $y = (A + Bx + x^2)e^{-\frac{1}{2}x}$.

2. (1) $y = Ae^{x} + Be^{-x} - \frac{1}{2}\cos x$.

(2) $y = A\cos\sqrt{2}x + B\sin\sqrt{2}\,x - \frac{2}{7}\sin 3x$.

(3) $y = Ae^{-ax} + \frac{1}{1+a^2}(\sin x + a\cos x)$. (4) $y = Ae^{ax} - \frac{1}{a}\cos ax$.

(5) $y = Ae^{-x} + Be^{2x} - \frac{3}{20}(\sin 2x + 3\cos 2x)$.

(6) $y = (Ae^{x} + 2\sqrt{2})\cos\sqrt{2}x + (Be^{x} + 1)\sin\sqrt{2}x$.

(7) $y = Ae^{\frac{1}{2}x} + Be^{x} + \frac{2}{5}(2\cos\frac{1}{2}x - \sin\frac{1}{2}x)$.

(8) $y = A + Be^{ax} - \frac{1}{\sqrt{2}\,a}\sin\left(ax + \frac{\pi}{4}\right)$.

(9) $y = A\cos ax + B\sin ax + \cos(bx + a)$.

(10) $y = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) - \frac{(r^2 - 1)\cos rx - r\sin rx}{r^4 - r^2 + 1}$.

(11) $y = A\cos x + (B + \frac{1}{2}x)\sin x$. (12) $y = A\cos 2x + (B + x)\sin 2x$.

(13) $y = A\sin\sqrt{2}x + (B - x)\cos\sqrt{2}x$.

(14) $y = A\sin\sqrt{k}x + (B - x)\cos\sqrt{k}x$.

(15) $y = (A - x)\cos\sqrt{k}x + (B + x)\sin\sqrt{k}x$.

3. (1) $y = x^2 - 2x + 2 + Ae^{-x}$.

(2) $y = x^{n} + nx^{n-1} + n(n-1)x^{n-2} + \ldots + n! + Ae^{x}$.

(3) $y = x + A\cos\sqrt{2}x + B\sin\sqrt{2}x$.

(4) $y = 4x^3 - 18x^2 + 42x - 45 + Ae^{-x} + Be^{-2x}$.

(5) $y = x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + A + Be^{2x}$.

(6) $y = x^2 - 2x + e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$.

(7) $y = Ae^{\frac{1}{3}x} + Be^{-\frac{1}{2}x} - (9x^2 + 42x + 134)$.

(8) $y = x^3 + 6x^2 + 18x + 24 + (A + Bx)e^x$.

4. (1) $y = -x + \frac{1}{3}e^{2x} + Ae^x + Be^{-x}$.

(2) $y = 2x^2 + 6x + 7 + \cos x - 3\sin x + Ae^x + Be^{2x}$.

(3) $y = -\frac{1}{6}\sin 4x + (A + \frac{1}{2}x)\cos 2x + B\sin 2x$.

5. (1) $y = Ax + Bx^2$. (2) $y = Ax + Bx^3 - x^2$.

(3) $y = A + B\log x + \frac{1}{2}a(\log x)^2$. (4) $y = A + B\log x + \frac{1}{6}(\log x)^3$.

(5) $y = Ax^{\frac{3}{2}} + Bx + \frac{1}{2}\{\cos(\log x) - 5\sin(\log x)\}$.

(6) $y = (A + \frac{2}{3}\log x)\sqrt{x} + \dfrac{B}{x}$.

(7) $y = \frac{1}{4}x + A\cos(\sqrt{3}\log x) + (B + \dfrac{1}{2\sqrt{3}}\log x)\sin(\sqrt{3}\log x)$.

6. (1) $y = \cosh x + A\cos x + B\sin x$.

(2) $x = \dfrac{1}{2a}\operatorname{cosec}^2\alpha(\cosh at - \cos\alpha\sinh at) + e^{-at\cos\alpha}\{A\cos(at\sin\alpha)$
$+ B\sin(at\sin\alpha)\}$.

CHAPTER XXXIV.

SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS--ORTHOGONAL TRAJECTORIES.

529. Simultaneous Equations.

Let x and y be functions of a third variable t.

Then the most general form of the linear equation of the first order involving both x and y is

$$a_1\frac{dx}{dt} + b_1\frac{dy}{dt} + c_1x + d_1y = f(t).$$

Or, dividing down by a, and using the operator $D(\equiv d/dt)$, we may write the equation in the form

$$(D + a)x + m(D + b)y = f_1(t).$$

We shall now proceed to solve the two simultaneous equations,

$$\left.\begin{array}{ll}(D + a)x + m(D + b)y = f_1(t) & (1)\\ (D + a')x + m'(D + b')y = f_2(t) & (2)\end{array}\right\}.$$

The method of solution is analogous to that adopted in ordinary simultaneous equations in algebra.

To eliminate y, operate on (1) with $m'(D+b')$, and on (2) with $m(D + b)$. We thus have

$$m'(D + b')(D + a)x + m'm(D + b')(D + b)y = m'(D + b')f_1(t),$$

$$m(D + b)(D + a')x + m'm(D + b)(D + b')y = m(D + b)f_2(t).$$

Since the coefficients of y are equal, we have, by subtraction,

$$\{m'(D + b')(D + a) - m(D + b)(D + a')\}x$$
$$= m'(D + b')f_1(t) - m(D + b)f_2(t),$$

which, after reduction, may be written in the form

$$(\alpha D^2 + \beta D + \gamma)x = \phi(t).$$

Solving this equation by the methods of the last chapter, we may substitute the value of x in either (1) or (2), and then find y; or, which is better, first eliminate Dy between (1) and (2), and then substitute for x its value, in the resulting equation, and so find y *without integration.*

Ex. 1. *Solve*
$$\left.\begin{aligned}\frac{dx}{dt}+2\frac{dy}{dt}+x-y&=2t+1 \quad (a)\\ 2\frac{dx}{dt}+\frac{dy}{dt}+x-y&=2t-1 \quad (b)\end{aligned}\right\}.$$

We have
$$(D+1)x+(2D-1)y=2t+1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$(2D+1)x+(D-1)y=2t-1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$
and $[(2)\times(2D-1)]-[(1)\times(D-1)]$ gives
$$\{(2D+1)(2D-1)-(D+1)(D-1)\}x=(2D-1)(2t-1)-(D-1)(2t+1);$$
whence, remembering that $Dt=1$,
$$3D^2x=4-(2t-1)-2+(2t+1)=4;$$
$$\therefore Dx=\tfrac{4}{3}t+c_1;$$
$$\therefore x=\tfrac{2}{3}t^2+c_1t+c_2.$$

Multiplying (2) by 2, and subtracting (1) from it, we have
$$\{2(2D+1)-(D+1)\}x+\{2(D-1)-(2D-1)\}y=2(2t-1)-(2t+1)$$
or
$$(3D+1)x-y=2t-3.$$
$$\begin{aligned}\therefore y&=3Dx+x-2t+3\\ &=4t+3c_1+\tfrac{2}{3}t^2+c_1t+c_2-2t+3\\ &=\tfrac{2}{3}t^2+(2+c_1)t+3c_1+c_2+3.\end{aligned}$$

NOTE.—Although c_1 and c_2 are arbitrary constants, yet their values in the expression for y must be the same as in the expression for x.

Otherwise:—In this particular example, it happens that we may adopt an alternative method, thus:—

(b) − (a) gives $\dfrac{dx}{dt}-\dfrac{dy}{dt}=-2$, whence $x-y=-2t+a_1$ . . (3)

(b) + (a) gives $3\left(\dfrac{dx}{dt}+\dfrac{dy}{dt}\right)+2(x-y)=4t,$

or
$$3\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=4t+4t-2a_1=8t-2a_1.$$
$$\therefore x+y=\tfrac{4}{3}t^2-\tfrac{2}{3}a_1t+a_2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3) and (4), $x = \frac{2}{3}t^2 - \left(\frac{a_1}{3} + 1\right)t + \frac{a_1 + a_2}{2}$,

$$y = \tfrac{2}{3}t^2 - \left(\frac{a_1}{3} - 1\right)t + \frac{a_2 - a_1}{2}.$$

Putting $-\left(\frac{a_1}{3} + 1\right) = c_1$ and $\frac{a_1 + a_2}{2} = c_2$, we can easily verify that

$$-\left(\frac{a_1}{3} - 1\right) = 2 + c_1 \text{ and } \frac{a_2 - a_1}{2} = 3c_1 + c_2 + 3;$$

in which case we obtain the same results as before.

Ex. 2. *Solve* $(2D + 3)x + (D - 1)y = \cos t + e^{2t}$ . . . . . . (1)

$$(D + 2)x + (D + 1)y = \sin t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$[(1) \times (D + 1)] - [(2) \times (D - 1)]$ gives

$\{2D^2 + 5D + 3 - (D^2 + D - 2)\}x = (D + 1)(\cos t + e^{2t}) - (D - 1)\sin t,$

or $(D^2 + 4D + 5)x = -\sin t + 2e^{2t} + \cos t + e^{2t} - (\cos t - \sin t) = 3e^{2t}.$

$$\therefore x = \frac{3}{2^2 + 4.2 + 5}e^{2t} + \text{the (c.f.)}$$

$$= \tfrac{3}{17}e^{2t} + e^{-2t}(A\cos t + B\sin t).$$

To find y, we have, on subtracting (2) from (1),

$$(D + 1)x - 2y = \cos t - \sin t + e^{2t}.$$

$$\therefore y = \tfrac{1}{2}(D + 1)[\tfrac{3}{17}e^{2t} + e^{-2t}(A\cos t + B\sin t)] - \tfrac{1}{2}(\cos t - \sin t + e^{2t})$$

$$= \tfrac{1}{2}[\tfrac{6}{17}e^{2t} + e^{-2t}\{-A\sin t + B\cos t - 2(A\cos t + B\sin t)\}]$$

$$+ \tfrac{1}{2}[\tfrac{3}{17}e^{2t} + e^{-2t}(A\cos t + B\sin t)] - \tfrac{1}{2}(\cos t - \sin t + e^{2t})$$

$$= \tfrac{9}{34}e^{2t} + \tfrac{1}{2}e^{-2t}\{(B - A)\cos t - (B + A)\sin t\} - \tfrac{1}{2}(\cos t - \sin t + e^{2t})$$

$$= \tfrac{1}{2}\{e^{-2t}(B - A) - 1\}\cos t - \tfrac{1}{2}\{e^{-2t}(B + A) - 1\}\sin t - \tfrac{4}{17}e^{2t}$$

Equations of higher orders than the first admit of a similar treatment, but we shall not discuss them here.

530. Orthogonal Trajectories.

Let $$f(x, y, a) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be a system, or family, of curves containing a single arbitrary parameter a. Then, if a second family of curves be drawn, each member of which cuts each member of (1) at a given angle α (or

according to some other fixed law), then every member of the second family is called a *trajectory* of the given system of curves.

If the angle α is a right angle, the trajectory is called *orthogonal.*

For example, the orthogonal trajectories of a system of concentric circles is evidently a system of straight lines through the origin.

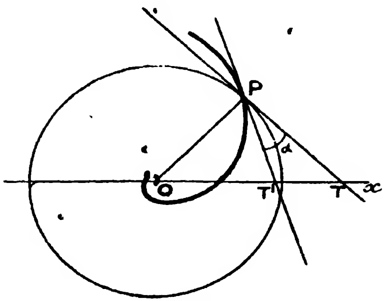

FIG. 127.

We can easily show that the *oblique* trajectories are a system of equiangular spirals, as follows:—

Let PT be a tangent to one of the circles, and PT' a tangent to one of the trajectories, the angle TPT' being constant and equal to α. Then, $\angle OPT' = \frac{\pi}{2} - \alpha$, which is constant; but this is the fundamental property of the equiangular spiral. Hence, the above statement.

Again, if on an ordnance map a system of contour lines be drawn, the orthogonal trajectories of the system will be the lines of greatest slope.

Or, again, suppose a plane lamina in the form of an irregular closed curve to be charged with electricity; then the system of equipotential lines in the plane of the lamina have for their orthogonal trajectories the lines of force in that plane.

531. To obtain the Equation to the Trajectory.

Since $f(x, y, a) = 0$ has only one arbitrary constant, we may obtain a differential equation of the first order in which a does not appear.

Let this be $$\phi(x, y, dy/dx) = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now consider a particular point (x, y) on one of the curves; let p be the value of dy/dx for this curve, and p' the value of dy/dx for the trajectory.

Then $p = \tan\psi$; $p' = \tan(\psi + \alpha)$, α being $+^{ve}$ or $-^{ve}$.

And to find the equation to the trajectory we must find a relation between x, y, and p'. Now (2) contains a relation between x, y, and p; and if we can express p in terms of p' we shall obtain the relation we require.

Thus $p = \tan\{(\psi+\alpha)-\alpha\} = \dfrac{\tan(\psi+\alpha)-\tan\alpha}{1+\tan(\psi+\alpha)\tan\alpha} = \dfrac{p'-\tan\alpha}{1+p'\tan\alpha}.$

Hence in (2) we have $\phi\left\{x, y, \dfrac{p'-\tan\alpha}{1+p'\tan\alpha}\right\} = 0$; and, now that there is no risk of confusion, we may write dy/dx for p', so that the equation to the oblique trajectory is

$$\phi\left\{x, y, \frac{dy/dx - \tan\alpha}{1 + dy/dx.\tan\alpha}\right\} = 0.$$

If $\alpha = \dfrac{\pi}{2}$ we have the orthogonal trajectory, whose equation is

$$\phi\left\{x, y, -\frac{dx}{dy}\right\} = 0.$$

By integration we obtain the *system* of trajectories.

532. Examples.

Ex. 1. *Find the oblique trajectories of the system of concentric circles* $x^2+y^2=c^2$.

The differential equation is $x + y\dfrac{dy}{dx} = 0$ . . . . (1)

Referring to the figure of Art. 530,

$$p = \tan PTx;$$
$$p' = \tan PT'x.$$

Expressing p in terms of p', we have

$$p = \tan(PT'x + \alpha) = \frac{p'+a}{1-p'a}, \text{ where } a \equiv \tan\alpha;$$

$$\therefore \text{ in (1) } x + y.\frac{p'+a}{1-p'a} = 0,$$

or
$$x + ay + \frac{dy}{dx}(y - ax) = 0.$$

Putting $y = vx$, $(1 + av)dx + (v - a)(vdx + xdv) = 0$,

or $$(1 + v^2)dx + (v - a)xdv = 0;$$

$$\therefore \frac{dx}{x} + \frac{v - a}{1 + v^2}dv = 0.$$

$$\therefore \log x + \tfrac{1}{2}\log(1 + v^2) - a\tan^{-1} v = C,$$

or $$\tfrac{1}{2}\log(x^2 + y^2) - a\tan^{-1}\frac{y}{x} = C.$$

Putting $$x = r\cos\theta, \quad y = r\sin\theta,$$

$$\log r = a\theta + C;$$

$$\therefore r = c_1 e^{a\theta} = c_1 e^{\theta \tan \alpha},$$

or, if $\angle OPT = \alpha'$, $r = c_1 e^{\theta \cot \alpha'}$,

which is the polar equation to a system of equiangular spirals, c_1 being an arbitrary constant.

Ex. 2. *Find the orthogonal trajectories of a system of confocal conics.*

Let their equations be

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \quad \ldots \ldots \quad (1)$$

Differentiating, $$\frac{x}{a^2 + \lambda} + \frac{yp}{b^2 + \lambda} = 0,$$

or $$\frac{a^2 + \lambda}{x} = \frac{b^2 + \lambda}{-yp} = \frac{a^2 - b^2}{x + yp} \text{ (by principle of ratios).}$$

$\therefore$ in (1), eliminating $a^2 + \lambda$ and $b^2 + \lambda$,

$$\frac{x^2(x + yp)}{x} + \frac{y^2(x + yp)}{-yp} = a^2 - b^2$$

or $$(xp - y)(x + yp) = p(a^2 - b^2) \quad \ldots \ldots \quad (2)$$

To find the orthogonal trajectory, change p into $-\dfrac{1}{p}$, and we have

$$\left(-\frac{x}{p} - y\right)\left(x - \frac{y}{p}\right) = -\frac{1}{p}(a^2 - b^2);$$

$$\text{i.e. } (xp - y)(x + yp) = p(a^2 - b^2) \quad \ldots \ldots \quad (3)$$

But (3) is the same equation as (2).

Hence the integral of (3) is (1). That is to say, the orthogonal trajectories are the members of the original system of confocal conics.

533. Polar Coordinates.

The method adopted in this case is very similar to that given above.

If ϕ be the angle between the tangent and radius vector, so that $\tan\phi = r\dfrac{d\theta}{dr}$; then $\phi + \alpha$ will be the corresponding angle for the trajectory.

If $p = \tan\phi$, and $p' = \tan(\phi + \alpha)$; we must express p in terms of p'; thus $p = \dfrac{p' - \tan\alpha}{1 + p'\tan\alpha}$.

Hence, if $f(r, \theta, a) = 0$ is the equation to the original system, and $f(r, \theta, dr/d\theta) = 0$ is the differential equation; we have to change $r\dfrac{d\theta}{dr}$ into $\dfrac{r\dfrac{d\theta}{dr} - \tan\alpha}{1 + r\dfrac{d\theta}{dr}\tan\alpha}$, when we shall obtain the required equation.

If $\alpha = \dfrac{\pi}{2}$, the second expression becomes $= -\dfrac{1}{r}\dfrac{dr}{d\theta}$.

Hence we must change $r\dfrac{d\theta}{dr}$ into $-\dfrac{1}{r}\dfrac{dr}{d\theta}$,† or $\dfrac{d\theta}{dr}$ into $-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, or $\dfrac{dr}{d\theta}$ into $-r^2\dfrac{d\theta}{dr}$.

Ex. *Find the orthogonal trajectories of the parabolas* $2a = r(1 + \cos\theta)$.

Differentiating, $\qquad 0 = -r\sin\theta + (1 + \cos\theta)\dfrac{dr}{d\theta}$.

The orthogonal trajectory is

$$-r\sin\theta - r^2(1 + \cos\theta)\frac{d\theta}{dr} = 0,$$

or

$$\frac{dr}{r} + \cot\frac{\theta}{2}d\theta = 0.$$

---

† This is a similar rule to that given in the case of cartesians; $r\dfrac{d\theta}{dr}$ is inverted and its sign changed.

Integrating, $$\log r + 2\log\sin\frac{\theta}{2} = C;$$

$$\therefore\ r\sin^2\frac{\theta}{2} = c,$$

$$\text{or } 2c = r(1 - \cos\theta),$$

a second family of parabolas having the same focus and axis, but all turned in the opposite direction.

EXAMPLES LXXIX.

1. Solve the simultaneous equations:—

(1) $\dfrac{dx}{dt} + ay = 0;$ $\quad\dfrac{dy}{dt} + ax = 0.$

(2) $\dfrac{dx}{dt} + 3x + 2y = 0;$ $\quad\dfrac{dy}{dt} + 2x + 3y = 0.$

(3) $\dfrac{dx}{dt} + y = e^t;$ $\quad\dfrac{dy}{dt} - x = e^{-t}.$

(4) $\dfrac{dx}{dt} + ay = t;$ $\quad\dfrac{dy}{dt} - ax = t.$

(5) $\dfrac{dx}{dt} + 2y = \cos t;$ $\quad\dfrac{dy}{dt} - x = -\sin t.$

(6) $\dfrac{dx}{dt} + \dfrac{dy}{dt} = 2(x + y);$ $\quad\dfrac{dy}{dt} = 3x + y$

(7) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2y = t;$ $\quad\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = 2t.$

(8) $\dfrac{dx}{dt} + \dfrac{dy}{dt} - y = t;$ $\quad\dfrac{dx}{dt} + 2\dfrac{dy}{dt} + x = e^t.$

(9) $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - y = \cos t;$ $\quad 3\dfrac{dx}{dt} - 2\dfrac{dy}{dt} - x - 5y = e^t.$

2. Find the orthogonal trajectories of the following curve-families:—

(1) $y = Cx$, C being a variable parameter.

(2) The conics $ax^2 + by^2 = 1$, a being variable.

(3) $y = x^c$, c being variable.

(4) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, a being variable.

(5) The parabolas $\sqrt{x} + \sqrt{y} = C$, C being variable.

(6) $y = a \cosh x$, a variable.

(7) $y = a \cosh \dfrac{x}{a}$, a variable.

(8) $p = r \cos \theta$, p being variable.

(9) $r = c \cos \theta$, a system of coaxial circles in which the radical axis is a tangent.

(10) $r = c(1 - \cos \theta)$, a family of cardioides.

(11) $l/r = 1 + e \cos \theta$, e being variable.

3. Find the orthogonal trajectories of a family of concentric conics of constant eccentricity.

Hence, show that if the conics are equilateral hyperbolas, the orthogonal trajectories are a second set of equilateral hyperbolas, the axes and asymptotes of the two sets being interchanged. Examine the cases in which $e = 1/\sqrt{2}$ and 0.

4. Show that the orthogonal trajectories of the system of curves

$$x^n/a^n + y^n/b^n = C, \ C \text{ being variable,}$$

are given by

$$a^n x^{2-n} - b^n y^{2-n} = C_1.$$

Examine the cases in which $n = 1, 2, 3$.

5. Show that the orthogonal trajectories of a family of similar concentric equiangular spirals are also equiangular spirals.

6. Show that the orthogonal trajectories of the system of coaxial circles

$$x^2 + y^2 - a^2 = kx, \ k \text{ being variable,}$$

are the system of coaxial circles given by

$$x^2 + y^2 + a^2 = k_1 y. \quad \text{[See Ans.]}$$

7. The polar equation of the above system being $r^2 - a^2 = kr \cos \theta$: find the orthogonal trajectories.

Answers.

1. (1) $x = Ae^{at} + Be^{-at}$; $y = Be^{-at} - Ae^{at}$.

(2) $x = e^{-3t}(A \cos 2t + B \sin 2t)$. $y = e^{-3t}(A \sin 2t - B \cos 2t)$.

(3) $x = \sinh t + A \cos t + B \sin t$; $y = \sinh t - B \cos t + A \sin t$.

(4) $x = (1 - at)/a^2 + A \cos at + B \sin at$; $y = (1 + at)/a^2 + A \sin at - B \cos at$.

(5) $x = \sin t + A\cos\sqrt{2}t + B\sin\sqrt{2}t$;
$y = \frac{1}{\sqrt{2}}(A\sin\sqrt{2}t - B\cos\sqrt{2}t)$.

(6) $x = Ae^{2t} + Be^{-2t}$;
$y = 3Ae^{2t} - Be^{-2t}$.

(7) $x = \frac{1}{2}(4t - 5) + Ce^{-3t}$;
$y = \frac{1}{4}(2t - 5) + \frac{1}{2}Ce^{-3t}$.

(8) $x = 2 + A\cos t + B\sin t$;
$y = \frac{1}{2}\{e^t - 2t - (A - B)\cos t - (A + B)\sin t - 2\}$.

(9) $x = \frac{1}{21}(2\sin t - 5\cos t) + Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t}$;
$y = \frac{1}{21}(3\sin t + \cos t) - \frac{1}{7}e^t + (\sqrt{2} - 1)Ae^{\sqrt{2}t} - (\sqrt{2} + 1)Be^{-\sqrt{2}t}$.

2. (1) $x^2 + y^2 = C^2$. (2) $b(x^2 + y^2) = 2(\log y + C)$.

(3) $x^2(2\log x - 1) + y^2(2\log y - 1) + C = 0$.

(4) $x^{\frac{4}{3}} - y^{\frac{4}{3}} = c^{\frac{4}{3}}$. (5) $x^{\frac{3}{2}} - y^{\frac{3}{2}} = c^{\frac{3}{2}}$. (6) $\log\sinh x + \frac{1}{2}y^2 = c$.

(7) $\log\tanh\frac{x}{2} + y = C$. (8) $p_1 = r\sin\theta$. (9) $r = c\sin\theta$.

(10) $r = c(1 + \cos\theta)$. (11) $\log r\sin\theta = r/l + C$.

3. $y = Cx^n$, where $n = 1/(1 - e^2)$.

4. If $n = 2$, the orthogonal trajectories are $x^{a^2} = cy^{b^2}$.

6. The differential equation of the orthogonal trajectories is

$$(x^2 - y^2 + a^2)dy - 2xy\,dx = 0$$

which may be written $\frac{y^2 - a^2}{y^2}dy + d\left(\frac{x^2}{y}\right) = 0.$

7. $r^2 + a^2 = k_1 r\sin\theta$.

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THE END.

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